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NUMERICAL SOLUTION OF A NONLINEAR PARABOLIC INVERSE PROBLEM USING OPTIMAL CONTROL AND FINITE ELEMENT METHODS

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Abstract. This paper deals with numerical approximations of an identification inverse problem governed by a highly nonlinear parabolic equation. In order to solve our problem, an optimal control formulation is proposed. The associated optimization problem is solved using a gradient-type method. The nonlinear state problem is discretized by the finite element scheme and solved by Newton's method. Finally, to demonstrate the efficiency of the proposed approach, numerical experiments are presented.

1. Introduction and preliminaries

Predicting fluid movement in unsaturated and saturated zones is of significant importance across a wide range of domains, including agriculture, hydrology, environmental science, civil engineering, and biomechanics [26, 25, 3, 29, 14]. Indeed, there are plenty of societal relevant applications of flow in porous media, e.g. water and soil pollution, CO2 storage, nuclear waste management, or enhanced oil recovery and other [4, 6]. The mathematical model of unsaturated porous media flow is usually expressed by the Richards equation [28], where the flow originates from pressure and gravitational forces. The theoretical and numerical studies of problems governed by such types of equations generate considerable difficulties [1, 16, 22]. The reason lies in the highly nonlinear and degenerate parabolic nature of the equation.

The frequent used formulation of Richards equation is known as the head-based and it is given by

$$C(\psi)\frac{\partial \psi}{\partial t} - \frac{\partial}{\partial z}(K(\psi)\frac{\partial \psi}{\partial z}) + \frac{\partial K(\psi)}{\partial z} = f, \quad (z,t) \in (0,1) \times (0,T), \tag{1.1}$$

where the real valued functions $K(\psi)$ and $C(\psi)$ represent respectively the hydraulic conductivity and hydraulic capacity. Here $K(\psi)$ describes the ease with

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which the water can move through pore spaces. It depends on the intrinsic permeability of the material, on the degree of saturation and on the density and viscosity of the fluid. These functions and the other parameters which define the model of Richards equation are, in the most cases, unknowns or badly measured. It is then necessary to identify them, we have then to solve a parameter identification inverse problem. This class of inverse problems is nowadays an active research area, see for example [9, 27].

When all the parameters are assumed to be known, the problem is usually referred to as a direct problem. In contrast, if at least one of the parameters is unknown, the problem is considered an inverse one.

At this stage, it must be mentioned, that the direct problem is extensively studied both theoretically and numerically (see for example [2, 10, 12, 11, 21, 30, 17, 18]).

In contrast, few mathematical studies have focused on its corresponding inverse problem. In fact, it is well known that the inverse problems are generally nonlinear and ill-posed in the sense of Hadamard [13]. Beside these difficulties, the considered inverse problem is governed by a highly nonlinear parabolic equation. Some numerical techniques for the approximation of this inverse problem, by using more observations data, were proposed. More precisely, Ngnepieba et al. in [23] used the finite difference method for the discretization of the inverse problem and propose a gradient descent method as a minimization approach of the infiltration rate. While in [5], the authors used the splines functions and the conjugate gradient method with two observations (the first one, in the edge and the other one on the infiltration flow). However, in these works, although many observational data points were used, their approaches cannot simultaneously identify all unknown parameters—a key challenge in solving inverse problems. To address this limitation, Johri et al. [14] proposed a first-approximation technique that enables simultaneous parameter identification using pressure observations at specific spatial points. We propose an optimal control formulation, where a finite difference scheme is used for discretizing the state problem and a genetic algorithm solves the resulting minimization task.

In this work, in order to propose a more efficient approximation method, for identifying the hydraulic conductivity $K(\psi)$ and hydraulic capacity $C(\psi)$ simultaneously in there general form based on a final time observation, we opt for an approach based on the reformulation of the problem into an adequate optimal control formulation, in where $K(\psi)$ and $C(\psi)$ are the control variables. Then, we solve the obtained optimization problem using technics based on the gradient-type method. So the gradient is derived via adjoint problem which is obtained thanks to the Lagrange multiplier approach. The approximation of the state problem is based on a semi-linearization of the temporal term and the finite elements scheme. The nonlinear discretized problem is then solved by Newton's method.

Our paper is organized as follows, in section 2, we present the setting of the inverse problem and our proposed optimal control formulation. In section 3, we give the variational problems and derived the gradient formula. The section 4 is devoted to the discretization of the optimal control problem and numerical results of model examples.

2. Setting of the problem

We are interest in a numerical realization, using the finite element method, of the identification problem governed by the following Richards equation:

$$\begin{cases} C(\psi)\dfrac{\partial\psi}{\partial t} - \dfrac{\partial}{\partial z}(K(\psi)\dfrac{\partial\psi}{\partial z}) + \dfrac{\partial K(\psi)}{\partial z} = f, \quad (z,t) \in (0,1)\times(0,T), \\ \psi(0,t) = \psi(1,t) = 0, \quad \forall\, t \in (0,T), \\ \psi(z,0) = \psi_0, \qquad \forall\, z \in (0,1). \end{cases} \tag{2.1}$$

The aim of this problem is to find the pressure ψ, the hydraulic conductivity $K(\psi)$ and the hydraulic capacity $C(\psi)$ based on the following additional observations on the pressure taken at the finale time

$$\psi(z,T) = g(z), \qquad z \in (0,1). \tag{2.2}$$

For a given $C(\psi)$ and $K(\psi)$, the direct problem (2.1) is well posed (see for example [20]). In order to solve the inverse problem (2.1)-(2.2), we assume that:

(H1): $C(\psi), K(\psi) \in C^1(\mathbb{R})$, bounded and uniformly Lipschitz continuous,
(H2): $g(z) \in L^2(0,1),\ f(z,t) \in L^2(0,T,L^2(0,1))$.

The above inverse problem can be reformulated as the following optimal control one:

$$\begin{cases} \text{Find } P^* \in U_{ad} \text{ solution of} \\ J(P^*) = \inf\limits_{P\in\mathcal{U}_{ad}} J(P) \\ \text{where } J(P) = \dfrac{1}{2}\displaystyle\int_\Omega |\psi(z,T) - g(z)|^2\, dz, \\ \text{and } \psi \text{ is the solution of :} \\ \begin{cases} C(\psi)\dfrac{\partial\psi}{\partial t} - \dfrac{\partial}{\partial z}(K(\psi)\dfrac{\partial\psi}{\partial z}) + \dfrac{\partial K(\psi)}{\partial z} = f, \quad (z,t) \in (0,1)\times(0,T), \\ \psi(0,t) = \psi(1,t) = 0, \quad \forall\, t \in (0,T), \\ \psi(z,0) = \psi_0, \qquad \forall\, z \in (0,1), \end{cases} \end{cases} \tag{2.3}$$

where $P = (K(\psi), C(\psi))$ is the parameter to be estimated in (2.3) and $\mathcal{U}_{ad}$ is the set of admissible controls defined by:

$$\begin{aligned} \mathcal{U}_{ad} =& \{P = (C(\psi), K(\psi)) \in C^1(\mathbb{R}) \times C^1(\mathbb{R})/0 < d_0 \le C(\psi) \le d_1 \text{ and } 0 < \lambda_0 \le K(\psi) \le \lambda_1 \\ & a.e. \text{ on } \Omega \text{ and } |C(s_1) - C(s_2)| \le L_0|s_1 - s_2| \text{ and } |K(s_1) - K(s_2)| \le L_1|s_1 - s_2|,\ \forall s_1, s_2 \in \mathbb{R}\}. \end{aligned}$$

$\mathcal{U}_{ad}$ defines the class of admissible functions that are both uniformly bounded and uniformly Lipschitz continuous.

Remark 2.1. If the problem (2.1)-(2.2) admits a solution $(\psi, P = (C, K))$ with $P \in U_{ad}$. Then, the problem (2.1)-(2.2) is equivalent to the problem (2.3). Note that our proposed optimal control formulation presents from a practical point of view, a precise mean for solving an inverse Richards problem, even if the equivalence result is not satisfied.

3. Variational problem

To introduce the concept of weak solution, we use the standard Sobolev spaces $H^1(0,1), H_0^1(0,1)$, [19, 32]. Further, for a Banach space B, we define

$$L^2(0,T;B) = \{u : u(t) \in B\, a.e.\ t \in (0,T) \text{ and } ||u||_{L^2(0,T;B)} < \infty\},$$

equipped with the norm

$$||u||^2_{L^2(0,T;B)} = \int_0^T ||u(t)||^2_B \, dt.$$

In the sequel, we shall use the space $W_0(0,T)$ defined as

$$W_0(0,T) = \{u : u \in L^2(0,T;H_0^1(0,1)), u_t \in L^2(0,T;H^{-1}(0,1))\},$$

equipped with the norm

$$||u||^2_{W_0(0,T)} = ||u||^2_{L^2(0,T;H_0^1(0,1))} + ||u_t||^2_{L^2(0,T;H^{-1}(0,1))}.$$

Then a weak solution in $W_0(0,T)$ of the direct problem is a function $\psi(z,t) \in W_0(0,T)$ satisfying the identity

$$\int_0^T < C(\psi)\frac{\partial \psi}{\partial t}, v >_{(H^{-1}(\Omega),H_0^1(\Omega))} dt + \int_0^T \int_0^1 K(\psi)\frac{\partial \psi}{\partial z}\frac{\partial v}{\partial z}\, dz\, dt + \int_0^T \int_0^1 \frac{\partial K(\psi)}{\partial z} v\, dz\, dt \tag{3.1}$$

$$= \int_0^T \int_0^1 f v \, dz\, dt,$$

$$\forall\, v \in L^2(0,T,H_0^1(0,1))$$

and

$$\psi(z,0) = \psi_0(z), \qquad z \in (0,1).$$

Note that the problem admits a unique solution in $W_0(0,T)$ [14, 15]. Furthermore, there is a positive constant c_d independent of $C(\psi)$, $K(\psi)$ and f such that

$$||\psi||_{W_0(0,T)} \le c_d.$$

Below we outline the key steps of the proof, while noting that a complete technical exposition will be presented in a forthcoming paper currently in submission.

From a theoretical standpoint, the existence and uniqueness of the state variable ψ are rigorously established within the functional space $W_0(0,T)$. The analysis is built upon a set of a priori estimates ensuring boundedness in appropriate norms, which are crucial for handling the nonlinearities of the problem. The existence result is obtained by constructing a fixed-point operator acting on a closed ball in $W_0(0,T)$ and applying Schauder's fixed-point theorem. Compactness is ensured via the Aubin–Lions lemma, and continuity follows from energy estimates. For uniqueness, a monotonicity argument based on the Riesz isomorphism and the uniform ellipticity of the diffusion operator, combined with a Lipschitz condition on the hydraulic conductivity, leads to a differential inequality that yields uniqueness via Gronwall's lemma. This rigorous functional analysis provides a solid foundation for the numerical scheme and justifies the well-posedness of the forward problem.

We denote the solution by $\psi(z,t;P)$ or $\psi(P)$, to emphasize its dependence on $P=(C(\psi),K(\psi))$. To identify P, we introduce the lagrangian operator:

$$L = J(\psi(z,t;P)) + \int_0^T\int_0^1 C(\psi)\frac{\partial\psi}{\partial t}v(z,t)\,dz\,dt + \int_0^T\int_0^1 K(\psi)\frac{\partial\psi}{\partial z}\frac{\partial v(z,t)}{\partial z}\,dz\,dt$$
$$+\int_0^T\int_0^1 \frac{\partial K(\psi)}{\partial z}v(z,t)\,dz\,dt - \int_0^T\int_0^1 f\,v(z,t)\,dz\,dt. \tag{3.2}$$

Next, by using the Gâteaux differentiability of L, we derive a formula for the gradient of J. For this, we suppose first that $C(\psi)$ and $K(\psi)$ belong to a specific known class of functions, such as the class of polynomials of degree at most k, namely

$$P_k = \{\sum_{i=0}^{k} a_i\psi^i | a_i \in \mathbb{R}, \quad i=\overline{0,k}\}.$$

The proposed approach reduces function estimation (an infinite-dimensional problem) to parameter estimation (a finite-dimensional one), also aims to improve the efficiency of the numerical approximation. It is also physical, since, for example, the authors in [24] employed a fourth-order power law to model the experimental data. This leads us to calculate the gradient with respect to the coefficients and to have indeed the following theorem.

Theorem 3.1. *The functional J is Gâteaux differentiable and its gradient ∇J at P has the form*

To solve the inverse problem, a Lagrangian functional is introduced:

$$\mathcal{L}(\psi_P, v, P) = J(\psi_P) + \int_0^T\int_0^1 v\left(-C(\psi_P)\frac{\partial\psi_P}{\partial t} + \frac{\partial}{\partial z}\left(K(\psi_P)\frac{\partial\psi_P}{\partial z}\right)\right.$$
$$\left. - \frac{\partial K(\psi_P)}{\partial z} + f dz\,dt, \right.$$

where v is a Lagrange multiplier enforcing the PDE constraint that links the control P and the state ψ_P.

Using Green's formula, the Lagrangian is rewritten in a weak form. Then, by computing the Gâteaux derivative of $\mathcal{L}$ with respect to ψ_P, the adjoint equation is derived:

$$\begin{cases} C(\psi_P)\frac{\partial v}{\partial t} + \frac{\partial}{\partial z}\left(K(\psi_P)\frac{\partial v}{\partial z}\right) + \frac{\partial K(\psi_P)}{\partial \psi_P}\frac{\partial v}{\partial z} = \psi_P(z,T) - g(z), & \text{in } Q_T, \\ v(0,t) = v(1,t) = 0, & t \in (0,T), \\ v(z,T) = 0, & z \in (0,1). \end{cases} \tag{3.3}$$

The adjoint equation is a backward parabolic PDE that encodes the sensitivity of the cost functional to perturbations in the state.

Next, by computing the derivative of $\mathcal{L}$ with respect to the control parameters $P = (q_i, s_i)$—which are the coefficients of polynomial representations of $K(\psi)$ and $C(\psi)$—the gradient of the cost functional with respect to the control is obtained:

$$\frac{\partial J}{\partial q_i} = \int_0^T \int_0^1 \left(-\frac{\partial K}{\partial q_i} \frac{\partial \psi}{\partial z} \frac{\partial v}{\partial z} + \frac{\partial K}{\partial q_i} \frac{\partial v}{\partial z} \right) dz\, dt, \tag{3.4}$$

$$\frac{\partial J}{\partial s_i} = \int_0^T \int_0^1 \frac{\partial C}{\partial s_i} \frac{\partial \psi}{\partial t} v\, dz\, dt. \tag{3.5}$$

where $K(\psi) = \sum_{i=0}^{k} q_i \psi^i$, $C(\psi) = \sum_{i=0}^{k} s_i \psi^i$, *$\psi$ is the solution of direct problem and v is the solution of the following adjoint state problem: for* $(z,t) \in (0,1)\times(0,T)$,

$$\begin{cases} -C(\psi)\dfrac{\partial v}{\partial t} - \dfrac{\partial}{\partial z}(K(\psi)\dfrac{\partial v}{\partial z}) + (\dfrac{\partial K(\psi)}{\partial \psi}\dfrac{\partial \psi}{\partial z})\dfrac{\partial v}{\partial z} + (\dfrac{\partial C(\psi)}{\partial \psi}\dfrac{\partial \psi}{\partial t} + \dfrac{\partial}{\partial \psi}(\dfrac{\partial K(\psi)}{\partial z}))v = 0, \\ v(0,t) = v(1,t) = 0, \quad \forall\, t \in (0,T), \\ v(z,T) = g(z), \qquad \forall\, x \in (0,1). \end{cases} \tag{3.6}$$

4. Main results

4.1. numerical aspects of forward and inverse problem.

Solving an identification inverse problem in a highly nonlinear parabolic equation numerically is not an easy task. Indeed, even if the direct problem is linear (for ψ), the inverse problem and the parameter-to-output map are usually nonlinear. The situation in our case is more complicated because the direct problem is also nonlinear. In our work, we adopt a new strategy in order to reduce the computational cost.

4.2. Discretization of the inverse problem.

The following section, we propose an approximation of (2.3), we will then discretize both the state problem (2.1), the adjoint problem (3.6), the admissible family $\mathcal{U}_{ad}$ and the gradient of J. Accordingly, we introduce $P1$-piecewise approximations of $\mathcal{U}_{ad}$.

Let us consider a uniform partition $(z_n)_{n=0}^{d}$ of $[0;1]$, such that $0 = z_0 < z_1 < ... < z_d = 1$, $z_n = n * h$, $h = 1/d$, $n = 0, ..., d$.

We discretize the time interval $(0;T]$ into a series of M boundary elements, namely

$$(0,T] = \cup_{j=1}^{M}(t_{j-1}, t_j], \quad \text{with} \quad \Delta t = t_j - t_{j-1}.$$

We note that in order to obtain a faster solution algorithm, a reduction technique can be used. As at each iteration of solving process of the inverse problem, we need to solve the direct problem (2.1), which is nonlinear, one can use some technique of linearization which reduce the computation complexity of Newton method. We decided then to modify the discretized-in-times model of direct problem (2.1) by solving a semi-implicit problem. Indeed, we offer a semi-linearization with respect to the capacity of the equation (2.1). The rest of the time discretization is done using an implicit Euler scheme:

$$(Pd)\begin{cases} C(\psi^j(z))\dfrac{\psi^{j+1}(z)-\psi^j(z)}{\Delta t}-\dfrac{\partial}{\partial z}[K(\psi^{j+1}(z))\dfrac{\partial\psi^{j+1}(z)}{\partial z}]+\dfrac{\partial K(\psi^{j+1}(z))}{\partial z}=f^{j+1}(z),\\ \psi_0^{j+1}=0,\\ \psi_d^{j+1}=0,\\ \psi^0(z)=\psi_0, \end{cases} \tag{4.1}$$

for $j=1,..,M$.
Where $C(\psi^j(z))$, $K(\psi^j(z))$ and $f^{j+1}(z)$ are respectively, the approximates functions of $C(\psi)$, $K(\psi)$, and f at the time t_j.

4.3. **Discretization of the variational direct problem.** We begin by discretizing the Sobolev spaces, enabling the discrete variational formulation of the direct problem.

Let us denote by

$$H_h([z_n,z_{n+1}])=\{v_h\in C([0,1])\mid v_h|_{[z_n,z_{n+1}]}\in P_1([z_n,z_{n+1}])\},$$

$$H_0^h([z_n,z_{n+1}])=\{v_h\in H_h([z_n,z_{n+1}])\mid v_h|_{z_n}=v_h|_{z_{n+1}}=0\},$$

the finite dimensional spaces associated respectively to $H^1(0,1)$ and $H_0^1(0,1)$.

We note that the finite element method used here is the conforming one [7]. Then for any $P_h\in\mathcal{U}_{ad}^h$, the approximation $\psi_h:=\psi_h(P_h)\in H_0^h$ of $\psi\in H_0^1(0,1)$ is given by: $\psi_h=\sum_{n=0}^{n=d}\psi_h(z_n)\varphi_n$, where d is the number of the nodes of triangulation, and $(\varphi_n)_{n=1}^d$ is a basis function of H_0^h.

The discrete state problem reads

$$\begin{cases} \text{Find } \psi_h\in H_0^h, \text{ such that } \forall\varphi\in H_0^h\\ \displaystyle\int_0^1 K(\psi_h^{j+1})\frac{\partial\psi_h^{j+1}}{\partial z}\frac{\partial\varphi}{\partial z}\,dz+\int_0^1\frac{\partial K(\psi_h^{j+1})}{\partial z}\varphi\,dz,\\ \displaystyle+\frac{1}{\Delta t}\int_0^1 C(\psi_h^j)\psi_h^{j+1}\varphi\,dz=\frac{1}{\Delta t}\int_0^1 C(\psi_h^j)\psi_h^j\varphi\,dz+\int_0^1 f^{j+1}\varphi\,dz, \end{cases} \tag{4.2}$$

where ψ_h^j denotes the numerical approximation of ψ at the discrete time t_j, for all $z\in[0,1]$.

Despite the semi-linearization, the problem remains nonlinear. Therefore, Newton's method is employed to solve it.

Let's now denote by

$$G(\psi_h^{j+1}) = \begin{cases} \displaystyle\int_0^1 K(\psi_h^{j+1}) \frac{\partial \psi_h^{j+1}}{\partial z} \frac{\partial \varphi}{\partial z}\, dz + \int_0^1 \frac{\partial K(\psi_h^{j+1})}{\partial z} \varphi\, dz \\ \displaystyle + \frac{1}{\Delta t} \int_0^1 C(\psi_h^j) \psi_h^{j+1} \varphi\, dz - \frac{1}{\Delta t} \int_0^1 C(\psi_h^j) \psi_h^j \varphi\, dz + \int_0^1 f^{j+1} \varphi\, dz, \end{cases} \tag{4.3}$$

then the equation (4.2) is of the form

$$G(\psi_h^{j+1}) = 0. \tag{4.4}$$

The Newton-Raphson method [8], is one of the standard method for solving nonlinear equations. It consists to replace G in (4.3) by its approximation of first order, i.e. $G(\psi_h^{j+1} + w) = G(\psi_h^{j+1}) + G'(\psi_h^{j+1})w$. This gives the following iterative method:

Algorithm 1 Newton method

$k = 0, (\psi_h^{j+1})^0$ is given,
 $k \geq 0, (\psi_h^{j+1})^k$ being known,
 Calculate w^k solution of $G'((\psi_h^{j+1})^k) w^k = -G((\psi_h^{j+1})^k)$,
 $(\psi_h^{j+1})^{k+1} = (\psi_h^{j+1})^k + w^k$.

4.4. **Discretization of the optimal control formulation.** For the discretization of the optimal control problem, let us define a family of discrete admissible controls:

$$\mathcal{U}_{ad}^h = \{(C_h, K_h) \in C([0,1])^2 /\ C_h|_{[z_n, z_{n+1}]} \text{ and } K_h|_{[z_n,z_{n+1}]} \in \mathbb{P}_1([z_n, z_{n+1}]),\ \forall\, n = 1, \dots, d\} \cap \mathcal{U}_{ad},$$

and approach the cost functional by the following discrete one

$$J_h(P_h(\psi_h)) = \frac{1}{2} \sum_{k=1}^{d} \int_{z_k}^{z_{k+1}} (\psi_k(T) - g_k)^2\, dz,$$

where $\psi_h(T)$ is the solution at final time T.

We state our discrete optimal controls problem as follows

$$(PO) \begin{cases} \displaystyle\inf_{P_h \in \mathcal{U}_{ad}^h} J_h(\psi_h(P_h); P_h(\psi_h)) \\ \text{where } \psi_h(P_h) \text{ is the solution of (4.2)}, \end{cases} \tag{4.5}$$

with $\psi_h = (\psi_n^j)_{1 \leq n \leq d}^{1 \leq j \leq M}$.

In order to determine the minimizer of (4.5), we use the gradient method (GM) [31] summarized in the following algorithm:

Algorithm 2 Gradient method

choose $\epsilon > 0$.
Step 1: Set $k = 0$, initiate P_0.
Step 2: Calculate $d_0 = -\nabla J(P_0)$.
Step 3: Set $P_1 = P_0 + \rho.d_0$.
Step 4: For $k = 1, 2, ...$, Calculate $d_k = -\nabla J(P_k)$
Update

$$P_{k+1} = P_k + \rho.d_k.$$

until $||\nabla J(P_k)|| < \epsilon$.

4.5. **Numerical Examples.** In order to show the performance of the proposed approach, we validate our numerical tests against exact solution.

Accordingly, we consider the following data:

$$K(\psi) = \psi^2 + \psi + 1, \qquad C(\psi) = \psi^2 + \psi + 1,$$

and

$$f = 0, \qquad \psi_0 = 0, \qquad g(z) = 60 * (z - z^2),$$

for which the exact solution is:

$$\psi = t * (z - z^2).$$

For all following numerical examples, we take these numerical data

$$T = 60, \qquad N = 30, \qquad M = 11.$$

Due to the complexity of the considered identification problem, we will investigate different configurations.

First setup

We seek to identify the coefficients q_0 and q_1 related to the hydraulic conductivity, while we keep known the capacity. We consider then these two situations **First setup**

$$\text{case } 1 : \begin{cases} K(\psi) = \psi^2 + \psi + q_0, \\ C(\psi) = \psi + 1, \end{cases} \quad \text{case } 2 : \begin{cases} K(\psi) = \psi^2 + q_1\psi + q_0, \\ C(\psi) = \psi + 1. \end{cases}$$

So the identification problem in these situations, is reduced to find the parameters q_0 and q_1.

Figure 1 illustrate the decrease of the cost functional J_h with respect to the iteration numbers for the approximation of q_0 in first and both q_0 and q_1 simultaneously in the second. We note that, for these two cases, the cost function decreases quickly by approximately 12 orders of magnitude" (cost from 10^0 to 10^{-12}).

The approached values of q_0 and q_1 are given in the following table:

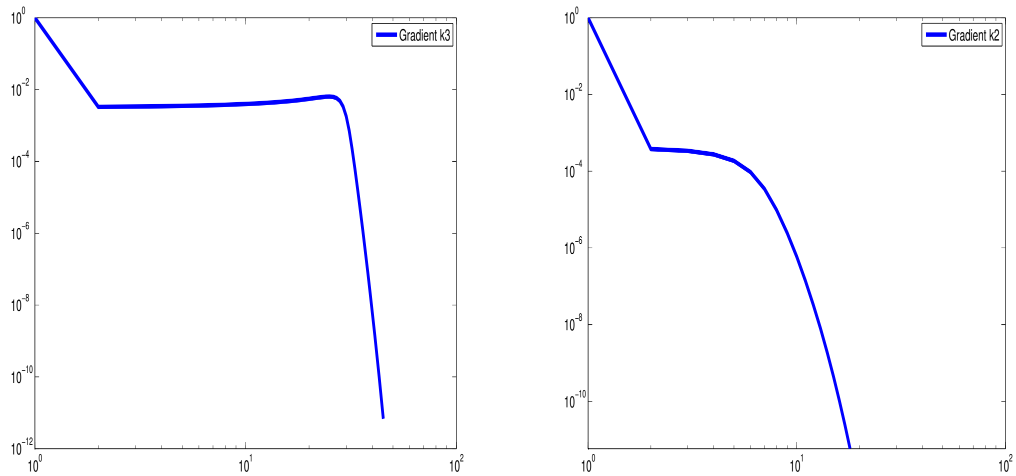

FIGURE 1. Gradient for q_1 and q_0

TABLE 1. The initial and optimal value of q_0 and q_1.

case 1 (q_0)	
q_0 initial	5.00
q_0 exact	1.00
q_0 optimal	1.0051
iteration	45
$\|\|\nabla J(P_k)\|\|$	6.8110e-012

case 2 (q_0, q_1)	
(q_0, q_1) initial	(3.00,5.00)
(q_0, q_1) exact	(1.00,1.00)
(q_0, q_1) optimal	(1.0447,1.0006)
iteration	300
$\|\|\nabla J(P_k)\|\|$	1.01e-06

We observe that we have obtained a good approximations of the parameters q_0 and q_1.

For more illustration of the effectiveness of our numerical approach, we present in Figure 2 the iterative convergence process from the initial conductivity K_{init} to the exact solution K_{exact}, for various numbers of iterations performed.

case 3 : Now, we focus on the identification of the most influential coefficient in the conductivity, namely the coefficient q_2. We consider then the following situation:

$$\text{case 3 :} \begin{cases} K(\psi) = q_2\psi^2 + \psi + 1, \\ C(\psi) = \psi + 1, \end{cases}$$

In Figure 3, we present the variation of the gradient of the cost function with respect to the iteration number for the approximate influencing term q_2. We note that this case requires a particular intention. In fact, the coefficient q_2 is the most sensitive one because it act on the nonlinear term. The empirical evidence from

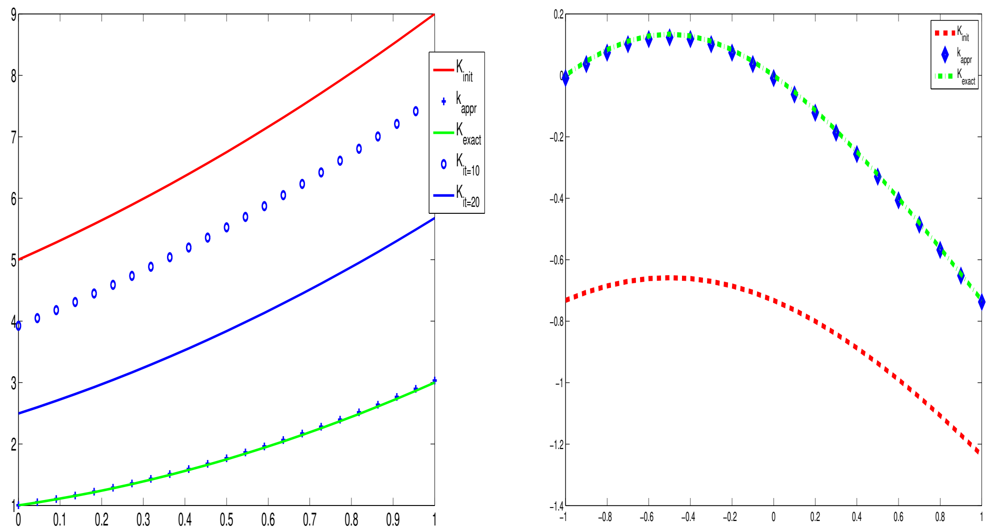

FIGURE 2. Representation of functions K_{init}, K_{exact} and K_{appr} by identifying q_0 and q_1

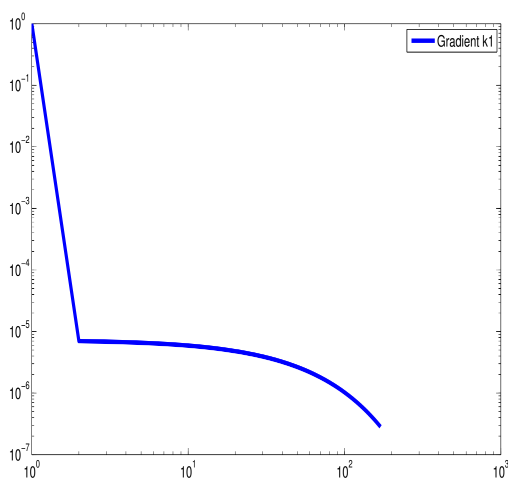

FIGURE 3. Gradient for q_2.

Figure 3 and Table 2 validates the algorithm's behavior. Indeed, we see that the algorithm spend 290 iterations to reach the optimal value of q_2.

The approached values of q_2 is given in Table 2.

Table 2. The initial and optimal value of q_2

case 3 (q_2)	
q_2 initial	2.00
q_2 exact	1.00
q_2 optimal	1.2264
iteration	300
$\|\|\nabla J(P_k)\|\|$	2.8343e-007

Second setup

The objective here is to identify the hydraulic capacity $C(\psi)$ by approximating there related coefficients s_0 and s_1, while we we keep known the conductivity. We consider then these two situations

$$\text{case } 1 : \begin{cases} K(\psi) = \psi^2 + \psi + 1, \\ C(\psi) = \psi + s_0, \end{cases} \quad \text{case } 2 : \begin{cases} K(\psi) = \psi^2 + \psi + 1, \\ C(\psi) = s_1\psi + s_0. \end{cases}$$

In the Figure 4, we present the decrease of the cost functional J_h with respect to the iteration number for the approximation of the hydraulic capacity C. We note that in these two cases, the functional J_h decreases quickly as in the previous configuration.

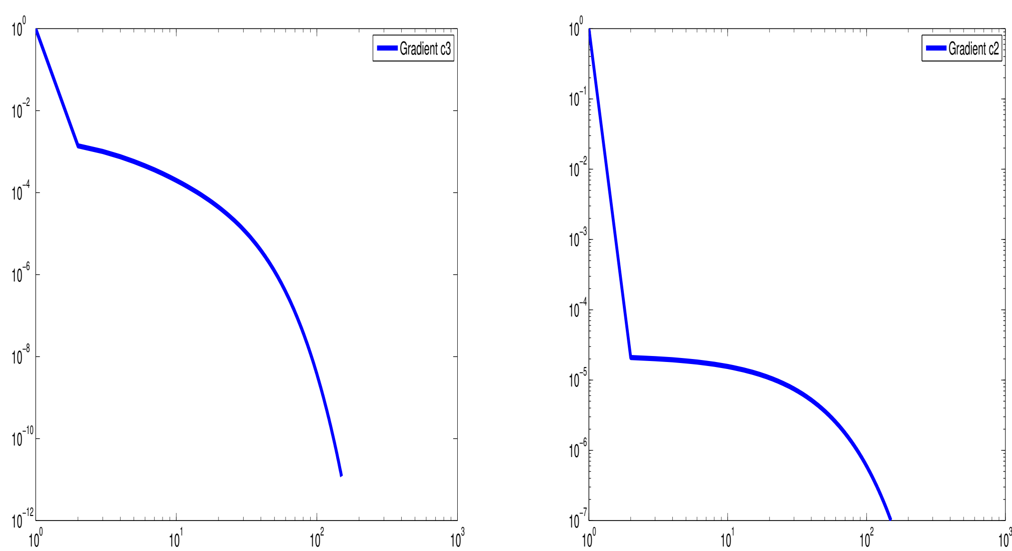

Figure 4. Gradient for s_0 and s_1

The approached values of s_0 and s_1 are given in Table 3.

TABLE 3. The initial and optimal value of s_0 and s_1

case 1 (s_0)	
s_0 initial	3.00
s_0 exact	1.00
s_0 optimal	1.0443
iteration	150
$\|\|\nabla J(P_k)\|\|$	1.1961e-011

case 2 (s_1, s_0)	
(s_1, s_0) initial	(3.00,5.00)
(s_1, s_0) exact	(1.00,1.00)
(s_1, s_0) optimal	(1.10 1.045)
iteration	300
$\|\|\nabla J(P_k)\|\|$	1.621e-06

Third setup

In this configuration, we assume that the hydraulic capacity and the hydraulic conductivity are quadratic. We seek then to identify simultaneously the coefficients q_0, q_1, q_2 and s_0, s_1, s_2 related respectively to the conductivity and the capacity. We consider then the situation:

$$\begin{cases} K(\psi) = q_2\psi^2 + q_1\psi + q_0, \\ C(\psi) = s_\psi^2 + s_1\psi + s_0. \end{cases}$$

In the Figure 5, we illustrate the iterative convergence process from the initial data to the exact solution, for various numbers of iteration performed for the both functions K and C.

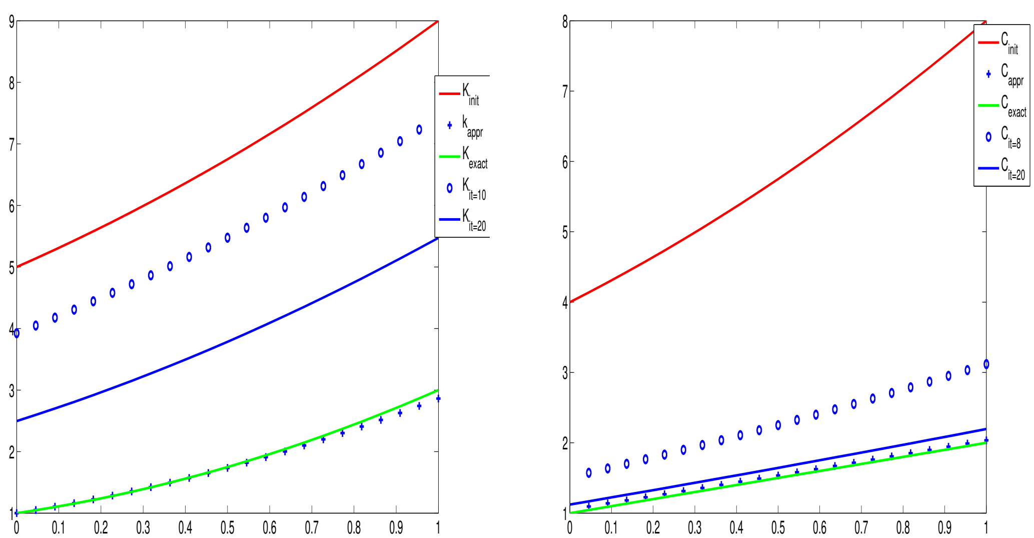

FIGURE 5. Illustration of $K(\psi)$ and $C(\psi)$ by estimating all coefficients

TABLE 4. The approximation of all coefficients q_i and s_i, for $i = 0, .., 2$

(q_2, q_1, q_0)	
(q_2, q_1, q_0) initial	(2.00, 3.00, 5.00)
(q_2, q_1, q_0) exact	(1.00, 1.00, 1.00)
(q_2, q_1, q_0) optimal	(0.7753,1.0768,1.0006)
iteration	300
$\|\|\nabla J(P_k)\|\|$	1.0e-006

(s_2, s_1, s_0)	
(s_2, s_1, s_0) initial	(1.00,3.00,4.00)
(s_2, s_1, s_0) exact	(0.00, 1.00, 1.00)
(s_2, s_1, s_0) optimal	(0.0291,0.9551,1.0513)
iteration	300
$\|\|\nabla J(P_k)\|\|$	1,78e-004

In this last case (Table 4), which aggregate all previous difficulties, it can be seen that the obtained results are in good agreement with the exact solutions.

Remark 4.1. Compared to our previous work [14], which employed finite difference discretization and a genetic algorithm for parameter identification, the present study introduces several important improvements. First, we adopt the finite element method (FEM), which provides greater flexibility and accuracy, particularly for handling complex geometries and boundary conditions. Second, the genetic algorithm is replaced by a gradient-based optimization method combined with an adjoint-state approach, which ensures faster convergence and allows for a rigorous theoretical analysis. Third, while [14] considered fixed functional forms for the hydraulic functions, the current formulation allows for their identification in a parametric form using low-degree polynomials. Finally, this work embeds the inverse problem within an optimal control framework, enabling a structured derivation of the adjoint equations and offering deeper insights into existence, uniqueness, and stability properties.

We conclude for the three considered numerical tests that, despite the complexity of the problem which lies in the strong nonlinearity not only of the direct problem but also in the dependence of the cost functional to the parameters, the numerical results remain good.

5. Numerical stability

This section is devoted to study the numerical stability of our proposed algorithm, its refers to how a malformed input affects the execution of the algorithm.

The identification reported so far used noiseless data input. Now we will suppose that input data are perturbed by a different random noise level. The proposed approach, was found to be more efficient is used. The results are reported in Table 5. They show a substantial change of coefficients, especially for low conductivities and a higher noise in the measurements.

TABLE 5. The optimal value of q_0, (q_1, q_0), and (q_2, q_1, q_0) for different noise levels.

Noise Level	q_0	(q_1, q_0)	(q_2, q_1, q_0)
1%	1.0185	(0.8576, 1.0326)	(1.1464, 0.8399, 1.0324)
10%	1.0187	(0.9893, 1.0198)	(1.1746, 0.9588, 1.0205)
15%	1.0189	(0.8582, 1.0330)	(1.1813, 0.9890, 1.0177)
20%	1.0190	(0.8585, 1.0331)	(1.1813, 0.9889, 1.0174)

6. CONCLUSION

This work presents a novel approach for approximating solutions to a hydrologic inverse problem governed by a highly nonlinear equation. The method reformulates the problem as an optimal control problem, discretized via finite elements and solved through Newton iteration. The discrete associated optimization problem is solved by gradient-type method. This approach has the advantage of identifying all parameters simultaneously with a good computational cost, even if we take only a single observation data ($N_{obs} = 1$).

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