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A NEW NUMERICAL SCHEME FOR SOLVING LINEAR OPTIMIZATION PROBLEMS

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ABSTRACT. In this work, we analyze a logarithmic penalty method for linear optimization problems based on a novel approximation function. This function enables the computation of the step-size with ease and without significant complications. Numerical results are presented to highlight the efficiency and enhanced performance of the proposed method compared to classical line search techniques.
Key words: Linear optimization, minorante function, penalty method.

1. Introduction

Linear optimization (LO) focuses on optimizing a linear objective function subject to a set of linear constraints. These problems arise frequently in various disciplines, including engineering, finance, operations research, and management science. The goal is typically to maximize profit or minimize cost while satisfying a collection of resource or operational constraints. Due to their structured nature, LO problems are fundamental in optimization theory and serve as a basis for more advanced models.

Numerous studies explore techniques to enhance constrained optimization processes, including those applicable to Linear optimization (LO). Forinstance, Manickam [16] investigated related optimization frameworks. Classical methods, such as those relying on line search techniques [18], face several challenges. Chief among them is the heavy dependence on step-size search procedures, which significantly increases computational cost and execution time, particularly for large-scale or structurally complex problems.

The idea of our new method is to solve a linear optimization problem using an innovative logarithmic barrier technique. Newton's method is employed to handle the corresponding perturbed equations in order to determine a descent direction. For the calculation of the step-size, we propose a new variant based on the minorant function technique. This idea contributes to the maximal reduction of the complexity of our algorithm in comparison to classical linear search methods.

Consider the canonical linear problem

$$(1): \ \min \ b^T y, \quad \text{Subject to:} \quad A^T y \geq c; y \in \mathbb{R}^m,$$

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with A a $\mathbb{R}^{m\times n}$-matrix of full rank, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ and its dual problem is written as

$$(2): \ \max\ c^T x, \quad \text{Subject to:} \quad Ax = b;\ x \geq 0.$$

The associated scalar product is given by

$$\langle u, v\rangle = u^T v = \sum_{i=1}^{n} u_i v_i,\ \forall u, v \in \mathbb{R}^n.$$

and the Euclidean norm of $u \in \mathbb{R}^n$ is defined by

$$\|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

Throughout the following, we assume that the set of strictly feasible primal-dual solutions is non-empty (assumption H1). Let (pr) represent the solutions of (1) and (2) perturbed problem without constraints associated to (1). (pr) is given by

$$(pr): \ \min_{y\in\mathbb{R}^m} h_r(y).$$

Here, τ denotes a strictly positive real number, for which we have

$$h_\tau(y) = \begin{cases} b^T y + n\tau \ln \tau - \tau \sum\limits_{i=1}^{n} \ln \left\langle A^T y - c, e_i \right\rangle & \text{if } \ A^T y - c > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

with $(e_1, e_2, ..., e_n)$ is the canonical base in $\mathbb{R}^n$.

The rest of the paper is structured as follows. In Section 2, we examine the existence and uniqueness of the optimal solution for the initial problem. We also demonstrate its convergence to initial problem. Section 3 contains the core of our work, a numerical study concerning the calculation of the step-size, which is performed using the minorant function technique. This study is supported by numerical tests presented at the next section. Finally, a conclusion closes the last section.

In the rest of this work, we focus on the resolution of a linear program using a logarithmic barrier method.

2. *Optimal Solution: Existence and Uniqueness*

This section is devoted to analyzing the existence and uniqueness of the optimal solution for problem (pr), and we give its convergence towards the problem (1). Given that h_τ is strictly convex, any optimal solution to the problem (pr), if it exists, is necessarily unique. In order to prove that the problem (pr) possesses an optimal solution, we show that the following recession cone amounts to zero

$$R_0((h_\tau)) = \{d \in \mathbb{R}^n, (h_\tau)_\infty(d) \leq 0\} = \{0\},$$

where

$$(h_\tau)_\infty(d) = \lim_{t\to+\infty} \frac{h_\tau(y + td) - h_\tau(y)}{t} = b^T d.$$

Lemma 2.1. [20] *Suppose that the assumption H1 holds and $A^T d > 0$, if $b^T d \leq 0$ then $d = 0$.*

It is known that the Hessian matrix $H = \bigtriangledown^2 h_\tau(y)$ is defined as positive, then the perturbed problem (pr) is strictly convex and if it admits an optimal solution, it is unique. We have

$$h_\tau(y) = b^T y + n\tau \ln \tau - \tau \sum_{i=1}^{n} \ln \langle e_i,\ A^T y - c\rangle,$$

then

$$\nabla h_\tau(y) = b - \tau \sum_{i=1}^{n} \frac{Ae_i}{\langle e_i,\ A^T y - c\rangle},$$

and

$$\bigtriangledown^2 h_\tau(y) = \tau \sum_{i=1}^{n} \frac{Ae_i (Ae_i)^T}{\langle e_i\ ,\ A^T y - c\rangle^2}.$$

The matrixes $B_i = Ae_i (Ae_i)^T \in \mathbb{R}^{m\times m}, i = 1, ..., n$ are defined as positive. Indeed $\forall y \in \mathbb{R}^m$ such as $y \neq 0$ and for all $i = 1, ..., n$

$$\left\langle Ae_i (Ae_i)^T y,\ y\right\rangle = \langle e_i e_i^T A^T y,\ A^T y\rangle = \langle e_i^T A^T y,\ e_i^T A^T y\rangle = \| e_i^T A^T y \|^2 .$$

Hence B_i are positive defined matrices (because A is of full rank), which gives positive defined H. Since h_τ is strictly convex and inf-compact, the problem (pr) has a unique optimal solution.

2.1. **Study of the convergence of** (pr).

Lemma 2.2. [20] *Let $\tau > 0$ and y_τ be an optimal solution of the problem (pr), then the feasible point y becomes optimal of the problem* (1) *where* $\lim_{\tau\to 0} y_\tau = y^*$.

3. Main Results

This section focuses on the numerical solution of problem (pr) by applying an interior-point method, specifically the logarithmic barrier approach. These methods are tailored to solve such problems by relying on optimality conditions that are both necessary and sufficient. y_τ is considered an optimal solution of (pr) if it meets the condition given below

$$\bigtriangledown h_\tau(y_\tau) = 0. \tag{3.1}$$

To solve (3.1), we apply Newton's method of finding a vector at each iteration $y_{\tau k} + d_k$ verifying the following linear system

$$\bigtriangledown^2 h_\tau(y_{\tau k})\ d_k = -\bigtriangledown h_\tau(y_{\tau k}). \tag{3.2}$$

As $H_{ess} = \bigtriangledown^2 h_\tau(y_{\tau k})$ is a symmetric matrix defined as positive the Cholesky and conjugate gradient methods are most appropriate to solve the system (3.2). To guarantee that the algorithm converges to the optimal solution y^* of (pr), it is important to ensure that all $y_{\tau k} + d_k$ iterations remain strictly achievable. For this, we introduce to each equation a step-size t_k that satisfies the condition

$$A^T(y_{\tau k} + t_k d_k) - c > 0.$$

For simplicity the presentation, we use y_k instead of $y_{\tau k}$ and y instead of y_τ.

In the following section, we compute the step-size using our novel approach.

3.1. **Calculation of the step-size t_k.** Line search methods are well-known for calculating the optimal step-size t_k. Just approximated the one-dimensional function

$$\varphi(t) = \min_{t>0} h_\tau(y + td).$$

The most widely used linear search methods are Goldstein-Armijo, Fibonnacci, and others. Unfortunately, these latter are enormously expensive and impossible in certain difficult problems among which those of semidefinite programming. To overcome this difficulty, we leverage the idea suggested by Merikhi et al. [8] which approximates the function

$$\varphi(t) = \frac{1}{\tau}\left(h_\tau(y + td) - h_\tau(y)\right),$$

By using simple majorant functions for each iteration k, and Leulmi et al. [11] which approximates, always, the function

$$\varphi(t) = \frac{1}{\tau}\left(h_\tau(y + td) - h_\tau(y)\right),$$

by simple minorant functions giving each iteration k,an explicit t_k step-size in a straightforward and considerably cheaper way than line search methods. Such that y and $(y + td)$ are strictly feasible.

Definition 3.1. A majorant of a function is another function that "bounds" or "dominates" it from above. Specifically, a function $g(x)$ is a majorant of a function $f(x)$ if, for all x,

$$f(x) \leq g(x).$$

The efficiency of this step-size can be measured by numerical tests established at the end of this work. Before introducing our new minorant function, we present, in the following lemma, an alternative simplified form of the function $\varphi(t)$.

Lemma 3.2. [20] *For any* $t \in \left[0, \widehat{t}\right]$, *where* $\widehat{t} = \sup\left\{t > 0, A^T(y_k + t_k d) - c > 0\right\} = \sup\{t > 0, 1 + tz_i > 0, \forall i\}$ *and* $z_i = \frac{\langle e_i,\, A^T d\rangle}{\langle e_i,\, A^T y - c\rangle}, \forall i = 1, ..., n$. *The function* $\varphi(t)$ *is well defined and is written as follows*

$$\varphi(t) = t \times \left(\sum_{i=1}^{n} z_i - \|z\|^2\right) - \sum_{i=1}^{n} \ln(1 + tz_i), \tag{3.3}$$

such that

$$\varphi(0) = 0,\ \varphi'(0) = -\varphi''(0) = \|z\|^2. \tag{3.4}$$

The result below is due to Wolkowicz et al. [17], see also Crouzeix et al. [8] for further findings.

Proposition 3.3. [17]

$$\begin{aligned} \overline{x} - \sigma_x\sqrt{n-1} &\leq \min_i x_i \leq \overline{x} - \frac{\sigma_x}{\sqrt{n-1}} \\ \overline{x} + \frac{\sigma_x}{\sqrt{n-1}} &\leq \max_i x_i \leq \overline{x} + \sigma_x\sqrt{n-1}, \end{aligned}$$

with

$$\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i\,,\quad \sigma_x^2=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\overline{x}^2.$$

The first value of $\widehat{t}$

$$\widehat{t}_0=\sup\left\{\ t\quad,1+t\alpha_0>0\ \right\}\ \text{with}\,\alpha_0=\overline{z}-\frac{\sigma_z}{\sqrt{n-1}},$$

another value $\widehat{t}_{opt}$ is due to the fact that $|z_i|\leq\|z\|$ for all i.

$$\widehat{t}_{new}=\sup\left\{\ t\quad,1+t\alpha_1>0\ \right\}\ \text{with}\ \alpha_1=-\frac{z+\sigma_z\sqrt{n-1}}{\overline{\|z\|^2}}.$$

3.2. **Approximate functions of φ.**

3.2.1. *Approximate function φ_0.* We seek an approximate function φ_0 of φ on the interval $\left[0,\widehat{t}\right]$ such that φ_0 satisfies condition

$$\varphi\left(0\right)=0,\ \varphi'\left(0\right)=-\varphi''\left(0\right)=\|z\|^2.$$

In [11], the authors applied Theorem 5 from [8] and proposed the following approximate function

$$\varphi_0\left(t\right)=t\gamma_0-\left(n-1\right)\ln\left(1+t\alpha_0\right)-\ln\left(1+t\beta_0\right),$$

such as: $\gamma_0=n\overline{z}-\|z\|^2,\ \ \alpha_0=\overline{z}-\frac{\sigma_z}{\sqrt{n-1}},\ \beta_0=\overline{z}+\sigma_z\sqrt{n-1}$ and

$$\begin{aligned}\overline{x}&=\frac{1}{n}\sum_{i=1}^{n}x_i\,,\\ \sigma_x^2&=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\overline{x}^2.\end{aligned}$$

When we set $x_i=1+tz_i$, so $\overline{x}=1+t\overline{z}$ and $\sigma_x=t\sigma_z$.

The approximate function φ_0 is defined and convex on $\left[0,\widehat{t}_1\right[$, where $\widehat{t}_1=\sup\left\{t>0:\ 1+t\alpha_0>0\right\}$ and we have

$$\varphi\left(t\right)\geq\varphi_0\left(t\right),$$

with

$$\varphi_0\left(0\right)=0,\ \varphi_0''\left(0\right)=-\varphi_0'\left(0\right)=\|z\|^2.$$

It is deduced that the function φ_0 attains its minimum at the point

$$\widetilde{t}_0=b\pm\sqrt{b^2-c},$$

such as

$$b=\frac{1}{2}\left(\frac{n}{\gamma_0}-\frac{1}{\alpha_0}-\frac{1}{\beta_0}\right)$$

and

$$c=\frac{-\|z\|^2}{\gamma_0\alpha_0\beta_0}.$$

3.3. **New approximate function** φ_{new}. We define another new function involving only one logarithm as follows

$$\varphi_{new}(t)=\left(\tau\frac{\|z\|^2}{\gamma}-\|z\|^2\right)t-\tau\ln\left(1+\frac{\|z\|^2}{\gamma}t\right).$$

with

$$\tau=\frac{\gamma^2}{\|z\|^2},\gamma=\overline{z}+\sigma_z\sqrt{n-1}$$

The logarithm is properly defined on $[0,\widehat{t}[$ with

$$\begin{aligned}\widehat{t}\ &=\ \begin{cases}\frac{\gamma}{\|z\|^2} & \text{if } \gamma>0,\\ +\infty & \text{if not,}\end{cases}\\ \varphi''_{new}(0)\ &=\ -\varphi'_{new}(0)=\|z\|^2.\end{aligned}$$

φ_{new} is convex and has a single minimum of $[0,\widehat{t}[$, which is obtained by solving the following equation

$$\varphi'_{new}(t)=0.$$

This minimum is

$$t_{opt}=\frac{\gamma^2}{\|z\|^2(\tau-\gamma)}.$$

In this lemma, we give the relation between φ and its approximations φ_0 and φ_{new}.

Lemma 3.4. *Let φ_0 and φ_{new} are strictly convex on the interval $[0,\widehat{t}[$ and satisfies*

$$\varphi_{new}(t)\le\varphi_0(t)\quad\forall\ t\in[0;\widehat{t}[,$$

with

$$\varphi''_{new}(0)=-\varphi'_{new}(0)=\|z\|^2.$$

Proof. Let $k(t)=\varphi_0(t)-\varphi_{new}(t)$, then,

$$k(t)=(n\overline{z}-\tau\frac{\|z\|^2}{\gamma})t-\sum_{i=1}^{n}\ln(1+tz_i)+\tau\ln\left(1+\frac{\|z\|^2}{\gamma}t\right).$$

So

$$k'(t)=n\overline{z}-\tau\frac{\|z\|^2}{\gamma}-\sum_{i=1}^{n}\frac{z_i}{1+tz_i}-\tau\frac{\frac{\|z\|^2}{\gamma}}{(1+\frac{\|z\|^2}{\gamma}t)}$$

and

$$k''(t)=\sum_{i=1}^{n}\frac{z_i^2}{(1+tz_i)^2}-\tau\frac{\frac{\|z\|^2}{\gamma^2}}{(1+\frac{\|z\|^2}{\gamma}t)^2},$$

then,

$$k''(t)=\sum_{i=1}^{n}\frac{z_i^2}{(1+tz_i)^2}-\frac{\|z\|^2}{\left(1+\frac{\|z\|^2}{\gamma}t\right)^2}.$$

To show that $k''(t) \geq 0$. It is sufficient to demonstrate that

$$\frac{1}{(1+tz_i)^2} \geq \frac{1}{\left(1+\frac{\|z\|^2}{\gamma}t\right)^2}.$$

As $z_i \leq \|z\|$ and $\gamma \leq \|z\|$, then, $k''(t) \geq 0$ and we have

$$k(0) = 0 \text{ and } k'(0) = 0.$$

On the other hand

$$\begin{aligned} \varphi'_{new}(t) &= \rho - \frac{\kappa^2}{\mu + \kappa^2 t}, \\ \varphi''_{new}(t) &= \frac{\kappa^4}{(\mu + \kappa^2 t)^2}. \end{aligned}$$

where $\kappa^2 = \|z\|^2$, $\mu = \bar{z}^2$ and $\rho = \frac{\bar{z}^2}{\|z\|^2}$, then, we oblain

$$\varphi'_{new}(0) = -\|z\|^2 \quad \text{and } \varphi''_{new}(0) = \|z\|^2.$$

□

Thus, the values t_{opt} and $\widetilde{t}_0$ are explicitly computed, and take values in the interval $\left(0, \widehat{t} - \varepsilon\right)$ and $\varphi'(t) < 0$, with $\varepsilon > 0$, is a fixed precision.

Remark 3.5. The computation of t_{opt} and $\widetilde{t}_0$ are performed by a dichotomous procedure, in the cases where t_{opt} and $\widetilde{t}_0 \notin \left(0, \widehat{t} - \varepsilon\right)$, and $\varphi'(t) > 0$, as follows
put $a = 0$ and $b = \widehat{t} - \varepsilon$
while $|b - a| > 10^{-4}$
if $\varphi\left(\frac{a+b}{2}\right) < 0$ then $b = \frac{a+b}{2}$
else $a = \frac{a+b}{2}$, so $t_{opt} = b$.

This procedure guarantees that the resulting step size remains within the interval $\left(0, \widehat{t} - \varepsilon\right)$ and satisfies the descent condition $\varphi'(t) < 0$, thus preserving strict feasibility and ensuring progress at each iteration.

Proposition 3.6. [10] *Let y_{k+1} and y_k two strictly feasible solutions of (pr), obtained respectively at the iteration $k+1$ and k, so we have $h_\tau(y_{k+1}) < h_\tau(y_k)$.*

3.4. **Algorithm.** In this section, we introduce the algorithm of our approach to find an optimal solution y^* to the problem (1)

Begin

Initialization: $y_0 \in \mathbb{R}^m$ such that $A^T y_0 > c$, ε given precision and $k = 0$.

- while $\left|b^T d_k\right| > \varepsilon$ do
  - Determine the solution to the system $H_{ess}\ d_k = -\bigtriangledown h_\tau(y_k)$, to calculate the direction d_k.
  - Calculate the step-size t_k using approximate functions φ_0 or φ_{new} or line search method.
    $\widetilde{t}_0 = b \pm \sqrt{b^2 - c}$, such as: $b = \frac{1}{2}\left(\frac{n}{\gamma_0} - \frac{1}{\alpha_0} - \frac{1}{\beta_0}\right)$ and $c = \frac{-\|z\|^2}{\gamma_0 \alpha_0 \beta_0}$. we take the positve one.
    $t_{new} = \frac{\gamma^2}{\|z\|^2(\tau - \gamma)}$.

- Take the new iteration $y_{k+1} = y_k + t_k d_k$.
- Set $k = k + 1$.

• End while.

End algorithm.

This approach minimizes both the number of iterations and the computation time. In the next section, we present numerical experiments and some commentaries.

4. Numerical Results.

This section is reserved for comparative tests of the obtained algorithm on different examples with fixed-dimension and variable-dimension of linear optimization problems [7, 11] using MATLAB R2016a on an Intel i5-8350 CPU (3.6 GHz) with 8 GB of RAM.

4.1. **Fixed-dimension *examples*.** We summarize in the table below the obtained results for different examples with fixed-dimension

	t_{opt}		t_0		Ls	
	K	CPU	K	CPU	K	CPU
Example 1	1	0.0001	2	0.0006	19	0.0099
Example 2	6	0.0018	6	0.0048	19	0.0311
Example 3	9	0.0028	9	0.0082	24	0.0499
Example 4	4	0.0026	4	0.0079	20	0.0344
Example 5	8	0.0064	8	0.0085	22	0.0501
Example 6	7	0.0056	7	0.0281	24	0.0311
Example 7	10	0.0089	10	0.0199	20	0.0456

Table 1. Numerical results for fixed-dimension examples

4.2. **Variable-dimension example.** Let

$$(1) \begin{cases} \min \ [2\,(y_1 + y_2 + ... + y_m)] \\ y_i - 1 \geq 0, \ \forall i = 1, ..., m \end{cases} \quad \text{and } n = 2m.$$

We summarize in the table below the obtained results for different dimensions (m, n).

	t_{opt}		t_0		Ls	
(m, n)	K	CPU	K	CPU	K	CPU
$(100, 200)$	1	0.0015	1	0.0007	21	2.7699
$(200, 400)$	1	0.0020	1	0.0015	22	4.6553
$(400, 800)$	2	0.0038	2	0.0028	22	19.7521
$(500, 1000)$	2	0.0064	2	0.0042	23	59.6754
$(600, 1200)$	2	0.0098	2	0.0045	24	287.091
$(1000, 2000)$	3	0.0176	3	0.0081	24	999.985
$(1500, 3000)$	3	0.0185	3	0.099	24	1655.4

Table 2. Numerical results for variable-dimension example

In the above tables, we take $\varepsilon = 10^{-4}$.
We also denote the following
(1) (K) iteration number,
(2) (CPU) necessary computation time,
(3) (Ls) classical Wolfe's line search.

4.2.1. *Comments on the obtained results.* Tables 1 and 2 support our claim that, in the numerical experiments conducted, the approximate functions achieved solutions to problem (1) within an acceptable number of iterations and in minimal computational time. Notably, the newly proposed approximation function φ_{new} demonstrated superior performance compared to the existing function φ_0 and the Wolfe line search method, as evidenced by the iteration counts and CPU times presented in Tables 1 and 2.

5. Conclusion

This study demonstrates that the proposed approximation functions provide an effective mechanism for determining step-sizes, thereby significantly enhancing the performance of penalty methods. By avoiding the computational burden associated with traditional line search techniques, particularly Wolfe's line search method, the new approach achieves notable improvements in both efficiency and robustness. Numerical experiments confirm that the proposed scheme leads to a considerable reduction in computation time and the number of iterations required to reach an optimal solution. These promising results suggest that the developed approximation functions can serve as a valuable alternative to conventional strategies, especially in large-scale and complex optimization problems. Future research directions may include extending the approach to nonlinear programming problems and exploring adaptive strategies for further improving step-size selection.

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