

ON THE GRAPH OF $\mathcal{C}$ - LATTICES

LAKPA SHERPA¹, VILAS KHARAT², MANISH AGALAVE³, RUPALI YEOLE⁴ and
NARAYAN PHADATARE^{5*}

ABSTRACT. Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. In this paper, we associate to each $\mathcal{C}$ -lattice $\mathcal{L}$, an undirected simple graph $\mathcal{P}(\mathcal{L})$ in which the vertex set is the collection of all non-trivial proper elements of $\mathcal{L}$ and two different members a and b of vertex set of $\mathcal{P}(\mathcal{L})$ are adjacent if and only if $a \vee b$ is a prime element of $\mathcal{L}$. Also, we obtain the interrelationship between some algebraic properties of $\mathcal{C}$ -lattice $\mathcal{L}$ and graph-theoretic properties of $\mathcal{P}(\mathcal{L})$.

1. INTRODUCTION AND PRELIMINARIES

In ([9], [17]), Dilworth and Ward introduced multiplicative lattice as an abstract framework for studying the ideal theory of commutative rings with identity. Let $(\mathcal{L}, \leq)$ be a complete lattice with lower element $0_{\mathcal{L}}$ and top element $1_{\mathcal{L}}$. A lattice $\mathcal{L}$ is *multiplicative lattice* with respect to a binary operation $*$ (called multiplication) satisfies the following properties:

- (1) $a * b \leq a \wedge b$.
- (2) $a * b = b * a$.
- (3) $(a * b) * c = a * (b * c)$.
- (4) $a * 1_{\mathcal{L}} = a$.
- (5) $a * (\bigvee_{\alpha \in I} b_{\alpha}) = \bigvee_{\alpha \in I} (a * b_{\alpha})$, where $a, b, b_{\alpha} \in \mathcal{L}$ and I is an indexing set.

Henceforth, we shall write $a * b = ab$.

Now, let's review some definitions. An element $x \in \mathcal{L}$ is called *proper* if $x < 1_{\mathcal{L}}$. A proper element $p \in \mathcal{L}$ is said to be *prime* if $ab \leq p$ implies that $a \leq p$ or $b \leq p$, where $a, b \in \mathcal{L}$. We will indicate by $\sigma(\mathcal{L})$ the collection of prime element of $\mathcal{L}$. A proper element $m \in \mathcal{L}$ is said to be *maximal*, if $m \leq x$ and x is proper implies that $m = x$. Let $Max(\mathcal{L})$ denote the collection of all maximal elements of $\mathcal{L}$. Note that if $Max(\mathcal{L}) = \{m\}$, then $\mathcal{L}$ is called *local*. In [10], Johnson and Sherette proved that, if $m \in Max(\mathcal{L})$, then $m \in \sigma(\mathcal{L})$. This implies that $Max(\mathcal{L}) \subseteq \sigma(\mathcal{L})$. An element $c \in \mathcal{L}$ is *compact* if for any $\{a_{\lambda}\}_{\lambda \in \Lambda} \subseteq \mathcal{L}$ and $c \leq \bigvee_{\lambda \in \Lambda} a_{\lambda}$, implies that $c \leq \bigvee_{i=1}^n a_{\lambda_i}$, where $n \in \mathbb{Z}^+$. If each element of $\mathcal{L}$ is the join of compact elements of $\mathcal{L}$ implies that $\mathcal{L}$ is *compactly generated*. In [1], a multiplicative lattice $\mathcal{L}$ is called $\mathcal{C}$ -lattice, if $\mathcal{L}$ possesses the following properties:

- (1) Top element $1_{\mathcal{L}}$ of $\mathcal{L}$ is compact and serves as the multiplicative identity.

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* Corresponding author.

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- (2) $\mathcal{L}$ is generated under join by a multiplicatively closed subset $\mathcal{C}$ of compact elements of $\mathcal{L}$.

An element $a \in \mathcal{L}$ is said to be *nilpotent* if $a^n = 0_{\mathcal{L}}$ for some $n \in \mathbb{Z}^+$. Note that if $0_{\mathcal{L}}$ of $\mathcal{L}$ is the only nilpotent then $\mathcal{L}$ is called *reduced* multiplicative lattice.

Naturally, the study of the intersection graphs $G(F)$ is more challenging when F is an algebraic structure. In the last few decades, many mathematicians accepted these challenges and studied such graphs on various algebraic structures. During these interdisciplinary investigations, many researchers obtained numerous characterizations and representations of special classes of algebraic structures in terms of graphs. In [6], Bosak took the initial move in this way in 1964. Numerous graph structures pertaining to the ring structure may be found in ([2], [3], [5], [8], [11], [12], [13] and [14]).

In [7], Chakrabarty et al. used a collection of nontrivial ideals of ring, say $L(R)$ to define and investigate the intersection graph $G(R)$ of given ring. The *intersection graph* $G(R)$ of $L(R)$ is the undirected simple graph whose vertices are all non-trivial left ideals of R and two distinct vertices are adjacent if and only if the corresponding left ideals have a nonzero intersection (see [7]). The work of Chakrabarty et al. in [7] is further extended by Akbari et al. in [4]. Akbari et al. used the same intersection graph $G(R)$ as in [7] and characterized all rings for which the intersection graphs of ideals are not connected. Among other outcomes, it is proved that, for a reduced ring R , the clique number of the graph $G(R)$ is equal to the chromatic number of $G(R)$.

In [15], Saha M. et al. used prime ideals of a commutative ring R with identity to define new graph called the *prime ideal sum graph* of R . The prime ideal sum graph $PIS(R)$ of R is an undirected graph in which vertices are non-trivial proper ideals of R and two distinct vertices I and J are adjacent if and only if $I + J$ is a prime ideal of R . Saha M. et al. used the graph $PIS(R)$ to study the connections between the graph-theoretic properties of this graph and some algebraic properties of rings R . Also, the study of the clique number, the chromatic number and the domination number of $PIS(R)$ for some classes of rings are obtained (see [15]).

In this paper, we introduce the graph $\mathcal{P}(\mathcal{L})$ of a $\mathcal{C}$ -lattice $\mathcal{L}$ as a generalization of the prime ideal sum graph $PIS(R)$ of R and obtain some connections between the graph-theoretic properties of this graph $\mathcal{P}(\mathcal{L})$ and algebraic properties of a $\mathcal{C}$ -lattice $\mathcal{L}$. It is noted that, under some circumstances the graph $\mathcal{P}(\mathcal{L})$ is complete. Also, we prove that the girth of $\mathcal{P}(\mathcal{L})$ regulate the number of distinct prime elements of $\mathcal{L}$.

A *graph* is an ordered pair $\mathbb{G} = (V, E)$ where V is the set of all vertices in the graph and E is the set of all edges joining the two vertices in the graph. If a and b are adjacent in the graph we denote it by $a \sim b$. If $a \sim b$ for all distinct $a, b \in V$ then we say that graph $\mathbb{G}$ is *complete*. If $E = \emptyset$ then $\mathbb{G}$ is empty and $\mathbb{G}$ is *null* graph if $V = \emptyset$. A chain of adjacent vertices $a \sim v_1 \sim v_2 \sim \dots \sim v_n \sim b$ between any two distinct vertices a and b is called *path*. Number of edges in the path is called *length of the path*. A *cycle* is a path that begins and end at the same vertex. A length of the shortest cycle of the graph $\mathbb{G}$ is called *girth* of $\mathbb{G}$. A

graph without cycle is said to be *acyclic*. For acyclic graph $\mathbb{G}$, $\text{girth}(\mathbb{G}) = \infty$. If for any two distinct vertices a and b there exists a path then that graph is said to be *connected* otherwise *disconnected*. The length of the shortest path from a to b is called *distance* from a to b and it is written as $d(a, b)$. If path between any two vertices a and b is absent then we write $d(a, b) = \infty$. Maximum distance in the graph $\mathbb{G}$ is called *diameter* of the graph $\mathbb{G}$. It is denoted by $\text{diam}(\mathbb{G})$. *Universal* vertex is a vertex in the graph $\mathbb{G}$ which is adjacent to every vertex in the graph. The *degree* of vertex a in $\mathbb{G}$ is number of edges incident to that vertex and it is denoted by $\text{deg}(a)$. If for some vertex $a \in V$, $\text{deg}(a) = 0$ then that vertex is known as *isolated* vertex.

For more information on graph theory, the reader may refer ([16]).

2. GRAPH $\mathcal{P}(\mathcal{L})$ OF $\mathcal{L}$

We begin this section with the definition of the graph $\mathcal{P}(\mathcal{L})$ of a $\mathcal{C}$ -lattice $\mathcal{L}$.

Definition 2.1. Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. The graph $\mathcal{P}(\mathcal{L})$ of $\mathcal{L}$ is a graph whose vertices are nonzero proper elements of $\mathcal{L}$ and two distinct vertices a and b are adjacent if and only if $a \vee b$ is prime element of $\mathcal{L}$.

Example 2.2. The multiplicative lattice $\mathcal{L}$ and its graph $\mathcal{P}(\mathcal{L})$ are shown below.

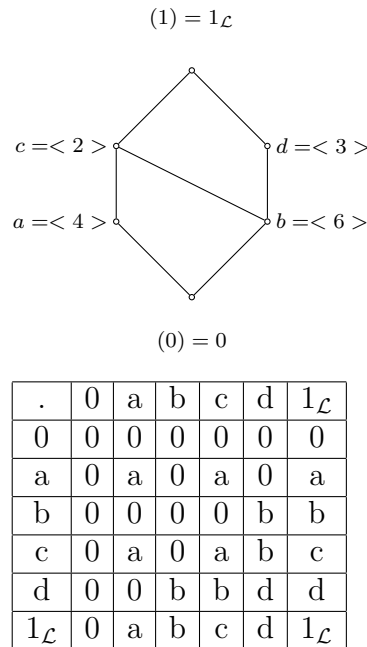
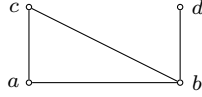


Fig.(2.1): Multiplicative Lattice $\mathcal{L}$

Note that, $\sigma(\mathcal{L}) = \{c, d\}$ is the collection of all prime element of $\mathcal{L}$.

Remark 2.3. [1] If $\mathcal{L}$ is a compactly generated multiplicative lattice with $1_{\mathcal{L}}$ compact, then maximal elements exist in $\mathcal{L}$, and every maximal element is a prime element.


 Fig.(2.2): Graph $\mathcal{P}(\mathcal{L})$

We need the following two Lemmas throughout the paper.

Lemma 2.4. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. Then meet of two maximal elements of $\mathcal{L}$ is not prime.*

Proof. Suppose that m_1 and m_2 are two maximal elements of $\mathcal{L}$ and p is a prime element of $\mathcal{L}$ such that $p = m_1 \wedge m_2$. Then $m_1 \cdot m_2 \leq m_1 \wedge m_2 = p$. Therefore by definition, we have $m_1 \leq p$ or $m_2 \leq p$, a contradiction to the maximality of m_1 and m_2 . $\square$

Lemma 2.5. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. If every proper element of $\mathcal{L}$ is nilpotent, then $\mathcal{L}$ has unique prime element.*

Proof. Suppose that each proper element of $\mathcal{L}$ is nilpotent. Then $\sqrt{0} = \vee\{a \in \mathcal{L} | a^n = 0, a \neq 1\}$ is the only maximal element of $\mathcal{L}$ and hence it is prime. Now, suppose that $\mathcal{L}$ has a prime element, say, p distinct from $\sqrt{0}$. Then by definition, we have $\sqrt{0} = \wedge\{p \in \sigma(\mathcal{L})\}$ and hence $\sqrt{0} \leq p$. Since $\sqrt{0}$ is maximal element of $\mathcal{L}$, we have $p = \sqrt{0}$. Consequently, $\mathcal{L}$ has unique prime element. $\square$

Theorem 2.6. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. Then the graph $\mathcal{P}(\mathcal{L})$ has a universal vertex if and only if one of the following condition holds:*

- (1) $\mathcal{L}$ is local.
- (2) $\mathcal{L}$ has exactly two maximal elements m_1 and m_2 such that $m_1 \wedge m_2$ is non-trivial minimal element and that there is no non-prime element, say x , such that $m_1 \wedge m_2 < x$.

Proof. Suppose $\mathcal{L}$ is local with unique maximal element m . Then for any $a \in \mathcal{L}$, we have $a \vee m = m$. Since m is a prime element $\mathcal{L}$, we have $a \sim m$. Consequently, vertex m is universal.

Now suppose that 2 holds. Let $a = m_1 \wedge m_2$ and b is a non-trivial element such that $a \neq b$. Assume that $b \leq m_1$. Since a is minimal, we have $a \leq a \vee b \leq m_1$. But it is given that, $a \vee b$ is prime element of $\mathcal{L}$ and hence a is a universal vertex.

Conversely, suppose that the graph $\mathcal{P}(\mathcal{L})$ has a universal vertex, say, a . If $\mathcal{L}$ is local, then we are through. Now, suppose that $\mathcal{L}$ possesses at least three maximal elements, viz, m_1, m_2, m_3 . It is clear that, a can not be a maximal element because two maximal elements are not adjacent. Since $\mathcal{L}$ is a $\mathcal{C}$ -lattice, there is a maximal element, say, m_1 such that $a \leq m_1$. If $a \not\leq m_2$, then $a \vee m_2 = 1$, because m_2 is maximal. This implies that, a is not adjacent to m_2 , contradiction to the fact that a is a universal vertex. Therefore, we have $a \leq m_1 \wedge m_2 \wedge m_3$.

Case-1: Let $a \neq m_1 \wedge m_2 \wedge m_3$. Then $a \vee (m_1 \wedge m_2 \wedge m_3) = m_1 \wedge m_2 \wedge m_3$. Since a is a universal vertex in $\mathcal{P}(\mathcal{L})$, by definition, $m_1 \wedge m_2 \wedge m_3$ is prime element in $\mathcal{L}$. Therefore $m_1 m_2 m_3 \leq m_1 \wedge m_2 \wedge m_3$ and $m_1 \wedge m_2 \wedge m_3$ is prime element of $\mathcal{L}$ implies that $m_i \leq m_j$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$, a contradiction.

Case 2: Let $a = m_1 \wedge m_2 \wedge m_3$. We assume that $b = m_1 \wedge m_2$. By Lemma 2.4, we have b is not a prime element of $\mathcal{L}$. Thus $a \vee b = b$ implies that a is not adjacent to b , which is a contradiction to the fact that a is a universal vertex in the graph $\mathcal{P}(\mathcal{L})$.

The above Cases implies that $\mathcal{L}$ has exactly two maximal elements say m_1 and m_2 . Now, we contend that $a = m_1 \wedge m_2$ is minimal. Suppose a non-trivial element $b \in \mathcal{L}$, such that $b \leq a = m_1 \wedge m_2$. Since a is a universal vertex, we have $b \vee a = a = m_1 \wedge m_2$ is prime, which is a contradiction. Consequently, a is minimal.

Now, suppose that b is a non prime element with $a = m_1 \wedge m_2 \leq b \leq m_i$ where $i = 1$ or 2 . Then $a \vee b = b$. Therefore $a \approx b$, i.e., a is not adjacent to b , a contradiction to the fact that a is a universal vertex. Consequently, there does not exist such b . $\square$

Lemma 2.7. *Let $\mathcal{L}$ be a non reduced $\mathcal{C}$ -lattice. Then $\sqrt{0}$ is adjacent to each $m \in \sigma(\mathcal{L})$.*

Proof. By definition, we have $\sqrt{0} = \bigwedge \{a \in \sigma(\mathcal{L})\}$. Therefore, for any $p \in \sigma(\mathcal{L})$, we have $\sqrt{0} \vee p = p$. This implies that, $\sqrt{0}$ is adjacent to each element of $\sigma(\mathcal{L})$. $\square$

Following Theorem, gives equivalent statement that the vertex $\sqrt{0}$ is universal.

Theorem 2.8. *Let $\mathcal{L}$ be a non-reduced $\mathcal{C}$ -lattice. Then the following are equivalent:*

- (1) *Each $x < 1_{\mathcal{L}}$ of $\mathcal{L}$ is nilpotent.*
- (2) *$\mathcal{L}$ has only one prime element.*
- (3) *$(\mathcal{L}, \sqrt{0})$ is a local.*
- (4) *$\sqrt{0}$ is a universal vertex in $\mathcal{P}(\mathcal{L})$.*

Proof. $1 \Rightarrow 2$ Suppose that any proper element of $\mathcal{L}$ is nilpotent. Then the only maximal element $\sqrt{0} = \bigvee \{a \in \mathcal{L} \mid a^n = 0, a \neq 1\}$ of $\mathcal{L}$ is also prime. Because, if $\mathcal{L}$ has another prime element, say, p other than $\sqrt{0}$. Then by definition of $\sqrt{0} = \bigwedge \{p \in \sigma(\mathcal{L})\}$, we have $\sqrt{0} \leq p$. Since $\sqrt{0}$ is maximal element of $\mathcal{L}$, we have $p = \sqrt{0}$. This implies that, $\mathcal{L}$ has unique prime element.

$2 \Rightarrow 3$ Suppose that $\mathcal{L}$ has unique prime element, say, p . Clearly, $\sqrt{0}$ is the only maximal element of $\mathcal{L}$. Consequently, $(\mathcal{L}, \sqrt{0})$ is local

$3 \Leftrightarrow 4$ Follows from Theorem 2.8.

$4 \Rightarrow 1$ Suppose that $\sqrt{0}$ is a universal vertex in $\mathcal{P}(\mathcal{L})$ and $a \in \mathcal{L}$ is non nilpotent. Then $a \not\leq \sqrt{0}$, therefore $a \vee \sqrt{0} = 1_{\mathcal{L}}$ which is not a prime and hence $a \approx \sqrt{0}$, a contradiction. Consequently, every proper element of $\mathcal{L}$ is nilpotent. $\square$

Theorem 2.9. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice and x is non-trivial proper element of $\mathcal{L}$. Vertex x of $\mathcal{P}(\mathcal{L})$ is isolated if and only if x is both maximal and minimal element.*

Proof. Suppose that $a \in \mathcal{L}$ is an isolated vertex in $\mathcal{P}(\mathcal{L})$ and suppose that a is not a maximal element of $\mathcal{L}$. Since $\mathcal{L}$ is $\mathcal{C}$ -lattice, there exist a maximal element m of $\mathcal{L}$ such that $a \leq m$. Therefore, $a \vee m = m$. This implies that a is adjacent

to m in $\mathcal{P}(\mathcal{L})$, a contradiction to the fact that a is an isolated vertex in $\mathcal{P}(\mathcal{L})$. Consequently, $a \in \mathcal{L}$ is maximal element.

Now, suppose that $a \in \mathcal{L}$ is not minimal. Since a is maximal element of $\mathcal{L}$, we have $b \in \mathcal{L}$ such that $0 < b < a$. This implies that $b \vee a = a$. But a is maximal and hence prime element of $\mathcal{L}$, therefore $a \sim b$ which is contradiction, consequently a is minimal element of $\mathcal{L}$.

Conversely, suppose that a is both maximal and minimal element of $\mathcal{L}$. Suppose that, vertex a is not isolated. Hence $\exists b (\neq 0) < 1_{\mathcal{L}}$ with $b \neq a$ such that $a \vee b$ is prime element. If $b \not\leq a$, then as a is maximal, we have $a \vee b = 1_{\mathcal{L}}$. Since $1_{\mathcal{L}}$ is not prime, we get a contradiction. Now, suppose that, $b \leq a$. As a is minimal element, we have, either $b = a$ or $b = 0$, again a contradiction. $\square$

In the following theorem we characterize the complete graph $\mathcal{P}(\mathcal{L})$.

Theorem 2.10. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. The graph $\mathcal{P}(\mathcal{L})$ of $\mathcal{L}$ is complete if and only if $\mathcal{L}$ is local and every proper element p of $\mathcal{L}$ which is not a prime is minimal.*

Proof. Suppose that the graph $\mathcal{P}(\mathcal{L})$ is complete. Then the graph $\mathcal{P}(\mathcal{L})$ has a universal vertex, say u . Therefore by Theorem 2.6, we have, either $\mathcal{L}$ is local or $\mathcal{L}$ has exactly two maximal elements, say, m_1 and m_2 with $m_1 \wedge m_2$ is non trivial minimal element and there does not exist a non prime element which properly contains $m_1 \wedge m_2$. Since two maximal elements are not adjacent in the graph $\mathcal{P}(\mathcal{L})$ of $\mathcal{L}$, we have no two maximal elements in $\mathcal{L}$, and hence $\mathcal{L}$ is local.

Now, suppose that q is proper element of $\mathcal{L}$ such that q is not prime. Then we contend that q is minimal, indeed if q is not a minimal element of $\mathcal{L}$. Then we have $b \in \mathcal{L}$ such that $0_{\mathcal{L}} < b < q$. Therefore $q \vee b = q$. Since q is not prime, we have q and b are not adjacent, we get a contradiction to the fact the graph $\mathcal{P}(\mathcal{L})$ is complete. Consequently, each proper element of $\mathcal{L}$ which is not a prime is minimal.

Conversely, suppose that $(\mathcal{L}, m)$ is a local $\mathcal{C}$ -lattice and every proper element q of $\mathcal{L}$ which is not a prime is minimal. Let a and b are two distinct proper elements of $\mathcal{L}$. Then it is clear that, $a, b \leq m$. If $a \vee b = c \leq m$ is not prime. Hence, c is minimal. Also, $a, b \leq c$ and a, b are non zero elements. Thus $a = b = c$, a contradiction. Thus $a \vee b$ is prime and hence $a \sim b$. Thus $\mathcal{P}(\mathcal{L})$ is complete. $\square$

Theorem 2.11. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice and for any two distinct prime elements m_1 and m_2 of $\mathcal{L}$, $m_1 \wedge m_2 \neq 0_{\mathcal{L}}$. Then the graph $\mathcal{P}(\mathcal{L})$ is connected with $\dim(\mathcal{P}(\mathcal{L})) \leq 2$.*

Proof. Suppose that x and y are non-zero proper elements of $\mathbb{L}$. If $x \wedge y$ is maximal element of $\mathcal{L}$, $x \sim y$ in $\mathcal{P}(\mathcal{L})$, because every maximal is prime element of $\mathcal{L}$. Otherwise, suppose that $x \wedge y$ is not a maximal. Since $\mathcal{L}$ is $\mathcal{C}$ -lattice, we have maximal elements say, p_1, k_1 , of $\mathcal{L}$ with $x \leq p_1$ and $y \leq k_1$. Now, we have the following cases:

Case – 1) If $p_1 = k_1$. Then there is a path $x \sim p_1 = k_1 \sim y$ of length 2.

Case – 2) If $p_1 \neq k_1$. Since K_1 is maximal element of $\mathcal{L}$, we have $k_1 \vee x = 1_{\mathcal{L}}$. Also, since $p_1 \wedge k_1 \neq 0_{\mathcal{L}}$, we have $x \vee (p_1 \wedge k_1) = x \vee p_1 \wedge x \vee k_1 = x \vee p_1 \wedge 1_{\mathcal{L}} = p_1$. As every maximal is prime element of $\mathcal{L}$, we have $x \sim p_1 \wedge k_1$. Similarly, we can

prove that $p_1 \wedge k_1 \sim y$. Consequently, there is a path $x \sim p_1 \wedge k_1 \sim y$ such that length equal to 2. $\square$

Theorem 2.12. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice and x, y are non-trivial proper elements of $\mathcal{L}$ such that $x \parallel y$. If x is adjacent to y in the graph $\mathcal{P}(\mathcal{L})$, then $\text{girth}(\mathcal{P}(\mathcal{L})) = 3$.*

Proof. Suppose that non-comparable elements $a, b \in \mathcal{L}$ are adjacent in the graph $\mathcal{P}(\mathcal{L})$. By definition, $a \vee b$ is a prime element of $\mathcal{L}$. Since a and b are non-comparable, we have $a, b < a \vee b$, and hence $a \sim a \vee b \sim b \sim a$ i.e., a, b and $a \vee b$ forms a triangle in $\mathcal{P}(\mathcal{L})$. Consequently, $\text{girth}(\mathcal{P}(\mathcal{L})) = 3$. $\square$

Following Corollary follows from the Theorem 2.12.

Corollary 2.13. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. If the graph $\mathcal{P}(\mathcal{L})$ contains no cyclic or the girth of $\mathcal{P}(\mathcal{L})$ is greater than 3. Then there are no non-trivial proper elements say, x, y of $\mathcal{L}$ such that $x \parallel y$ adjacent in $\mathcal{P}(\mathcal{L})$. Furthermore, for any non-trivial proper elements of $\mathcal{L}$ are adjacent if they are comparable.*

Theorem 2.14. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. If $\text{girth}(\mathcal{P}(\mathcal{L})) = n$, then the number of different prime elements of $\mathcal{L}$ are at least $\lfloor n/2 \rfloor$.*

Proof. From the Theorem 2.12, for any non-trivial proper elements $x, y \in \mathcal{L}$ such that $x \parallel y$ are adjacent, then girth of $\mathcal{P}(\mathcal{L})$ is 3. Therefore by definition, the join of any two non-comparable elements is a prime element of $\mathcal{L}$ and hence $\mathcal{L}$ contains at least $\lfloor 3/2 \rfloor = 1$ prime element.

Now suppose that $\text{girth}(\mathcal{P}(\mathcal{L})) > 3$. Then by Corollary 2.13, any non-trivial proper comparable elements are adjacent. Let $a_1 \sim a_2 \sim a_3 \sim \dots \sim a_n \sim a_1$ is the cycle of length n in the graph $\mathcal{P}(\mathcal{L})$. It is clearly observed that neither $a_1 < a_2 < a_3 < \dots < a_n < a_1$ nor $a_1 > a_2 > a_3 > \dots > a_n > a_1$ can hold. Without loss of generality, we have $a_1, a_3 < a_2$ and a_2 is a prime element of $\mathcal{L}$. We have the following two cases:

Case 1: Let $a_1, a_3 < a_2$ and $a_3 < a_4$. Then a_2, a_4 are prime elements.

Case 2: Let $a_1, a_3 < a_2$ and $a_4 < a_3$. Then a_2, a_3 are prime elements.

Thus in any case, we get at least 2 elements to be prime among a_1, a_2, a_3 and a_4 . That is, among any four consecutive elements of the cycle, two of them must be prime. This process gives that the number of prime elements of $\mathcal{L}$ are at least $\lfloor n/2 \rfloor$. $\square$

Following Corollary follows from the Theorem 2.14.

Corollary 2.15. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. If $|\sigma(\mathcal{L})| = k$. Then $\mathcal{P}(\mathcal{L})$ contains no cyclic or girth of $\mathcal{P}(\mathcal{L})$ is less than or equal to $2k$.*

Theorem 2.16. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice and $\text{Max}(\mathcal{L})$ be the set of all maximal elements of $\mathcal{L}$. Then $\text{Max}(\mathcal{L})$ is the smallest dominating set of $\mathcal{P}(\mathcal{L})$ and $\gamma(\mathcal{P}(\mathcal{L})) \leq |\text{Max}(\mathcal{L})|$. Furthermore, $\gamma(\mathcal{P}(\mathcal{L})) = 1$ if and only if $\mathcal{L}$ is local or $\text{Max}(\mathcal{L}) = \{m_1, m_2\}$ such that $m_1 \wedge m_2$ is non-trivial minimal element of $\mathcal{L}$ and there is no $p \notin \sigma(\mathcal{L})$ with $m_1 \wedge m_2 \leq p$.*

Proof. As $\mathcal{L}$ is a $\mathcal{C}$ -lattice, for $a \in \mathcal{L}$ we have a maximal elements say, m , of $\mathcal{L}$ with $a \leq m$. So $a \vee m = m$. But every maximal is prime, therefore a is adjacent

to m . This implies that any vertex of $\mathcal{P}(\mathcal{L})$ is in $\text{Max}(\mathcal{L})$ or has a neighbor in $\text{Max}(\mathcal{L})$ and hence $\text{Max}(\mathcal{L})$ dominates $\mathcal{P}(\mathcal{L})$. Also, it is clearly observed that, for $m \in \text{Max}(\mathcal{L})$, $\text{Max}(\mathcal{L}) \setminus m$ does not dominates $\mathcal{P}(\mathcal{L})$. Therefore, $\text{Max}(\mathcal{L})$ is the smallest dominating set for $\mathcal{P}(\mathcal{L})$ and so $\gamma(\mathcal{P}(\mathcal{L})) \leq |\text{Max}(\mathcal{L})|$. $\square$

Corollary 2.17. *Let $\mathcal{L}$ be a $\mathcal{C}$ -lattice. If $\mathcal{L}$ has exactly two minimal elements which does not satisfy the conditions of Theorem 2.16, then $\gamma(\mathcal{P}(\mathcal{L})) = 2$.*

Proof. Assume that $\mathcal{L}$ contains exactly two minimal elements which does not satisfies the conditions of Theorem 2.16. Then $\gamma(\mathcal{P}(\mathcal{L})) \leq 2$. By Theorem 2.6, $\gamma(\mathcal{P}(\mathcal{L})) \neq 1$. Hence $\gamma(\mathcal{P}(\mathcal{L})) = 2$. $\square$

Conclusion

In this paper, we have investigated the graph associated with a multiplicative lattice and explored how its structure reflects underlying algebraic properties. By analyzing the interplay between lattice elements and their graphical representation, we established meaningful connections between algebraic behavior and graph-theoretic characteristics such as connectivity, diameter, and clique structure. These results contribute to a deeper understanding of how algebraic systems can be studied through combinatorial and graphical frameworks, opening avenues for further research in both lattice theory and algebraic graph theory.

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¹ DEPARTMENT OF MATHEMATICS, SAVITRIBAI PHULE PUNE UNIVERSITY, PUNE, INDIA
Email address: csherpaap@gmail.com

² DEPARTMENT OF MATHEMATICS, SAVITRIBAI PHULE PUNE UNIVERSITY, PUNE, INDIA
Email address: laddoo1@yahoo.com

³ DEPARTMENT OF MATHEMATICS, FERGUSSON COLLEGE(AUTONOMUS), PUNE, INDIA
Email address: manishagalave@gmail.com

⁵ HOPE FOUNDATION'S INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY, PUNE, INDIA
Email address: rautrupali8685@gmail.com

⁴ BHARATI VIDYAPEETH DEEMED TO BE UNIVERSITY COLLEGE OF ENGINEERING, PUNE, INDIA
Email address: nmphadatare@bvucoep.edu.in