

ON SOME PROPERTIES OF SOLUTIONS OF CERTAIN Q-DIFFERENCE EQUATIONS IN ULTRAMETRIC FIELDS

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ABSTRACT. Let K be a complete, algebraically closed ultrametric field, and let $\mathcal{M}(K)$ denote the field of meromorphic functions over K . In this article, we investigate the properties of solutions to certain ultrametric q -difference equations. Specifically, we examine the growth behavior of ultrametric meromorphic functions f that satisfy these equations. Furthermore, we establish necessary conditions on the coefficients for which a difference equation of the form $\sum_{i=0}^s A_i(x)f(x+i) = B(x)$, where $B(x), A_0(x), \dots, A_s(x)$ (with $s \geq 1$) is a polynomial, admits a meromorphic solution.

1. INTRODUCTION

Recently, a significant number of studies (see, for example [1, 2, 4, 5, 6, 7, 11, 12, 13, 15]) have focused on difference equations and q -difference equations, yielding many meaningful results. The main objective of this paper is to extend some of these results, originally established over the complex field $\mathbb{C}$, to the setting of meromorphic solutions in an ultrametric field. Throughout this paper, we use the ultrametric Nevanlinna theory (see, e.g., [9, 10]). So we first have to recall some basic notions of this theory.

Let K be an algebraically closed field complete for an ultrametric absolute value. We denote by $\mathcal{A}(K)$ the K -algebra of entire functions in K and by $\mathcal{M}(K)$ the field of meromorphic functions in K , i.e. the field of fractions of $\mathcal{A}(K)$. Given $R > 0$, we denote by $D^-(0, R)$ the open disk $\{x \in K : |x| < R\}$ and by $D(0, R)$ the closed disk $\{x \in K : |x| \leq R\}$. Similarly, we denote by $\mathcal{A}(D^-(0, R))$ the K -algebra of analytic functions inside the disk $D^-(0, R)$, i.e. the set of power series converging inside $D^-(0, R)$ and by $\mathcal{M}(D^-(0, R))$ the field of meromorphic functions in $D^-(0, R)$, i.e. the field of fractions of $\mathcal{A}(D^-(0, R))$. For every $r \in]0, R[$, we define a multiplicative norm $|\cdot|(r)$ on $\mathcal{A}(D^-(0, R))$ by $|f|(r) = \sup_{n \geq 0} |a_n|r^n$ for

every function $f(x) = \sum_{n \geq 0} a_n x^n$ of $\mathcal{A}(D^-(0, R))$. We extend this to $\mathcal{M}(D^-(0, R))$

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by setting $|f|(r) = \frac{|g|(r)}{|h|(r)}$ for every element $f = \frac{g}{h}$ of $\mathcal{M}(D^-(0, R))$. Finally, for every $f \in \mathcal{M}(D^-(0, R)) \setminus \{0\}$ and every $\alpha \in D^-(0, R)$, we denote by $\omega_\alpha(f) \in \mathbb{Z}$ the integer such that $f(x) = \sum_{k \geq \omega_\alpha(f)} a_k(x - \alpha)^k$ and $a_{\omega_\alpha(f)} \neq 0$.

2. PRELIMINARIES

The following proposition, which is an immediate consequence of the above definitions, is in fact the starting point of the ultrametric Nevanlinna Theory. (see [10] for instance)

Proposition 2.1. [14] *Let $R > 0$ and let $f \in \mathcal{M}(D^-(0, R))$ be such that 0 is neither a zero nor a pole of f . Then, for every $r \in]0, R[$, we have*

$$\log |f|(r) = \log |f(0)| + \sum_{|\alpha| \leq r} \omega_\alpha(f) \log \frac{r}{|\alpha|}. \quad (2.1)$$

Let $f \in \mathcal{M}(D^-(0, R))$ be such that 0 is neither a zero nor a pole of f . For every $r \in]0, R[$, we denote by $Z(r, f)$ the counting function of zeros of f in the disk $D(0, r)$, counting multiplicity, i.e., we set

$$Z(r, f) = \sum_{\substack{\omega_\alpha(f) > 0 \\ |\alpha| \leq r}} \omega_\alpha(f) \log \frac{r}{|\alpha|}. \quad (2.2)$$

In the same way, we set $N(r, f) = Z(r, \frac{1}{f})$, to denote the counting function of poles of f in $D(0, r)$, counting multiplicity.

Using the notation $\log^+(x) = \max(0, \log x)$, (where $x > 0$ and $\log$ is the real logarithm function), we put for $r \in]0, R[$, $m(r, f) = \log^+ |f|(r)$. We finally set:

$$T(r, f) = \max\{Z(r, f), N(r, f)\}. \quad (2.3)$$

The function $r \rightarrow T(r, f)$ is called the Nevanlinna function (also called the characteristic function of f).

With the above notations, using the fact that $\log x = \log^+ x - \log^+ \frac{1}{x}$, for $x > 0$, we can rewrite Proposition 2.1 as follows (see [10] for instance):

Proposition 2.2. *Let $f \in \mathcal{M}(D^-(0, R))$ be such that 0 is neither a zero nor a pole of f . Then, for every $r \in]0, R[$, we have*

$$T(r, \frac{1}{f}) = T(r, f) + O(1). \quad (2.4)$$

The following propositions are easy to check (see, for instance, [8, 9, 10]).

Proposition 2.3. *Let $f_1, \dots, f_k \in \mathcal{M}(D^-(0, R))$. We assume that the functions f_i , ($1 \leq i \leq k$), $f_1 + \dots + f_k$, and $f_1 \dots f_k$ have no zero and no pole at the origin. Then, for every $r \in]0, R[$, we have*

$$\begin{aligned} m(r, f_1 + \dots + f_k) &\leq m(r, f_1) + \dots + m(r, f_k), \\ m(r, f_1 \dots f_k) &\leq m(r, f_1) + \dots + m(r, f_k), \end{aligned}$$

and

$$\begin{aligned} N(r, f_1 + \dots + f_k) &\leq N(r, f_1) + \dots + N(r, f_k), \\ N(r, f_1 \dots f_k) &\leq N(r, f_1) + \dots + N(r, f_k), \end{aligned}$$

and

$$\begin{aligned} T(r, f_1 + \dots + f_k) &\leq T(r, f_1) + \dots + T(r, f_k), \\ T(r, f_1 \dots f_k) &\leq T(r, f_1) + \dots + T(r, f_k). \end{aligned}$$

Proposition 2.4. *Let $f \in \mathcal{M}(K)$ such that $f(0) \neq 0$ and $f(0) \neq \infty$, we have the following equivalences*

- (i) f is a constant $\Leftrightarrow T(r, f) = o(\log r)$, $r \rightarrow +\infty$,
- (ii) f belongs to $K(x)$ $\Leftrightarrow T(r, f) = O(\log r)$, $r \rightarrow +\infty$,
- (iii) f is non constant $\Leftrightarrow$ there is $C \in \mathbb{R}$ and $A > 0$ such that

$$T(r, f) \geq \log r + C, \text{ pour } r > A.$$

3. q -DIFFERENCE EQUATIONS

Let us now return to the main problem. We consider the following q -difference equation

$$\sum_{i=0}^s A_i(x) f(q^i x) = g(x), \quad (3.1)$$

where $q \in K$ satisfies $0 < |q| < 1$, the coefficients $A_0, \dots, A_s$ ($s \geq 1$) are elements of the field K , and $g(x)$ is an element of $\mathcal{M}(K)$. In a previous contribution by J. Heittokangas and al (as referenced in [15]), the authors considered a special case of the equation (3.1), namely when the upper-hand side $g(x)$ takes the rational form $\frac{P(x)}{x^l}$, where $p(x)$ is a polynomial with complex coefficients and l is a positive integer. Their analysis was primarily concerned with the behavior of meromorphic solutions under this specific forcing term structure. Building upon their framework, the goal of the present work is to broaden the scope of that result. We aim to investigate the behavior of solutions under more general forms of $g(x)$, to derive deeper insights into the nature of meromorphic and rational solutions, and to establish new structural theorems for q -difference equations over ultrametric fields

Theorem 3.1. *Suppose that, in Equation (3.1), the function $g(x)$ is of the reduced form $g(x) = \frac{h(x)}{x^l}$, where $l \in \mathbb{N}$ and $h(x)$ is entire function (i.e., $h(x) \in \mathcal{A}(K)$), with $h(0) \neq 0$. If $f \in \mathcal{M}(K)$ is a non constant solution of Equation (3.1), then f is of the reduced form $f(x) = \frac{F(x)}{x^p}$, where $F(x)$ is an entire function and $p \geq l$.*

Proof. By hypothesis, we have

$$\sum_{i=0}^s A_i f(q^i x) = \frac{h(x)}{x^l}, \quad (3.2)$$

where $A_0, \dots, A_s \in K$ such that $A_0 A_s \neq 0$ and $h(x) \in \mathcal{A}(K)$, $h(0) \neq 0$, $l \in \mathbb{N}$. Suppose that Equation (3.2) possesses a meromorphic solution f and let $\alpha \neq 0$ be a pole of f . Then Equation (3.2) ensures that, for at least one index i_1 among $1, \dots, s$, the element $\alpha_1 = q^{i_1} \alpha$ of K is a pole of f . Applying this reasoning again to α_1 , we deduce that there exist $i_2 \in \{1, \dots, s\}$ such that $\alpha_2 = q^{i_2} \alpha_1 = q^{i_1+i_2} \alpha$ is a pole of f , etc... Hence we construct a sequence $\alpha_1, \alpha_2, \dots$ of poles of f tending to 0. This yields a contradiction because, a meromorphic function on K cannot possess a sequence of poles accumulating at a point in K . Hence f has no poles different from 0. Suppose now that f has a pole of order p at the origin. It follows immediately from Equation (3.2) that $p \geq l$. Then $h(x) = x^p f(x)$ is entire function. $\square$

Next, we give the following result, which is a generalization of Theorem 2.3. (in [15]) obtained in the complex case to ultrametric case.

Theorem 3.2. *Suppose that in Equation (3.1), the function $g(x)$ is meromorphic. If $g(x)$ has exactly one non-zero pole, then all meromorphic solutions f of Equation (3.1) have infinitely many poles.*

Proof. Let $g(x)$ be a meromorphic function in K having exactly one non zero pole at α . Assume that Equation (3.1) admits a meromorphic solution $f(x)$. From Equation (3.1), it is clear that at least one index i among $0, \dots, s$, the element $q^i \alpha$ is a pole of f . Suppose that $i \geq 1$. Substitute $q^i \alpha$ for x in Equation (3.1), then there exists at least an index $i_1 \in \{1, \dots, s\}$ such that $q^{i_1+i} \alpha$ is a pole of f . As in the proof of Theorem 3.1, we obtain a sequence of poles of f accumulating to the origin, which is a contradiction. Hence $i = 0$ and α is a pole of f . Let us set $\beta = q^{-s} \alpha$. From Equation (3.1), we have

$$\sum_{i=0}^s A_i f(q^i \beta) = g(\beta),$$

and $\alpha = q^s \beta$ is a pole of f . So there exists an index $i_1 \in \{0, \dots, s-1\}$ such that $\alpha_1 = q^{i_1} \beta = q^{i_1-s} \alpha$ is a pole of f . Repeating this reasoning for α_1 , we obtain an index $i_2 \in \{0, \dots, s-1\}$ and an other pole α_2 of f such that $\alpha_2 = q^{i_2} \beta = q^{i_2-s} \alpha_1 = q^{i_2+i_1-2s} \alpha$. In this way, we construct a sequence $(\alpha_n)_{n \geq 0}$ of poles of f satisfying, for every $n \geq 0$,

$$\left| \frac{\alpha_{n+1}}{\alpha_n} \right| = \frac{|q|^{i_{n+1}+i_n+\dots+i_1-(n+1)s} \alpha}{|q|^{i_n+\dots+i_1-n s} \alpha} = |q|^{i_{n+1}-s} \geq |q|^{-1} > 1. \quad (3.3)$$

Hence f has infinitely many poles. $\square$

Remark 3.3. If $g(x)$ has more than one non zero pole, then f may have finitely many poles.

Example 3.4. Consider the functional equation:

$$f(x) + f\left(\frac{x}{2}\right) = \frac{1}{(x-1)(x-2)}$$

where $q = \frac{1}{2}$, and the right-hand side has two non-zero poles at $x = 1$ and $x = 2$. We can solve this equation recursively using the relation

$$f(x) = g(x) - f\left(\frac{x}{2}\right), \quad \text{where } g(x) = \frac{1}{(x-1)(x-2)}.$$

By repeated substitution, we get

$$f(x) = g(x) - g\left(\frac{x}{2}\right) + g\left(\frac{x}{4}\right) - g\left(\frac{x}{8}\right) + \cdots$$

Assuming $\lim_{n \rightarrow \infty} f\left(\frac{x}{2^n}\right) = 0$, the solution is $f(x) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{\left(\frac{x}{2^k}-1\right)\left(\frac{x}{2^k}-2\right)}$. This series converges when $|q| < 1$, and it defines a function $f(x)$ with only finitely many poles. This confirms Remark 3.3.

4. DIFFERENCE EQUATIONS

Let us now consider the difference equation

$$\sum_{i=0}^s A_i(x) f(x+i) = B(x), \quad (4.1)$$

where $B(x), A_0(x), \dots, A_s(x)$ ($s \geq 1$) are elements of $K[x]$ such that $A_0(x)A_s(x) \neq 0$.

In the referenced work [13], B. Q. Chen and al investigated the necessary conditions imposed on the coefficients of a class of complex q -difference equations, particularly the form presented in Equation (3.1). Their study focused on identifying the conditions under which such equations possess meromorphic solutions when the modulus of q equals 1, i.e., $|q| = 1$. In this setting, the coefficients $B(x), A_0(x), \dots, A_s(x)$ were assumed to be polynomial functions, and their analysis provided foundational insights into the behavior of such equations within the complex domain. Motivated by their approach, the present article shifts the focus toward a related yet distinct framework—namely, the *ultrametric* context. Here, we consider the ultrametric analog of the difference equation, denoted as Equation (4.1), and explore the existence and nature of solutions under similar coefficient constraints. The aim is to generalize the previous findings and adapt the analytic techniques to the non-Archimedean setting, which demands careful consideration of valuation theory and p -adic function spaces. The results we obtain, derived under these ultrametric conditions, are detailed in the forthcoming sections and form the core contributions of this study.

We will need the following Lemma whose proof is straightforward.

Lemma 4.1. [7]

For every $f \in \mathcal{M}(K)$ and every $r > 0$ we have:

$$1) \quad |f \circ \tau|(r) = |f|(r),$$

$$2) \quad m(r, f \circ \tau) = m(r, f),$$

$$3) N(r, f \circ \tau) = N(r, f) + O(1),$$

$$4) T(r, f \circ \tau) = T(r, f) + O(1), \text{ where } \tau(x) = x + 1.$$

Theorem 4.2. *Let $f \in \mathcal{A}(K)$ be a non-constant solution of Equation (4.1).*

- i) *If the coefficients $B(x), A_0(x), \dots, A_s(x)$ are all constants with $A_0 A_s \neq 0$. and $f(x)$ is a non constant entire function. Then each $k \in \{0, \dots, s\}$ satisfies*

$$|A_k| \leq \max_{i \in \{0, \dots, s\} \setminus \{k\}} |A_i|.$$

- ii) *If $B(x), A_0(x), \dots, A_s(x)$, and $f(x)$ are polynomial, then there exists some $k \in \{0, \dots, s\}$ such that*

$$\deg A_k \geq \deg B(x) - \deg f(x).$$

- iii) *If $f(x)$ is a transcendental entire function, then there are at least two numbers $k, l \in \{0, \dots, s\}$ such that*

$$\deg A_k = \deg A_l = \max_{0 \leq i \leq s} \{\deg A_i\}.$$

Proof. i) Let $f \in \mathcal{A}(K)$ be a non-constant solution to Equation 4.1 According to the assumptions, all coefficients $B(x), A_0(x), \dots, A_s(x)$ are constant elements in K . and in particular, $A_0 A_s \neq 0$. From the functional equation

$$\sum_{i=0}^s A_i f(x+i) = B,$$

we can isolate any term in the summation, say $A_k f(x+k)$ and rewrite it as

$$A_k f(x+k) = - \sum_{\substack{i=0 \\ i \neq k}}^s A_i f(x+i) + B,$$

for each $k \in \{0, 1, \dots, s\}$ This follows directly from rearranging the terms in Equation 4.1. Now, let $r > 0$. Applying Lemma 4.1 (1), which states that $|f \circ \tau^i|(r) = |f|(r)$, and noting that the coefficients are constant, we can estimate:

$$|A_k| |f|(r) = |A_k f(x+k)|(r) \leq \max \{|A_i| |f|(r) / i \in \{0, \dots, s\}, i \neq k, |B|\}.$$

Thus, for sufficiently large r , this inequality implies that

$$|A_k| \leq \max_{i \in \{0, \dots, s\} \setminus \{k\}} |A_i|.$$

This estimation, combined with the fact that f is a non-constant entire function and the coefficients are constants, will be used to further analyze the growth behavior of f and derive additional properties in part ii of the theorem.

- ii) From Equation (4.1), we have

$$B(x) = \sum_{i=0}^s A_i(x) \cdot f(x+i)$$

Since $f(x)$ is a non-constant entire function over a non-Archimedean field K , the degree of each term $A_i(x)f(x+i)$ is given by

$$\deg(A_i(x)) + \deg(f(x)),$$

because the shift $f(x+i)$ does not change the degree of $f(x)$. That is

$$\deg(f(x+i)) = \deg(f(x)) \quad \text{for all } i.$$

Therefore, we obtain:

$$\deg\left(\sum_{i=0}^s A_i(x)f(x+i)\right) \leq \max_{0 \leq i \leq s} \{\deg(A_i(x)) + \deg(f(x))\}.$$

Since the left-hand side equals $\deg(B(x))$, we conclude

$$\deg(B(x)) \leq \max_{0 \leq i \leq s} \{\deg(A_i(x)) + \deg(f(x))\}.$$

Subtracting $\deg(f(x))$ from both sides gives

$$\deg(B(x)) - \deg(f(x)) \leq \max_{0 \leq i \leq s} \deg(A_i(x)).$$

Thus, there exists at least one $k \in \{0, \dots, s\}$ such that

$$\deg(A_k(x)) \geq \deg(B(x)) - \deg(f(x))$$

iii) We are given the functional equation

$$\sum_{i=0}^s A_i(x)f(x+i) = B(x),$$

and assume that $f \in \mathcal{M}(K)$ is a transcendental function. We suppose that

– There exists a unique index k such that

$$\deg A_k = \max_{i=0, \dots, s} \deg A_i$$

– Define $d_i = \deg A_i$ for all $i = 0, \dots, s$, and let

$$d = \max_{i \neq k} d_i \quad \text{so that} \quad d < d_k.$$

Now, for all $i \neq k$, we estimate the growth of the polynomials $A_i(x)$ as follows

$$|A_i|(r) \leq r^{d_i+1/2} < r^{d+1/2} < r^{d_k-1/4} < |A_k|(r), \quad \text{for sufficiently large } r$$

This inequality comes from the standard growth estimate for polynomials, with an added margin (e.g., $1/2$ or $1/4$) for analytic purposes. Next, using the main equation and assuming $|f(x+i)|(r) = |f|(r)$ for each i , we get

$$\begin{aligned} r^{d+\frac{1}{2}}|f|(r) &< r^{d_k-\frac{1}{4}}|f|(r) \leq |A_k|(r)|f|(r), \\ &\leq \max \left\{ \left(\max_{i \in \{0, \dots, s\} \setminus \{k\}} |A_i|(r) \right) |f|(r), |B|(r) \right\}, \\ &\leq \max \{ r^{d_k-1/4}|f|(r), |B|(r) \}. \end{aligned}$$

However, this implies that the growth of f is bounded by a polynomial rate, contradicting the assumption that f is transcendental (which should grow faster than any polynomial). Therefore, we obtain a contradiction. This contradiction shows that the assumption that only one coefficient A_k has the maximum degree cannot be true if f is transcendental. This critical step contributes to the full proof of Theorem 4.3. $\square$

Theorem 4.3. *Let $f \in \mathcal{M}(K)$ be a non-constant solution of Equation (4.1).*

i) *If $f(x)$ is a rational function of the form*

$$f(x) = \frac{p_0 + p_1x + \dots + p_mx^m}{q_0 + q_1x + \dots + q_tx^t},$$

then $t - m \leq \max_{0 \leq i \leq s} \{\deg A_i\} - \deg B(x)$.

ii) *If $f(x)$ is a transcendental meromorphic function and there is some $k \in \{0, \dots, s\}$ such that*

$$\deg A_k(x) > \max_{i \in \{0, \dots, s\} \setminus \{k\}} \{\deg A_i(x)\}.$$

Then $f(x)$ has infinitely many poles.

Proof. i) Let $f(x) = \frac{P(x)}{Q(x)}$, where

$$P(x) = p_0 + p_1x + \dots + p_mx^m, \quad Q(x) = q_0 + q_1x + \dots + q_tx^t.$$

Then by Equation (4.1), we have

$$\sum_{i=0}^s A_i(x) f(x+i) = B(x).$$

Multiplying both sides by

$$\prod_{j=0}^s Q(x+j),$$

we get

$$\sum_{i=0}^s \left(A_i(x) P(x+i) \prod_{0 \leq j \leq s, j \neq i} Q(x+j) \right) = B(x) \prod_{j=0}^s Q(x+j).$$

Hence, we have

$$\begin{aligned} \deg B(x) + (s+1)t &= \deg \left(B(x) \prod_{j=0}^s Q(x+j) \right) \\ &= \deg \left(\sum_{i=0}^s \left(A_i(x) P(x+i) \prod_{0 \leq j \leq s, j \neq i} Q(x+j) \right) \right) \\ &\leq \max_{0 \leq i \leq s} (\deg A_i) + m + st. \end{aligned}$$

Thus

$$t - m \leq \max_{0 \leq i \leq s} \{\deg A_i\} - \deg B(x).$$

- ii) Let $f(x)$ be a transcendental meromorphic function satisfying the difference equation

$$\sum_{i=0}^s A_i(x) f(x+i) = B(x), \quad (4.2)$$

where $A_i(x)$ and $B(x)$ are polynomials. Suppose there exists an index $k \in \{0, \dots, s\}$ such that $\deg A_k(x) > \max_{i \neq k} \deg A_i(x)$.

We aim to prove that under these assumptions, the function $f(x)$ must have infinitely many poles. Assume, for the sake of contradiction, that $f(x)$ has only finitely many poles. Then it can be expressed as $f(x) = \frac{g(x)}{b(x)}$, where $g(x)$ is an entire function and $b(x)$ is a polynomial that collects all the poles of $f(x)$, which are denoted as $x_0, \dots, x_p$ (counting multiplicity). Substituting this form into Equation 4.2 gives

$$\sum_{i=0}^s A_i(x) \frac{g(x+i)}{b(x+i)} = B(x).$$

Multiplying both sides by the common denominator $\prod_{j=0}^s b(x+j)$ to eliminate all denominators yields

$$\sum_{i=0}^s A_i(x) g(x+i) \prod_{j \neq i} b(x+j) = B(x) \prod_{j=0}^s b(x+j).$$

Now, we obtain the equation

$$\sum_{i=0}^s b_i(x) g(x+i) = B_1(x),$$

where $B_1(x), b_0(x), \dots, b_s(x)$ are polynomials and $g(x)$ is a transcendental entire function. However, by the assumption $\deg A_k(x) > \max_{i \neq k} \deg A_i(x)$,

we have

$$\begin{aligned} \deg b_k(x) &= \deg A_k(x) + (p+1)s \geq \max_{i \neq k} \{\deg A_i(x)\} + (p+1)s \\ &= \max_{i \neq k} \{\deg b_i(x)\}. \end{aligned}$$

By Theorem 4.2 (iii), this is impossible. Therefore, $f(x)$ must have infinitely many poles. Thus Theorem 4.3 is proved. $\square$

Corollary 4.4. *If in Equation (4.1) there is some $k, k \in \{0, \dots, s\}$ such that*

$$\deg A_k(x) > \max_{i \in \{0, \dots, s\} \setminus \{k\}} \{\deg A_i(x)\}.$$

Then,

- i) *Every entire solution of Equation (4.1) is a polynomial.*
- ii) *Every meromorphic solution of Equation (4.1) which has finitely many poles is a rational function.*

We give some examples to show that there exist some equations of the form (4.1), which satisfy Corollary 4.4.

Example 4.5. i) $f(x) = x + 1$ is a polynomial and satisfies the equation

$$(1 - x)y(x) + (2x - 1)y(x + 1) + x^2y(x + 2) = x^3 + 3x + 5.$$

ii) $f(x) = \frac{1}{x+1}$ is a rational function and satisfies the equation

$$(x + 1)y(x) + x^2(x + 2)y(x + 1) = x^2 + 1.$$

5. CONCLUSION

In this work, we have explored the nature of meromorphic solutions to certain ultrametric q -difference equations over a complete, algebraically closed ultrametric field K . By analyzing the growth behavior of these solutions within the field $\mathcal{M}(K)$, we established connections between the structure of the equation and the order of growth of its solutions.

Furthermore, we derived necessary conditions on the polynomial coefficients $A_0(x), \dots, A_s(x)$ and $B(x)$ under which the equation

$$\sum_{i=0}^s A_i(x)f(x+i) = B(x)$$

admits a meromorphic solution in $\mathcal{M}(K)$. These findings contribute to a deeper understanding of the analytic properties of ultrametric difference equations and pave the way for further investigations into their applications and generalizations.

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