

DECAY ESTIMATES OF A HIGHER-ORDER VISCOELASTIC WAVE EQUATION WITH GENERAL STRONG DAMPING AND SOURCE TERMS

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ABSTRACT. In this paper, we consider the initial-boundary value problem for a higher-order viscoelastic wave equation with general strong damping and general strong source terms

$$u_{tt} + (-\Delta)^m u - \int_0^t h(t-s)(-\Delta)^m u(s) ds - \mu(t) g_1(\Delta u_t) - g_2(\Delta u) = 0$$

in a bounded domain. Under appropriate conditions on μ, g_1, g_2 and on the kernel h , we establish the existence of global solutions by using the energy method combined with Faedo-Galarkin procedure. Furthermore, we utilize a direct method based on a the multipliers technique and Martinez's inequality to show a decay estimate for the energy of our problem, which depends on the initial energy and μ .

1. INTRODUCTION

In this work, we treat the global existence and decay properties of solutions for the following boundary value problem of a nonlinear higher-order viscoelastic wave equation with general strong damping and general strong source terms

$$\begin{cases} u_{tt} + (-\Delta)^m u - \int_0^t h(t-s)(-\Delta)^m u(s) ds \\ -\mu(t) g_1(\Delta u_t) - g_2(\Delta u) = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ \frac{\partial^i u}{\partial \nu^i} = \Delta u = 0, \quad i = 0, 1, \dots, m-2, & \text{on } \partial \Omega \times \mathbb{R}_+, \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $(\cdot)_t = \frac{\partial(\cdot)}{\partial t}$, $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and $m \in \mathbb{N}^* - \{1\}$, Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with a smooth boundary $\partial \Omega$, ν is unit outward normal vector on $\partial \Omega$ and $\frac{\partial^i u}{\partial \nu^i}$ denotes the i -order normal derivation of u , h, μ, g_1, g_2 are one-variable functions satisfying some conditions to be sepified later.

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The equation presented in the problem (1.1) is a versatile model that describes a wide range of phenomena in physics, engineering, and biology. It combines the terms of fractional derivative, memory effect and nonlinear dynamics to obtain complex interactions in systems with spatial and temporal dependencies. Its applications are broad and span areas like viscoelastic materials (such as modeling the mechanical behavior of polymeric materials), anomalous diffusion in complex systems (for example modeling diffusion of molecules in cells), nonlinear control systems (like modeling the dynamics of robots), heat transfer and diffusion in complex media (e.g. geothermal systems, heat flow in porous rocks), and nonlinear elasticity and wave propagation (for instance rubber, suspended structures) (see [3, 8, 9, 13, 15, 17]).

We will present some publications through which we treated this problem. In 1968, Sattinger [14] studied the existence of global weak solutions of

$$u_{tt} - \Delta u = f(u), \quad x \in \Omega, t > 0,$$

with initial and Dirichlet boundary conditions. Furthermore, Lasiecka and Toundykov [6] developed an intrinsic approach to derivation of energy decay rates for the semilinear wave equation with localized interior nonlinear monotone damping $g(u_t)$ and a source term $f(u)$ of the following problem

$$\begin{cases} u_{tt} - \Delta u + \chi g(u_t) = f(u), & \text{in } \Omega \times [0, T], \\ u(0) = u_0, \\ u_t(0) = u_1. \end{cases}$$

Amroun and Benaissa [1] investigated the global solvability and energy decay of the equation :

$$u'' - \Delta_x^2 u + g(u') = bu|u|^{p-2} \quad \text{in } \Omega \times [0, +\infty[, \quad (1.2)$$

with initial and Dirichlet boundary conditions, under the following hypothesis $b > 0$, $2 < p \leq +\infty$ if $n = 1, 2, 3, 4$ or $2 < p \leq \frac{2n-2}{n-4}$ if $n \geq 5$, g is an increasing $C^1(\mathbb{R})$ function satisfying

$$\begin{aligned} c_1|s| \leq |g(s)| \leq c_2|s|^r \quad \text{if } |s| \geq 1 \quad \text{with } 1 \leq r \leq +\infty \quad \text{if } n = 1, 2, 3, 4 \\ \text{or } 1 \leq r \leq \frac{n+4}{n-4} \quad \text{if } n \geq 5, \end{aligned}$$

where c_1, c_2 are positive constants. For the equation (1.2) when $g(s) = as|s|^{m-2}$ ($a > 0, m > 2$) Messaoudi [11] showed that the solutions with any initial data continue to exist globally in time if $m \geq p$ and blow up in finite time if $m < p$ and the initial energy is negative. To prove the global existence he used a new method introduced by Todorova [18] based on the fixed point theorem.

In 2015, Ye [20] studied the global existence and blow-up of solutions for higher-order viscoelastic wave equation with a nonlinear source term of the following

problem

$$\begin{cases} u_{tt} + (-\Delta)^m u - \int_0^t g(t-s)(-\Delta)^m u(s) ds = |u|^{p-2}u, & \text{in } \Omega \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \\ \frac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0, 1, \dots, m-1, & \text{on } \partial\Omega \times \mathbb{R}_+, \end{cases}$$

where $m \geq 1$ is a natural number, $p > 2$ is a real number. The existence of global weak solutions for this problem is established by using the Faedo-Galerkin method. Meanwhile, under suitable conditions on relaxation function g and the positive initial energy as well as non-positive initial energy, it is proved that the solution blows up in the finite time and the lifespan estimates of solutions are also given.

Moreover, Guo and Yuan, Lin [4] consider an initial-boundary value problem for a nonlinear viscoelastic wave equation with strong damping, nonlinear damping and source terms as follows

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s) ds - \varepsilon_1 \Delta u_t \\ + \varepsilon_2 |u_t|^{m-2} u_t = \varepsilon_3 |u|^{p-2} u, & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $m \geq 2, p > 2$, $\varepsilon_1, \varepsilon_2 \geq 0$, $\varepsilon_3 > 0$, they proved a blow up result for the solution with negative initial energy if $p > m$, and a global result for $p \leq m$ of the above problem. In addition to that, Tebou [16] studied the following problem

$$\begin{cases} y_{tt} + \Delta^2 y - a(x)|\Delta y_t|^{p-2} \Delta y_t = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ u = \Delta y = 0, & \text{on } \partial\Omega \times \mathbb{R}_+, \\ y|_{t=0} = y^0, \quad y_t|_{t=0} = y^1, & \text{in } \Omega. \end{cases} \quad (1.3)$$

The author proved existence, uniqueness and smoothness results for (1.3), by using an appropriate perturbed energy coupled with multiplier technique, he directly proved exponential and polynomial decay estimates for the underlying energy.

The rest of the paper is organized as follows. In the next section, we give some assumptions and notations and lemmas which will be used later. In section 3, we present the main results, where we use the Faedo-Galerkin approximation scheme to prove the global existence and multiplier technique to obtain the energy decay estimate.

2. PRELIMINARIES AND ASSUMPTIONS

In this section, we shall give the precise hypotheses on h, μ and g_1, g_2 , some notations and definition, and important lemmas which will be needed in this paper.

(H1) $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a decreasing function and $h \in C^2(\mathbb{R}_+)$, satisfying

$h', h'' \in L^\infty(\mathbb{R}_+)$,

$$\int_0^{+\infty} h(s) ds = a < 1. \quad (2.1)$$

(H2) $\mu : \mathbb{R}_+ \rightarrow [1, +\infty[$ is a C^1 -class function such that μ' and $\frac{\mu}{\mu'}$ are bounded,

$$\int_0^{+\infty} \mu(t) dt = +\infty.$$

(H3) $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ is a increasing continuous function satisfying $g_1 \in C^{m-1}(\mathbb{R})$, $g_1^{(k)}$ is bounded for all $0 \leq k \leq m-1$ and there exists $c_i > 0$, $i = 1, 2, 3, 4$ such that

$$c_1|x|^p \leq |g_1(x)| \leq c_2|x|^{1/p} \quad (|x| \leq 1), \quad c_3|x| \leq |g_1(x)| \leq c_4|x|^q \quad (|x| > 1), \quad (2.2)$$

where $p \geq 1$, $1 \leq q \leq +\infty$ if $n = 1, 2$, or $1 \leq q \leq \frac{n+2}{n-2}$ if $n \geq 3$.

(H4) $g_2 : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying $g_2 \in C^{m-1}(\mathbb{R})$, $g_2^{(k)}$ is bounded for all $0 \leq k \leq m-1$ and there exists $c_5 > 0$ such that

$$\forall s \in \mathbb{R} : |g_2(s)| \leq c_5|s| \text{ and } 0 \leq G(s) \leq sg_2(s), \quad (2.3)$$

where $G(s) = \int_0^s g_2(t) dt$.

Lemma 2.1. (Sobolev-Poincare inequality) Let q a number with

$2 \leq q \leq +\infty$ if $n = 1, 2$, or $2 \leq q \leq \frac{2n}{n-2}$ if $n \geq 3$, then there exists a constant $C_s = C_s(\Omega, q)$ such that

$$\|u\|_{L^q(\Omega)} \leq C_s \|\nabla u\|_{L^2(\Omega)} \quad \text{for all } u \in H_0^1(\Omega).$$

Lemma 2.2. For all $u \in H_0^1(\Omega) \cap H^2(\Omega)$, we have

$$\|\nabla u\|_{L^2(\Omega)} \leq c \|\Delta u\|_{H^{-1}(\Omega)} \leq c' \|\Delta u\|_{L^2(\Omega)},$$

where $H^{-1}(\Omega) = (H_0^1(\Omega))'$.

Lemma 2.3. Let Ω' be a bounded open set of $\mathbb{R}_x^n \times \mathbb{R}_t$, $1 < q < \infty$, $g \in L^q(\Omega')$ and (g_m) a sequence of functions of $L^q(\Omega')$ which satisfy

$$\exists C > 0, \forall m \in \mathbb{N} : \|g_m\|_{L^q(\Omega')} \leq C, \quad g_m \rightarrow g \text{ a.e. on } \Omega',$$

then

$$g_m \rightharpoonup g \text{ weakly in } L^q(\Omega').$$

Lemma 2.4. (Martinez inequality) Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a decreasing function, and $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a strictly increasing function of class C^1 such that

$$\varphi(0) = 0, \quad \varphi(t) \mapsto +\infty \text{ as } t \mapsto +\infty.$$

Assume that there exists $\alpha \geq 0$ and $C > 0$ such that for all $S \geq 0$ we have

$$\int_S^{+\infty} E^{\alpha+1}(t) \varphi'(t) dt \leq \frac{1}{C} E^\alpha(0) E(S),$$

then E has the following property : for all $t \in \mathbb{R}_+$

$$E(t) \leq E(0)e^{1-C\varphi(t)} \quad \text{if } \alpha = 0,$$

$$E(t) \leq E(0) \left[\frac{\alpha + 1}{1 + \alpha C\varphi(t)} \right]^{\frac{1}{\alpha}} \quad \text{if } \alpha > 0.$$

Let us introduce two Hilbert spaces H_m, V_m by setting

$$H_m = \{v \in H^{m+1}(\Omega) / v = \Delta v = 0 \text{ on } \partial\Omega\}, \quad \|v\|_{H_m} = \|\nabla^{m-1}\Delta v\|_{L^2(\Omega)},$$

$$V_m = \{v \in H^{2m+1}(\Omega) / v = \Delta v = \Delta^m v = 0 \text{ on } \partial\Omega\}, \quad \|v\|_{V_m} = \|\nabla \Delta^m v\|_{L^2(\Omega)},$$

we have

$$V_m \subset H_m \subset H_0^1(\Omega) \subset H'_m \subset V'_m,$$

with dense and compact imbedding.

If $v \in L^2(\Omega)$, we denote by $\|v\|_{L^2(\Omega)} = \|v\|$.

Definition 2.5. We define the energy associated with the solution of the problem (1.1) by the following formula

$$E(t) = \frac{1}{2} \left[\|\nabla u_t\|^2 + \left(1 - \int_0^t h(s) ds\right) \|u\|_{H_m}^2 + h \diamond \nabla^{m-1} \Delta u(t) + 2 \int_{\Omega} G(\Delta u) dx \right],$$

where $\diamond$ denote the binary operator defined by

$$h \diamond \nabla^{m-1} \Delta u(t) = \int_0^t h(t-s) \|\nabla^{m-1} \Delta u(t) - \nabla^{m-1} \Delta u(s)\|^2 ds.$$

3. MAIN RESULTS

3.1. Global existence. In this subsection, we establish the existence of global weak solution for the problem (1.1) by using the Faedo-Galerkin method.

Theorem 3.1. Suppose that (H1)-(H4) hold. Assume further that $(u_0, u_1) \in V_m \times H_m$. Then the problem (1.1) admits a global weak solution u satisfying

$$u \in L^\infty(0, T; H^{2m}(\Omega) \cap H_m), \quad u_t \in L^\infty(0, T; H_m),$$

and

$$u_{tt} \in L^\infty(0, T; H_0^1(\Omega)),$$

for $T > 0$.

Proof. Step 1 : Approximation solutions. Let $T > 0$ be fixed and let $\{w_j; j \in \mathbb{N}\}$ be a basis of $H_0^1(\Omega)$, H_m and V_m , i.e. the space generated by $\mathcal{W}_k = \{w_1, w_2, \dots, w_k\}$ is dense in $H_0^1(\Omega)$, H_m and V_m . We construct approximate solutions $u^{(k)}$, $k = 1, 2, 3, \dots$, in the form

$$u^{(k)}(t) = \sum_{j=1}^k c_{j,k}(t) w_j(x),$$

where $c_{j,k}$ is determined by the following approximate problem corresponding to (1.1)

$$0 = \int_{\Omega} \left[u_{tt}^{(k)} + (-\Delta)^m u^{(k)} - \int_0^t h(t-s)(-\Delta)^m u^{(k)}(s) ds - \mu(t) g_1(\Delta u_t^{(k)}) - g_2(\Delta u^{(k)}) \right] w dx \text{ for any } w \text{ in } \mathcal{W}_k, \quad (3.1)$$

with initial conditions

$$u^{(k)}(0) = u_0^{(k)} = \sum_{j=1}^k \langle u_0, w_j \rangle w_j \rightarrow u_0 \text{ in } V_m \text{ as } k \rightarrow +\infty, \quad (3.2)$$

$$u_t^{(k)}(0) = u_1^{(k)} = \sum_{j=1}^k \langle u_1, w_j \rangle w_j \rightarrow u_1 \text{ in } H_m \text{ as } k \rightarrow +\infty. \quad (3.3)$$

Replacing $u^{(k)}(t)$ by $\sum_{j=1}^k c_{j,k}(t)w_j(x)$ in (3.1), we obtain a system of ordinary differential equations, since $\mathcal{W}_k$ is a base, therefore $\det\{w_i, w_j\} \neq 0$, then according to the general results of the differential equations we are assured of the existence a solution of (3.1)-(3.3) in an interval $[0, t_k]$, $0 < t_k < T$. In the next steps, we shall prove that these solutions are global.

Step 2 : A priori estimates.

a) A priori estimate 1. Taking $w = -2\Delta u_t^{(k)}$ in (3.1), we get

$$\frac{\partial E^{(k)}(t)}{\partial t} = -\mu(t) \int_{\Omega} \Delta u_t^{(k)} g_1(\Delta u_t^{(k)}) dx - \frac{1}{2} h(t) \|u^{(k)}\|_{H_m}^2 + \frac{1}{2} h' \diamond \nabla^{m-1} \Delta u^{(k)}(t)$$

Integrating over $[0, t]$ where $t < t_k$, exploiting the assumptions (H1) and (H2), $V_m \subset H_m \subset H_0^1(\Omega)$, (3.2) and (3.3), we arrive at

$$\begin{aligned} & \|\nabla u_t^{(k)}\|^2 + (1-a) \|u^{(k)}\|_{H_m}^2 + 2 \int_0^t \int_{\Omega} \mu(s) \Delta u_t^{(k)} g_1(\Delta u_t^{(k)}) dx ds \\ & \leq 2 E^{(k)}(t) + 2 \int_0^t \int_{\Omega} \mu(s) \Delta u_t^{(k)}(s) g_1(\Delta u_t^{(k)}(s)) dx ds \\ & \leq 2 \|\nabla u_1^{(k)}\|^2 + 2 \|\nabla^{m-1} \Delta u_0^{(k)}\|^2 \\ & \leq 2 C_{H_m} \|u_1^{(k)}\|_{H_m}^2 + 2 C_{V_m} \|u_0^{(k)}\|_{V_m}^2 \\ & \leq C_1, \end{aligned} \quad (3.4)$$

where C_1 is a positive constant independent of k and $t \in [0, T]$, the estimate (3.4) implies that

$$u^{(k)} \text{ is bounded in } L^\infty(0, T; H_m), \quad (3.5)$$

$$\mu \Delta u_t^{(k)} g_1(\Delta u_t^{(k)}) \text{ is bounded in } L^1([0, T] \times \Omega). \quad (3.6)$$

It follows from the above formula and the assumption (H2) and (2.2) that

$$\mu g_1(\Delta u_t^{(k)}) \text{ is bounded in } L^2([0, T] \times \Omega). \quad (3.7)$$

Using (2.3), the Lemma 2.1 and the embedding $L^\infty(0, T; H_m) \hookrightarrow L^2(0, T; H_m)$, we get

$$\begin{aligned} \int_0^t \int_\Omega \Delta u^{(k)}(s) g_2(\Delta u^{(k)}(s)) dx ds &\leq c_5 \int_0^t \|\Delta u^{(k)}(s)\|^2 ds \\ &\leq c_5 C_s^{2(m-1)} \int_0^t \|\nabla^{m-1} \Delta u^{(k)}(s)\|^2 ds \\ &\leq c_5 C_s^{2(m-1)} \|u^{(k)}\|_{L^2(0, T; H_m)}^2 \\ &\leq c_5 C_m C_s^{2(m-1)} \|u^{(k)}\|_{L^\infty(0, T; H_m)}^2, \end{aligned}$$

we deduce from (3.5) and (2.3) and the above formula that

$$\Delta u^{(k)} g_2(\Delta u^{(k)}) \text{ is bounded in } L^1([0, T] \times \Omega), \quad (3.8)$$

$$g_2(\Delta u^{(k)}) \text{ is bounded in } L^2([0, T] \times \Omega). \quad (3.9)$$

b) A priori estimate 2. Differentiating the first equation in (1.1) with respect to x and we substitute u by $u^{(k)}$, we multiply the result by $\nabla u_{tt}^{(k)}$ and integrating over Ω , choosing $t = 0$, we obtain

$$\begin{aligned} \|\nabla u_{tt}^{(k)}(0)\|^2 + \int_\Omega \nabla(-\Delta)^m u_0^{(k)} \nabla u_{tt}^{(k)}(0) dx + \mu(0) \int_\Omega \nabla \Delta u_1^{(k)} \nabla u_{tt}^{(k)}(0) g_1'(\Delta u_1^{(k)}) dx \\ + \int_\Omega \nabla \Delta u_0^{(k)} \nabla u_{tt}^{(k)}(0) g_2'(\Delta u_0^{(k)}) dx = 0. \end{aligned}$$

By Cauchy's inequality with ε , from the assumptions (H3), (H4), the Lemmas 2.1 and 2.2, we have

$$\begin{aligned} \left[1 - (1 + \mu(0) \|g_1'\|_{L^\infty(\mathbb{R})} + \|g_2'\|_{L^\infty(\mathbb{R})} \varepsilon) \right] \|\nabla u_{tt}^{(k)}(0)\|^2 &\leq \frac{1}{4\varepsilon} \left[\|\nabla \Delta^m u_0^{(k)}\|^2 \right. \\ &\left. + \mu(0) \|g_1'\|_{L^\infty(\mathbb{R})} C_s^{2(m-2)} \|\nabla^{m-1} \Delta u_1^{(k)}\|^2 + (c' C_s)^{2(m-1)} \|g_2'\|_{L^\infty(\mathbb{R})} \|\nabla \Delta^m u_0^{(k)}\|^2 \right], \end{aligned}$$

choosing $\varepsilon = \frac{1}{2(1 + \mu(0) \|g_1'\|_{L^\infty(\mathbb{R})} + \|g_2'\|_{L^\infty(\mathbb{R})})}$, we get

$$\begin{aligned} \|\nabla u_{tt}^{(k)}(0)\|^2 &\leq (1 + \mu(0) \|g_1'\|_{L^\infty(\mathbb{R})} + \|g_2'\|_{L^\infty(\mathbb{R})}) \left[(1 + (c' C_s)^{2(m-1)}) \right. \\ &\quad \left. \times \|g_2'\|_{L^\infty(\mathbb{R})} \|u_0^{(k)}\|_{V_m}^2 + \mu(0) \|g_1'\|_{L^\infty(\mathbb{R})} C_s^{2(m-2)} \|u_1^{(k)}\|_{H_m}^2 \right], \end{aligned} \quad (3.10)$$

taking into account (3.2) and (3.3), we conclude from (3.10) that

$$u_{tt}^{(k)}(0) \text{ is bounded in } H_0^1(\Omega). \quad (3.11)$$

c) A priori estimate 3. Differentiating the first equation in (1.1) with respect to t and we substitute u by $u^{(k)}$, we multiply the result by $-2 \Delta u_{tt}^{(k)}(t)$ and

integrating over Ω , we obtain

$$\begin{aligned}
0 = & \frac{\partial}{\partial t} (\|\nabla u_{tt}^{(k)}\|^2 + \|u_t^{(k)}\|_{H_m}^2) \\
& + 2 \int_0^t \int_{\Omega} h'(t-s) (-\Delta)^m u^{(k)}(s) \Delta u_{tt}^{(k)}(t) dx ds \\
& + 2 h(0) \int_{\Omega} (-\Delta)^m u^{(k)} \Delta u_{tt}^{(k)} dx + 2 \mu'(t) \int_{\Omega} \Delta u_{tt}^{(k)} g_1(\Delta u_t^{(k)}) dx \\
& + 2 \mu(t) \int_{\Omega} (\Delta u_{tt}^{(k)})^2 g_1'(\Delta u_t^{(k)}) dx + 2 \int_{\Omega} \Delta u_t^{(k)} \Delta u_{tt}^{(k)} g_2'(\Delta u_t^{(k)}) dx.
\end{aligned} \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}
& \int_0^t \int_{\Omega} h'(t-s) (-\Delta)^m u^{(k)}(s) \Delta u_{tt}^{(k)}(t) dx ds + h(0) \int_{\Omega} (-\Delta)^m u^{(k)} \Delta u_{tt}^{(k)} dx \\
& = - \int_0^t \int_{\Omega} h'(t-s) \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_{tt}^{(k)}(t) dx ds - h(0) \int_{\Omega} \nabla^{m-1} \Delta u^{(k)} \nabla^{m-1} \Delta u_{tt}^{(k)} dx \\
& = - \frac{\partial}{\partial t} \left[\int_0^t \int_{\Omega} h'(t-s) \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(t) dx ds + h(0) \int_{\Omega} \nabla^{m-1} \Delta u^{(k)} \nabla^{m-1} \Delta u_t^{(k)} dx \right] \\
& \quad + h(0) \|u_t^{(k)}\|_{H_m}^2 + h'(0) \int_{\Omega} \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(t) dx \\
& \quad + \int_0^t \int_{\Omega} h''(t-s) \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(t) dx ds.
\end{aligned}$$

Inserting the above equality into (3.12), we obtain

$$\begin{aligned}
& \frac{\partial}{\partial t} \left[\|\nabla u_{tt}^{(k)}\|^2 + \|u_t^{(k)}\|_{H_m}^2 - 2 \int_0^t \int_{\Omega} h'(t-s) \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(t) dx ds \right. \\
& \quad \left. - 2 h(0) \int_{\Omega} \nabla^{m-1} \Delta u^{(k)} \nabla^{m-1} \Delta u_t^{(k)} dx \right] \\
& = - 2 \int_0^t \int_{\Omega} h''(t-s) \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(t) dx ds - 2 h(0) \|u_t^{(k)}\|_{H_m}^2 \\
& \quad - 2 h'(0) \int_{\Omega} \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(t) dx - 2 \mu'(t) \int_{\Omega} \Delta u_{tt}^{(k)} g_1(\Delta u_t^{(k)}) dx \\
& \quad - 2 \int_{\Omega} \Delta u_t^{(k)} \Delta u_{tt}^{(k)} g_2'(\Delta u_t^{(k)}) dx - 2 \mu(t) \int_{\Omega} (\Delta u_{tt}^{(k)})^2 g_1'(\Delta u_t^{(k)}) dx.
\end{aligned}$$

Integrating the above equality over $[0, t]$ where $t < t_k$, yields

$$\begin{aligned}
& \|\nabla u_{tt}^{(k)}\|^2 + \|u_t^{(k)}\|_{H_m}^2 \\
& \leq -2h'(0) \int_0^t \int_\Omega \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(s) dx ds \\
& \quad - 2 \int_0^t \int_0^\tau \int_\Omega h''(\tau - s) \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(\tau) dx ds d\tau \\
& \quad + 2 \int_0^t \int_\Omega h'(t - s) \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(t) dx ds \\
& \quad + 2h(0) \int_\Omega \left[\nabla^{m-1} \Delta u^{(k)} \nabla^{m-1} \Delta u_t^{(k)} - \nabla^{m-1} \Delta u_0^{(k)} \nabla^{m-1} \Delta u_1^{(k)} \right] dx \\
& \quad - 2 \int_0^t \int_\Omega \mu'(s) \Delta u_{tt}^{(k)}(s) g_1(\Delta u_t^{(k)}(s)) dx + \|\nabla u_{tt}^{(k)}(0)\|^2 + \|u_1^{(k)}\|_{H_m}^2 \\
& \quad - 2 \int_0^t \int_\Omega \Delta u_t^{(k)}(s) \Delta u_{tt}^{(k)}(s) g_2'(\Delta u^{(k)}(s)) dx.
\end{aligned} \tag{3.13}$$

Utilizing Cauchy's inequality and the embedding $L^\infty(0, T; H_m) \hookrightarrow L^2(0, T; H_m)$, the assumption (H1) and (3.5), we get

$$\begin{aligned}
& 2h'(0) \int_0^t \int_\Omega \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(s) dx ds \\
& \leq -h'(0) \int_0^t \|u^{(k)}\|_{H_m}^2 ds - h'(0) \int_0^t \|u_t^{(k)}\|_{H_m}^2 ds \\
& \leq -h'(0) \left[\|u^{(k)}\|_{L^2(0, T; H_m)}^2 + \int_0^t \|u_t^{(k)}\|_{H_m}^2 ds \right] \\
& \leq C_2 - h'(0) \int_0^t \|u_t^{(k)}\|_{H_m}^2 ds.
\end{aligned} \tag{3.14}$$

And

$$\begin{aligned}
& -2 \int_0^t \int_0^\tau \int_\Omega h''(\tau - s) \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(\tau) dx ds d\tau \\
& \leq 2 \int_0^t \int_\Omega \left| \nabla^{m-1} \Delta u_t^{(k)}(\tau) \int_0^\tau h''(\tau - s) \nabla^{m-1} \Delta u^{(k)}(s) ds \right| dx d\tau \\
& \leq \int_0^t \int_\Omega \left[\int_0^\tau h''(\tau - s) \nabla^{m-1} \Delta u^{(k)}(s) ds \right]^2 dx d\tau + \int_0^t \|u_t^{(k)}\|_{H_m}^2 ds \\
& \leq \|h''\|_{L^\infty(\mathbb{R})}^2 \int_0^t \int_\Omega \int_0^\tau (\nabla^{m-1} \Delta u^{(k)}(s))^2 ds dx d\tau + \int_0^t \|u_t^{(k)}\|_{H_m}^2 ds \\
& \leq T \|h''\|_{L^\infty(\mathbb{R})}^2 \int_0^t \|u^{(k)}\|_{H_m}^2 ds + \int_0^t \|u_t^{(k)}\|_{H_m}^2 ds \\
& \leq C_3 + \int_0^t \|u_t^{(k)}\|_{H_m}^2 ds.
\end{aligned} \tag{3.15}$$

By Cauchy's inequality with ε and the embedding $L^\infty(0, T; H_m) \hookrightarrow L^2(0, T; H_m)$, the assumption (H1) and (3.5) we see that

$$\begin{aligned}
& 2 \int_0^t \int_\Omega h'(t-s) \nabla^{m-1} \Delta u^{(k)}(s) \nabla^{m-1} \Delta u_t^{(k)}(t) dx ds \\
& \leq 2\varepsilon \|u_t^{(k)}\|_{H_m}^2 + \frac{1}{2\varepsilon} \int_\Omega \left[\int_0^t h'(t-s) \nabla^{m-1} \Delta u^{(k)}(s) ds \right]^2 dx \\
& \leq 2\varepsilon \|u_t^{(k)}\|_{H_m}^2 + \frac{c \|h'\|_{L^\infty(\mathbb{R})}^2}{2\varepsilon} \int_0^t \|u^{(k)}\|_{H_m}^2 ds \\
& \leq 2\varepsilon \|u_t^{(k)}\|_{H_m}^2 + C_1(\varepsilon).
\end{aligned} \tag{3.16}$$

Using Cauchy's inequality with ε and $V_m \subset H_m$, taking into account (3.2), (3.3), (3.5), we get easily

$$\begin{aligned}
& 2h(0) \int_\Omega \nabla^{m-1} \Delta u^{(k)} \nabla^{m-1} \Delta u_t^{(k)} - \nabla^{m-1} \Delta u_0^{(k)} \nabla^{m-1} \Delta u_1^{(k)} dx \\
& \leq 2h(0) \left[\varepsilon \|u_t^{(k)}\|_{H_m}^2 + \frac{\|u^{(k)}\|_{H_m}^2}{2\varepsilon} + C_{H_m} \|u_0^{(k)}\|_{V_m}^2 + \|u_1^{(k)}\|_{H_m}^2 \right] \\
& \leq 2h(0)\varepsilon \|u_t^{(k)}\|_{H_m}^2 + C_2(\varepsilon).
\end{aligned} \tag{3.17}$$

Applying the Green's formula and Cauchy's inequality, employing the Lemma 2.1, the assumptions (H2)-(H4) and (3.5), we obtain

$$\begin{aligned}
& -2 \int_0^t \int_\Omega \left[\mu'(s) g_1(\Delta u_t^{(k)}(s)) + \Delta u_t^{(k)}(s) g_2'(\Delta u^{(k)}(s)) \right] \Delta u_{tt}^{(k)}(s) dx ds \\
& = 2 \int_0^t \int_\Omega \mu'(s) \nabla u_{tt}^{(k)}(s) \nabla \Delta u_t^{(k)}(s) g_1'(\Delta u_t^{(k)}(s)) dx ds \\
& \quad + 2 \int_0^t \int_\Omega \nabla \Delta u^{(k)}(s) \nabla u_{tt}^{(k)}(s) \Delta u_t^{(k)}(s) g_2''(\Delta u^{(k)}(s)) + \nabla \Delta u_t^{(k)}(s) \nabla u_{tt}^{(k)}(s) g_2'(\Delta u^{(k)}(s)) dx ds \\
& \leq \|\mu'\|_{L^\infty(\mathbb{R}_+)} \|g_1'\|_{L^\infty(\mathbb{R})} \int_0^t \|\nabla u_{tt}^{(k)}(s)\|^2 + \|\nabla \Delta u_t^{(k)}(s)\|^2 ds \\
& \quad + \|g_2''\|_{L^\infty(\mathbb{R})} \int_0^t \|\nabla \Delta u^{(k)}(s)\|^2 + \|\nabla u_{tt}^{(k)}(s)\|^2 + \|\Delta u_t^{(k)}(s)\|^2 ds \\
& \quad + \|g_2'\|_{L^\infty(\mathbb{R})} \int_0^t \|\nabla \Delta u_t^{(k)}(s)\|^2 + \|\nabla u_{tt}^{(k)}(s)\|^2 ds \\
& \leq \left[\|\mu'\|_{L^\infty(\mathbb{R}_+)} \|g_1'\|_{L^\infty(\mathbb{R})} + \|g_2''\|_{L^\infty(\mathbb{R})} + \|g_2'\|_{L^\infty(\mathbb{R})} \right] \int_0^t \|\nabla u_{tt}^{(k)}(s)\|^2 ds \\
& \quad + \left[(\|\mu'\|_{L^\infty(\mathbb{R}_+)} \|g_1'\|_{L^\infty(\mathbb{R})} + \|g_2'\|_{L^\infty(\mathbb{R})}) C_s^{2(m-2)} + \|g_2''\|_{L^\infty(\mathbb{R})} C_s^{2(m-1)} \right] \\
& \quad \times \int_0^t \|\nabla^{m-1} \Delta u_t^{(k)}(s)\|^2 ds + \|g_2''\|_{L^\infty(\mathbb{R})} C_s^{2(m-2)} \int_0^t \|\nabla^{m-1} \Delta u^{(k)}(s)\|^2 ds,
\end{aligned}$$

$$\begin{aligned}
& -2 \int_0^t \int_{\Omega} \left[\mu'(s) g_1(\Delta u_t^{(k)}(s)) + \Delta u_t^{(k)}(s) g_2'(\Delta u^{(k)}(s)) \right] \Delta u_{tt}^{(k)}(s) dx ds \\
& \leq C_4 \int_0^t \|\nabla u_{tt}^{(k)}(s)\|^2 ds + C_5 \int_0^t \|u_t^{(k)}(s)\|_{H_m}^2 ds + C_6.
\end{aligned} \tag{3.18}$$

Combining (3.13)-(3.18), we deduce that

$$\begin{aligned}
& \left[1 - (2h(0) + 2)\varepsilon \right] \|u_t^{(k)}(t)\|_{H_m}^2 + \|\nabla u_{tt}^{(k)}(t)\|^2 \leq (-h'(0) + 1 + C_5) \int_0^t \|u_t^{(k)}(s)\|_{H_m}^2 ds \\
& + C_4 \int_0^t \|\nabla u_{tt}^{(k)}(s)\|^2 ds + \|\nabla u_{tt}^{(k)}(0)\|^2 + \|u_1^{(k)}\|_{H_m}^2 + C_7 + C(\varepsilon),
\end{aligned}$$

choosing $\varepsilon = \frac{1}{2(2 + 2h(0))}$ and taking into account (3.3) and (3.11), the previous inequality implies

$$\begin{aligned}
& \|u_t^{(k)}(t)\|_{H_m}^2 + \|\nabla u_{tt}^{(k)}(t)\|^2 \leq 2 \max(-h'(0) + 1 + C_5, C_4) \int_0^t \|u_t^{(k)}(s)\|_{H_m}^2 ds \\
& + \|\nabla u_{tt}^{(k)}(s)\|^2 ds + C_8,
\end{aligned}$$

applying Gronwall's Lemma on the above estimate, we deduce that

$$u_t^{(k)} \text{ is bounded in } L^\infty(0, T; H_m), \tag{3.19}$$

$$u_{tt}^{(k)} \text{ is bounded in } L^\infty(0, T; H_0^1(\Omega)). \tag{3.20}$$

d) A priori estimate 4. Taking $w = (-\Delta)^m u_t^{(k)}$ in (3.1), we have

$$\begin{aligned}
0 &= \int_{\Omega} u_{tt}^{(k)} (-\Delta)^m u_t^{(k)} dx + \int_{\Omega} (-\Delta)^m u^{(k)} (-\Delta)^m u_t^{(k)} dx \\
& - \int_0^t \int_{\Omega} h(t-s) (-\Delta)^m u^{(k)}(s) (-\Delta)^m u_t^{(k)}(t) dx ds - \mu(t) \int_{\Omega} g_1(\Delta u_t^{(k)}) (-\Delta)^m u_t^{(k)} dx \\
& - \int_{\Omega} g_2(\Delta u^{(k)}) (-\Delta)^m u_t^{(k)} dx,
\end{aligned}$$

applying the Green's formula on the above equality, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial t} \left[\|\nabla^m u_t^{(k)}\|^2 + \left(1 - \int_0^t h(s) ds \right) \|\Delta^m u^{(k)}\|^2 + h \diamond (-\Delta)^m u^{(k)} \right] = \frac{1}{2} h' \diamond (-\Delta)^m u^{(k)} \\
& - \frac{1}{2} h(t) \|\Delta^m u^{(k)}\|^2 - \mu(t) \int_{\Omega} \nabla^{m-1} g_1(\Delta u_t^{(k)}) \nabla^{m-1} \Delta u_t^{(k)} dx - \int_{\Omega} \nabla^{m-1} g_2(\Delta u^{(k)}) \nabla^{m-1} \Delta u_t^{(k)} dx,
\end{aligned}$$

integrating the last equality over $[0, t]$ where $t < t_k$ and using Cauchy's inequality, we obtain under the assumptions (H1)-(H4)

$$\begin{aligned}
& \|\nabla^m u_t^{(k)}\|^2 + \left(1 - \int_0^t h(s) ds \right) \|\Delta^m u^{(k)}\|^2 + h \diamond (-\Delta)^m u^{(k)} \leq \|\nabla^m u_1^{(k)}\|^2 + \|\Delta^m u_0^{(k)}\|^2 \\
& + \|\mu\|_{L^\infty(\mathbb{R}_+)} C_m \int_0^t \|\nabla^{m-1} \Delta u_t^{(k)}(s)\|^2 ds + C_m^* \int_0^t \|\nabla^{m-1} \Delta u^{(k)}(s)\|^2 ds,
\end{aligned}$$

utilizing (2.1), the Lemmas 2.1 and 2.2, the embedding $L^\infty(0, T; H_m) \hookrightarrow L^2(0, T; H_m)$, the last estimate implies

$$(1-a)\|\Delta^m u^{(k)}\|^2 \leq C_9 \|u_1^{(k)}\|_{H_m}^2 + C_{10} \|u_0^{(k)}\|_{V_m}^2 + C_{11} \|u_t^{(k)}\|_{L^\infty(0, T; H_m)}^2 + C_{12} \|u^{(k)}\|_{L^\infty(0, T; H_m)}^2,$$

taking into account (3.2) and (3.3), (3.5) and (3.19), we deduce from the above inequality that

$$u^{(k)} \text{ is bounded in } L^\infty(0, T; H^{2m}(\Omega)). \quad (3.21)$$

Step 3 : Passage to the limit. Now we easily conclude from (3.5), (3.19), (3.20) and (3.21) that there exists a subsequence of $\{u^{(k)}\}$, still represented by the same notation, such that

$$u^{(k)} \rightharpoonup u \text{ weak star in } L^\infty(0, T; H_m \cap H^{2m}(\Omega)), \quad (3.22)$$

$$u_t^{(k)} \rightharpoonup u_t \text{ weak star in } L^\infty(0, T; H_m), \quad (3.23)$$

$$u_{tt}^{(k)} \rightharpoonup u_{tt} \text{ weak star in } L^\infty(0, T; H_0^1(\Omega)). \quad (3.24)$$

As $u^{(k)}$ is bounded in $L^\infty(0, T; H_m)$ (see (3.5)) and the injection of H_m in $H_0^1(\Omega)$ is compact, we have

$$u^{(k)} \rightarrow u \text{ strongly in } L^2(0, T; H_0^1(\Omega)). \quad (3.25)$$

It follows from (3.22), (3.24) and (3.25) that for each fixed $w \in L^2(0, T; H_0^1(\Omega))$, we have

$$\begin{aligned} & \int_0^T \int_\Omega \left[u_{tt}^{(k)} + (-\Delta)^m u^{(k)} - \int_0^t h(t-s)(-\Delta)^m u^{(k)}(s) ds \right] w dx dt \longrightarrow \\ & \int_0^T \int_\Omega \left[u_{tt} + (-\Delta)^m u - \int_0^t h(t-s)(-\Delta)^m u(s) ds \right] w dx dt \text{ as } k \rightarrow +\infty \end{aligned}$$

It remains to show that for each fixed $w \in L^2(0, T; H_0^1(\Omega))$, we have

$$\begin{aligned} & \int_0^T \int_\Omega \left[\mu(t)g_1(\Delta u_t^{(k)}) + g_2(\Delta u^{(k)}) \right] w dx dt \longrightarrow \\ & \int_0^T \int_\Omega \left[\mu(t)g_1(\Delta u_t) + g_2(\Delta u) \right] w dx dt \text{ as } k \rightarrow +\infty. \end{aligned} \quad (3.26)$$

We conclude from $H_m \subset H_0^1(\Omega)$ and (3.5), (3.19) that $u^{(k)}$ and $u_t^{(k)}$ are bounded in $L^\infty(0, T; H_0^1(\Omega))$, $u_{tt}^{(k)}$ is bounded in $L^\infty(0, T; H_0^1(\Omega))$ (see (3.20)). In the other hand, we have $L^\infty(0, T; H_0^1(\Omega)) \subset L^2(0, T; H_0^1(\Omega))$, then $u^{(k)}$, $u_t^{(k)}$ and $u_{tt}^{(k)}$ are bounded in $L^2(0, T; H_0^1(\Omega))$. Consequently $u^{(k)}$ and $u_t^{(k)}$ are bounded in $H^1([0, T] \times \Omega)$. By the Rellich-Kondrachoff Theorem's, the injection of $H^1([0, T] \times \Omega)$ in $L^2([0, T] \times \Omega)$ is compact, then there exists a subsequence $\{u^{(k)}\}$, still represented by the same notation, such that

$$(u^{(k)}, u_t^{(k)}) \rightarrow (u, u_t) \text{ strongly in } (L^2([0, T] \times \Omega))^2,$$

which implies

$$(u^{(k)}, u_t^{(k)}) \rightarrow (u, u_t) \text{ a.e. on } ([0, T] \times \Omega)^2. \quad (3.27)$$

Since μ, g_1, g_2 are continuous, from (3.27) we have

$$(g_2(\Delta u^{(k)}), \mu g_1(\Delta u_t^{(k)})) \rightarrow (g_2(\Delta u), \mu g_1(\Delta u_t)) \text{ a.e. on } ([0, T] \times \Omega)^2, \quad (3.28)$$

$$\begin{aligned}\Delta u^{(k)} g_2(\Delta u^{(k)}) &\rightarrow \Delta u g_2(\Delta u) \quad \text{a.e. on } [0, T] \times \Omega, \\ \mu \Delta u_t^{(k)} g_1(\Delta u_t^{(k)}) &\rightarrow \mu \Delta u_t g_1(\Delta u_t) \quad \text{a.e. on } [0, T] \times \Omega.\end{aligned}\tag{3.29}$$

It follows from (3.6), (3.8), (3.29) and the Fatou's Lemma that

$$(\Delta u g_2(\Delta u), \mu \Delta u_t g_1(\Delta u_t)) \in (L^1([0, T] \times \Omega))^2,$$

exploiting the last formula and (2.2), (2.3), we have

$$(g_2(\Delta u), \mu g_1(\Delta u_t)) \in (L^2([0, T] \times \Omega))^2.\tag{3.30}$$

By (3.7), (3.9), (3.28), (3.30) and the Lemma 2.3, we get

$$(g_2(\Delta u^{(k)}), \mu g_1(\Delta u_t^{(k)})) \rightharpoonup (g_2(\Delta u), \mu g_1(\Delta u_t)) \text{ weakly in } (L^2([0, T] \times \Omega))^2,$$

which implies (3.26). Hence, for all $w \in L^2(0, T; H_0^1(\Omega))$:

$$\int_0^T \int_{\Omega} \left[u_{tt} + (-\Delta)^m u - \int_0^t h(t-s)(-\Delta)^m u(s) ds - \mu(t) g_1(\Delta u_t) - g_2(\Delta u) \right] w dx dt = 0.$$

□

3.2. Energy decay estimate. In this subsection, we shall investigate the asymptotic behavior of the problem (1.1). To achieve this, we will use the multiplied method.

Lemma 3.2. *Let u be a solutions of the problem (1.1). Then the energy functional defined by the Definition 2.5 satisfies*

$$E'(t) = -\mu(t) \int_{\Omega} \Delta u_t g_1(\Delta u_t) dx - \frac{1}{2} h(t) \|u\|_{H_m}^2 + \frac{1}{2} h' \diamond \nabla^{m-1} \Delta u(t) \leq 0.$$

Proof. Multiplying the first equation in (1.1) by $-\Delta u_t$ and integrating over Ω , we get

$$\begin{aligned}\frac{1}{2} \frac{\partial}{\partial t} \left[\|\nabla u_t\|^2 + \left(1 - \int_0^t h(s) ds\right) \|u\|_{H_m}^2 + h \diamond \nabla^{m-1} \Delta u(t) + 2 \int_{\Omega} G(\Delta u) dx \right] \\ = -\mu(t) \int_{\Omega} \Delta u_t g_1(\Delta u_t) dx - \frac{1}{2} h(t) \|u\|_{H_m}^2 + \frac{1}{2} h' \diamond \nabla^{m-1} \Delta u(t).\end{aligned}$$

Since g_1 is increasing, then if $\Delta u_t \leq 0$ we obtain $g_1(\Delta u_t) \leq g_1(0)$, according to the first formula of (2.2), we have $g_1(0) = 0$, consequently $\Delta u_t g_1(\Delta u_t) \geq 0$. In the same way we get $\Delta u_t g_1(\Delta u_t) \geq 0$ when $\Delta u_t \geq 0$, therefore $\int_{\Omega} \Delta u_t g_1(\Delta u_t) dx \geq 0$. By exploiting (H1) and (H2), we conclude that $E'(t) \leq 0$. □

Theorem 3.3. *Assume that (H1)-(H4) hold. Let $(u_0, u_1) \in V_m \times H_m$. Then for the global weak solution of the problem (1.1), there exists a positive constant C such that*

$$\forall t \in \mathbb{R}_+; E(t) \leq E(0) e^{1-C \int_0^t \mu(s) ds} \quad \text{if } P = 1,$$

$$\forall t \in \mathbb{R}_+; E(t) \leq E(0) \left[\frac{P+1}{2 + C(P-1) \int_0^t \mu(s) ds} \right]^{\frac{2}{P-1}} \quad \text{if } P > 1.$$

To prove this theorem, we need to prove the following three lemmas.

Lemma 3.4. *For all $0 \leq S < T < +\infty$, we have the following estimate*

$$\begin{aligned}
& \int_S^T E^{\frac{P+1}{2}} \mu(t) dt \leq \int_S^T \int_{\Omega} E^{\frac{P-1}{2}} \mu(t) \Delta u g_2(\Delta u) dx dt \\
& + \frac{1}{2} \int_S^T \left[E^{\frac{P-1}{2}} \mu(t) \|u\|_{H_m}^2 \int_0^t h(s) ds \right] dt \\
& + \frac{1}{2} \int_S^T \left[E^{\frac{P-1}{2}} \mu(t) \int_0^t h(s) \|u(t-s)\|_{H_m}^2 ds \right] dt \\
& + \left[E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} \Delta u u_t dx \right]_S^T \\
& - \int_S^T \left[\left(\frac{P-1}{2} E^{\frac{P-3}{2}} E' \mu(t) + E^{\frac{P-1}{2}} \mu'(t) \right) \int_{\Omega} \Delta u u_t dx \right] dt \\
& + \int_S^T E^{\frac{P-1}{2}} \left[\int_{\Omega} 2 \mu(t) |\nabla u_t|^2 - \mu^2(t) \Delta u g_1(\Delta u_t) dx \right] dt.
\end{aligned} \tag{3.31}$$

Proof. We define φ with the form $\varphi(t) = \int_0^t \mu(s) ds$, we multiply the first equation in (1.1) by $-\mu(t) \Delta u E^{\frac{P-1}{2}}$ and integrating over $[S, T] \times \Omega$, using integration by parts, we obtain

$$\begin{aligned}
& \int_S^T -E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} \Delta u \left[u_{tt} + (-\Delta)^m u - \int_0^t h(t-s) (-\Delta)^m u(s) ds - \mu(t) g_1(\Delta u_t) - g_2(\Delta u) \right] dx dt \\
& = - \left[E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} \Delta u u_t dx \right]_S^T - \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) h \diamond \nabla^{m-1} \Delta u(t) dt \\
& - \int_S^T E^{\frac{P-1}{2}} \left[\int_{\Omega} 2 \mu(t) |\nabla u_t|^2 - \mu^2(t) \Delta u g_1(\Delta u_t) dx \right] dt \\
& + \int_S^T \left[\left(\frac{P-1}{2} E^{\frac{P-3}{2}} E' \mu(t) + E^{\frac{P-1}{2}} \mu'(t) \right) \int_{\Omega} \Delta u u_t dx \right] dt - \int_S^T \int_{\Omega} E^{\frac{P-1}{2}} \mu(t) \Delta u g_2(\Delta u) dx dt \\
& + \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\|\nabla u_t\|^2 + \left(1 - \int_0^t h(s) ds \right) \|u\|_{H_m}^2 + h \diamond \nabla^{m-1} \Delta u(t) + 2 \int_{\Omega} \Delta u g_2(\Delta u) dx \right] dt \\
& - \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\|u\|_{H_m}^2 \int_0^t h(s) ds + \int_0^t h(s) \|u(t-s)\|_{H_m}^2 ds \right] dt \\
& = 0,
\end{aligned}$$

it follows from (2.3) that

$$\begin{aligned}
2 \int_S^T E^{\frac{P+1}{2}} \mu(t) dt &\leq \int_S^T \int_{\Omega} E^{\frac{P-1}{2}} \mu(t) \Delta u g_2(\Delta u) dx dt \\
&\quad + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \|u\|_{H_m}^2 \left[\int_0^t h(s) ds \right] dt \\
&\quad + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\int_0^t h(s) \|u(t-s)\|_{H_m}^2 ds \right] dt \\
&\quad + \left[E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} \Delta u u_t dx \right]_S^T + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) h \diamond \nabla^{m-1} \Delta u(t) dt \\
&\quad - \int_S^T \left[\left(\frac{P-1}{2} E^{\frac{P-3}{2}} E' \mu(t) + E^{\frac{P-1}{2}} \mu'(t) \right) \int_{\Omega} \Delta u u_t dx \right] dt \\
&\quad + \int_S^T E^{\frac{P-1}{2}} \left[\int_{\Omega} 2 \mu(t) |\nabla u_t|^2 - \mu^2(t) \Delta u g_1(\Delta u_t) dx \right] dt,
\end{aligned}$$

we know that $\frac{1}{2} h \diamond \nabla^{m-1} \Delta u(t) \leq E(t)$, therefore (3.31) is obtained. $\square$

Lemma 3.5. *For all $0 \leq S < T < +\infty$, we have the following inequality*

$$\begin{aligned}
& - \int_S^T \left[\left(\frac{P-1}{2} E^{\frac{P-3}{2}} E' \mu(t) + E^{\frac{P-1}{2}} \mu'(t) \right) \int_{\Omega} \Delta u u_t dx \right] dt \\
& + \left[E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} \Delta u u_t dx \right]_S^T + \int_S^T \int_{\Omega} E^{\frac{P-1}{2}} \mu(t) \Delta u g_2(\Delta u) dx dt \\
& + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\int_0^t h(s) ds \right] \|u\|_{H_m}^2 dt \\
& + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\int_0^t h(s) \|u(t-s)\|_{H_m}^2 ds \right] dt \\
& \leq C_{13} E^{\frac{P-1}{2}}(0) E(S).
\end{aligned} \tag{3.32}$$

Proof. Using Cauchy's inequality and the Lemma 2.1, the decreasing of E , for all $t \in \mathbb{R}_+$, we get

$$\begin{aligned}
& - \int_S^T \left(\frac{P-1}{2} E^{\frac{P-3}{2}} E' \mu(t) + E^{\frac{P-1}{2}} \mu'(t) \right) \int_{\Omega} \Delta u u_t dx dt \\
& \leq - \frac{1}{2} \int_S^T \left(\frac{P-1}{2} E^{\frac{P-3}{2}} E' \mu(t) + E^{\frac{P-1}{2}} \mu'(t) \right) \left[C_s^{2(m-1)} \frac{1 - \int_0^t h(s) ds}{1 - \int_0^t h(s) ds} \|u\|_{H_m}^2 + C_s'^2 \|\nabla u_t\|^2 \right] dt \\
& \leq - \int_S^T \frac{(P-1) C_s^{2(m-1)}}{2(1-a)} E^{\frac{P-1}{2}} E' \mu(t) + C_s'^2 E^{\frac{P+1}{2}} \mu'(t) dt \\
& \leq \frac{\|\mu\|_{L^\infty(\mathbb{R}_+)} (P-1) C_s^{2(m-1)}}{(1-a)(P+1)} \left[- E^{\frac{P+1}{2}} \right]_S^T - C_s'^2 E^{\frac{P+1}{2}}(S) \int_S^T \mu'(t) dt,
\end{aligned}$$

the above estimate implies

$$\begin{aligned}
& - \int_S^T \left(\frac{P-1}{2} E^{\frac{P-3}{2}} E' \mu(t) + E^{\frac{P-1}{2}} \mu'(t) \right) \int_{\Omega} \Delta u u_t dx dt \\
& \leq \left[\frac{(P-1)C_s^{2(m-1)}}{(1-a)(P+1)} + 2C_s'^2 \right] \|\mu\|_{L^\infty(\mathbb{R}_+)} E^{\frac{P-1}{2}}(0) E(S).
\end{aligned} \tag{3.33}$$

Similarly, we have

$$\begin{aligned}
E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} \Delta u u_t dx & \leq E^{\frac{P-1}{2}}(0) \|\mu\|_{L^\infty(\mathbb{R}_+)} \int_{\Omega} |\Delta u| |u_t| dx \\
& \leq \frac{1}{2} \|\mu\|_{L^\infty(\mathbb{R}_+)} E^{\frac{P-1}{2}}(0) \left[C_s^{2(m-1)} \frac{1 - \int_0^t h(s) ds}{1 - \int_0^t h(s) ds} \|u\|_{H_m}^2 + C_s'^2 \|\nabla u_t\|^2 \right] \\
& \leq \left[\frac{C_s^{2(m-1)}}{1-a} + C_s'^2 \right] \|\mu\|_{L^\infty(\mathbb{R}_+)} E^{\frac{P-1}{2}}(0) E(t),
\end{aligned}$$

hence

$$\left[E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} \Delta u u_t dx \right]_S^T \leq \left[\frac{C_s^{2(m-1)}}{1-a} + C_s'^2 \right] \|\mu\|_{L^\infty(\mathbb{R}_+)} E^{\frac{P-1}{2}}(0) (E(S) + E(t)),$$

therefore

$$\left[E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} \Delta u u_t dx \right]_S^T \leq \left[\frac{2C_s^{2(m-1)}}{1-a} + 2C_s'^2 \right] \|\mu\|_{L^\infty(\mathbb{R}_+)} E^{\frac{P-1}{2}}(0) E(S). \tag{3.34}$$

We obtain from (2.3) and the Lemma 2.1, (2.1), (H2), the decreasing of E that

$$\begin{aligned}
& \int_S^T \int_{\Omega} E^{\frac{P-1}{2}} \mu(t) \Delta u g_2(\Delta) dx dt + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\int_0^t h(s) ds \right] \\
& \times \|u\|_{H_m}^2 dt + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\int_0^t h(s) \|u(t-s)\|_{H_m}^2 ds \right] dt \\
& \leq c_5 C_s^{2(m-1)} \int_S^T E^{\frac{P-1}{2}} \mu(t) \frac{1 - \int_0^t h(s) ds}{1 - \int_0^t h(s) ds} \|u\|_{H_m}^2 dt \\
& + a \int_S^T E^{\frac{P-1}{2}} \mu(t) \frac{1 - \int_0^t h(s) ds}{1 - \int_0^t h(s) ds} \|u\|_{H_m}^2 dt \\
& + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\int_0^t h(t-s) \frac{1 - \int_0^t h(s) ds}{1 - \int_0^t h(s) ds} \|u(s)\|_{H_m}^2 ds \right] dt
\end{aligned}$$

$$\begin{aligned}
& \int_S^T \int_{\Omega} E^{\frac{P-1}{2}} \mu(t) \Delta u g_2(\Delta) dx dt + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\int_0^t h(s) ds \right] \\
& \times \|u\|_{H_m}^2 dt + \frac{1}{2} \int_S^T E^{\frac{P-1}{2}} \mu(t) \left[\int_0^t h(s) \|u(t-s)\|_{H_m}^2 ds \right] dt \\
& \leq \frac{2c_5 C_s^{2(m-1)} + 2a}{1-a} E^{\frac{P-1}{2}}(0) E(S) \int_S^T \mu(t) dt \\
& + E^{\frac{P-1}{2}}(S) \int_S^T \mu(t) \left[\int_0^t h(t-s) \frac{E(0)}{1-a} ds \right] dt \\
& \leq \frac{2c_5 C_s^{2(m-1)} + 3a}{1-a} \left\| \frac{\mu}{\mu'} \right\|_{L^\infty(\mathbb{R}_+)} E^{\frac{P-1}{2}}(0) E(S) \int_S^T \mu'(t) dt \\
& \leq \frac{4c_5 C_s^{2(m-1)} + 6a}{1-a} \|\mu\|_{L^\infty(\mathbb{R}_+)} \left\| \frac{\mu}{\mu'} \right\|_{L^\infty(\mathbb{R}_+)} E^{\frac{P-1}{2}}(0) E(S).
\end{aligned} \tag{3.35}$$

From (3.33)-(3.35), we obtain (3.32), where

$$C_{13} = \left[\frac{4c_5 C_s^{2(m-1)} + 6a}{1-a} \left\| \frac{\mu}{\mu'} \right\|_{L^\infty(\mathbb{R}_+)} + \frac{2C_s^{2(m-1)}}{1-a} + \frac{(P-1)C_s^{2(m-1)}}{(1-a)(P+1)} + 4C_s'^2 \right] \|\mu\|_{L^\infty(\mathbb{R}_+)}.$$

□

Lemma 3.6. For all $0 \leq S < T < +\infty$, we have

$$\begin{aligned}
& \int_S^T E^{\frac{P-1}{2}} \left[\int_{\Omega} 2\mu(t) |\nabla u_t|^2 - \mu^2(t) \Delta u g_1(\Delta u_t) dx \right] dt \\
& \leq C_{14} \varepsilon \int_S^T E^{\frac{P+1}{2}} \mu(t) dt + C(\varepsilon) E^{\frac{P-1}{2}}(0) E(S),
\end{aligned} \tag{3.36}$$

where $C_{14}, \varepsilon, C(\varepsilon)$ will be determined later.

Proof. For $t \in \mathbb{R}_+$ fixed arbitrarily, we set

$$\Omega_1 = \{x \in \Omega; |\Delta u_t| \leq 1\}, \text{ and } \Omega_2 = \{x \in \Omega; |\Delta u_t| > 1\}.$$

By employing the Lemma 2.2 and the embedding $L^{P+1}(\Omega) \hookrightarrow L^2(\Omega)$ and the first inequality in (2.2), we obtain

$$\begin{aligned}
E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} 2 |\nabla u_t|^2 dx & \leq E^{\frac{P-1}{2}} \mu(t) \left[c'^2 \int_{\Omega_1} |\Delta u_t|^2 dx + \int_{\Omega_2} |\Delta u_t|^2 dx \right] \\
& \leq E^{\frac{P-1}{2}} \mu(t) \left[c_{\Omega_1} c'^2 \left(\int_{\Omega_1} |\Delta u_t|^{P+1} dx \right)^{\frac{2}{P+1}} + \frac{1}{c_3} \int_{\Omega_2} |\Delta u_t| |g_1(\Delta u_t)| dx \right] \\
& \leq E^{\frac{P-1}{2}} \left[\frac{c_{\Omega_1} c'^2}{c_1^{\frac{2}{P+1}}} \mu^{\frac{P-1}{P+1}}(t) \left(\int_{\Omega_1} \mu(t) |\Delta u_t| |g_1(\Delta u_t)| dx \right)^{\frac{2}{P+1}} + \frac{1}{c_3} |E'| \right] \\
& \leq \frac{c'^2 c_{\Omega_1}}{c_1^{\frac{2}{P+1}}} \mu^{\frac{P-1}{P+1}}(t) E^{\frac{P-1}{2}} |E'|^{\frac{2}{P+1}} + \frac{1}{c_3} E^{\frac{P-1}{2}} |E'|,
\end{aligned}$$

since $\mu(t) \geq 1$ and $\frac{P-1}{P+1} \leq 1$, then $\mu^{\frac{P-1}{P+1}}(t) \leq \mu(t)$, applying Young's inequality with ε on $E^{\frac{P-1}{2}} |E'|^{\frac{2}{P+1}}$, exploiting the decreasing of E , the above inequality

implies

$$\begin{aligned} E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} 2 |\nabla u_t|^2 dx &\leq \frac{c^2 c_{\Omega_1}}{c_1^{\frac{2}{P+1}}} \varepsilon \mu(t) E^{\frac{P+1}{2}} \\ &\quad - \left[\frac{c^2 c_{\Omega_1}}{c_1^{\frac{2}{P+1}}} C_{\varepsilon} \|\mu\|_{L^{\infty}(\mathbb{R}_+)} + \frac{1}{c_3} E^{\frac{P-1}{2}}(0) \right] E'. \end{aligned} \quad (3.37)$$

Utilizing the Holder's inequality and the second inequality in (2.2), the embedding $L^2(\Omega) \hookrightarrow L^{\frac{P+1}{P}}(\Omega)$, the Lemma 2.1 and the assumption (H2), we get

$$\begin{aligned} &- E^{\frac{P-1}{2}} \mu(t) \int_{\Omega} \Delta u g_1(\Delta u_t) dx \\ &\leq E^{\frac{P-1}{2}} \mu(t) \left[\int_{\Omega_1} |\Delta u| |g_1(\Delta u_t)| dx + \int_{\Omega_2} |\Delta u| |g_1(\Delta u_t)| dx \right] \\ &\leq E^{\frac{P-1}{2}} \mu(t) \left[\|\Delta u\|_{L^{\frac{P+1}{P}}(\Omega_1)} \left(\int_{\Omega_1} |g_1(\Delta u_t)|^{P+1} dx \right)^{\frac{1}{P+1}} \right. \\ &\quad \left. + \|\Delta u\|_{L^{q+1}(\Omega_2)} \left(\int_{\Omega_2} |g_1(\Delta u_t)|^{\frac{q+1}{q}} dx \right)^{\frac{q}{q+1}} \right] \\ &\leq E^{\frac{P-1}{2}} \mu(t) \left[C_s'^{m-1} C_{\Omega_1}' c_2^{\frac{P}{P+1}} \|u\|_{H_m} \left(\int_{\Omega} |\Delta u_t| |g_1(\Delta u_t)| dx \right)^{\frac{1}{P+1}} \right. \\ &\quad \left. + c_4^{\frac{1}{q+1}} C_s^{m-1} \|u\|_{H_m} \left(\int_{\Omega} |\Delta u_t| |g_1(\Delta u_t)| dx \right)^{\frac{q}{q+1}} \right] \\ &\leq \frac{\sqrt{2} C_s'^{m-1} C_{\Omega_1}' c_2^{\frac{P}{P+1}}}{\sqrt{1-a}} \mu^{\frac{P}{P+1}}(t) E^{\frac{P}{2}} |E'|^{\frac{1}{P+1}} + \frac{\sqrt{2} C_s^{m-1} c_4^{\frac{1}{q+1}}}{\sqrt{1-a}} \mu^{\frac{1}{q+1}}(t) E^{\frac{P}{2}} |E'|^{\frac{q}{q+1}} \\ &\leq \frac{\sqrt{2} C_s'^{m-1} C_{\Omega_1}' c_2^{\frac{P}{P+1}} \|\mu\|_{L^{\infty}(\mathbb{R}_+)}^{\frac{P}{P+1}}}{\sqrt{1-a}} E^{\frac{P}{2}} |E'|^{\frac{1}{P+1}} + \frac{\sqrt{2} C_s^{m-1} c_4^{\frac{1}{q+1}} \|\mu\|_{L^{\infty}(\mathbb{R}_+)}^{\frac{1}{q+1}}}{\sqrt{1-a}} E^{\frac{P}{2}} |E'|^{\frac{q}{q+1}}, \end{aligned}$$

applying Young's inequality with ε on $E^{\frac{P}{2}} |E'|^{\frac{1}{P+1}}$ and $E^{\frac{P}{2}} |E'|^{\frac{q}{q+1}}$, exploiting the decreasing of E , the last formula implies

$$\begin{aligned} &- E^{\frac{P-1}{2}} \mu^2(t) \int_{\Omega} \Delta u g_1(\Delta u_t) dx \leq \left[\frac{\sqrt{2} C_s'^{m-1} C_{\Omega_1}' c_2^{\frac{P}{P+1}} \|\mu\|_{L^{\infty}(\mathbb{R}_+)}^{\frac{P}{P+1}}}{\sqrt{1-a}} \right. \\ &\quad \left. + \frac{\sqrt{2} C_s^{m-1} c_4^{\frac{1}{q+1}} \|\mu\|_{L^{\infty}(\mathbb{R}_+)}^{\frac{1}{q+1}}}{\sqrt{1-a}} E^{\frac{Pq-1}{2}}(0) \right] \varepsilon \mu(t) E^{\frac{P+1}{2}} \\ &\quad - \left[\frac{\sqrt{2} C_s'^{m-1} C_{\Omega_1}' c_2^{\frac{P}{P+1}} \|\mu\|_{L^{\infty}(\mathbb{R}_+)}^{\frac{2P+1}{P+1}}}{\sqrt{1-a}} C_{\varepsilon}' + \frac{\sqrt{2} C_s^{m-1} c_4^{\frac{1}{q+1}} \|\mu\|_{L^{\infty}(\mathbb{R}_+)}^{\frac{q+2}{q+1}}}{\sqrt{1-a}} C_{\varepsilon}'' \right] E'. \end{aligned} \quad (3.38)$$

We conclude from (3.37) and (3.38) that

$$\int_S^T E^{\frac{P-1}{2}} \left[\int_{\Omega} 2 \mu(t) |\nabla u_t|^2 - \mu^2(t) \Delta u g_1(\Delta u_t) dx \right] dt \leq C_{14} \varepsilon \int_S^T \mu(t) E^{\frac{P+1}{2}} dt + C(\varepsilon) E^{\frac{P-1}{2}}(0) E(S),$$

$$\text{where } C_{14} = \frac{c'^2 c_{\Omega_1}}{c_1^{\frac{2}{P+1}}} + \frac{\sqrt{2} C_s'^{m-1} C_{\Omega_1}' c_2^{\frac{P}{P+1}} \|\mu\|_{L^\infty(\mathbb{R}_+)}^{\frac{P}{P+1}}}{\sqrt{1-a}} + \frac{\sqrt{2} C_s'^{m-1} c_4^{\frac{1}{q+1}} \|\mu\|_{L^\infty(\mathbb{R}_+)}^{\frac{1}{q+1}}}{\sqrt{1-a}} E^{\frac{Pq-1}{2}}(0),$$

$$C(\varepsilon) = \frac{1}{c_3} + \left(\frac{c'^2 c_{\Omega_1} \|\mu\|_{L^\infty(\mathbb{R}_+)}}{c_1^{\frac{2}{P+1}}} C_\varepsilon + \frac{\sqrt{2} C_s'^{m-1} C_{\Omega_1}' c_2^{\frac{P}{P+1}} \|\mu\|_{L^\infty(\mathbb{R}_+)}^{\frac{2P+1}{P+1}}}{\sqrt{1-a}} C_\varepsilon' + \frac{\sqrt{2} C_s'^{m-1} c_4^{\frac{1}{q+1}} \|\mu\|_{L^\infty(\mathbb{R}_+)}^{\frac{q+2}{q+1}}}{\sqrt{1-a}} C_\varepsilon'' \right) \times E^{\frac{1-P}{2}}(0).$$

Proof of the Theorem 3.3. By choosing $\varepsilon < \frac{1}{C_{14}}$, we deduce from (3.31), (3.32) and (3.36) that

$$\int_S^T E^{\frac{P+1}{2}} \mu(t) dt \leq \frac{1}{C} E^{\frac{P-1}{2}}(0) E(S),$$

i.e.

$$\int_S^T E^{\frac{P-1}{2}+1} \mu(t) dt \leq \frac{1}{C} E^{\frac{P-1}{2}}(0) E(S),$$

where $C = \frac{1 - C_{14} \varepsilon}{C_{13} + C(\varepsilon)}$. We set $\varphi(t) = \int_0^t \mu(s) ds$ (see page 14), then $\varphi'(t) = \mu(t)$, taking $\alpha = \frac{P-1}{2}$ (we have $\alpha = 0$ when $P = 1$, and $\alpha > 0$ when $P > 1$), applying Martinez inequality on the last estimate, we get the result of Theorem 3.3. $\square$

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