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MIXED WAVE - DIFFUSION - WAVE EQUATION: SOLVABILITY OF AN INITIAL-BOUNDARY PROBLEM

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Abstract. We analyze the unique solvability of the direct problem for the wave-diffusion-wave equation, considering initial, boundary, and transmission conditions. This problem is associated with a mathematical model of gas flow in a channel, governed by a combination of wave and diffusion equations. Our primary method of investigation is the separation of variables, with proof of the uniform convergence of the resulting series as a central aspect of our study. To ensure this uniform convergence, we impose specific conditions on the given functions. A key component of the solution is the bivariate Mittag-Leffler-type function, whose properties play a crucial role in our analysis. Although the operator used in the spatial variables is relatively simple, our approach remains applicable to more general operators and domains.

1. Introduction and auxiliary information

In this section, we justify the actuality of the chosen topic, present a comparison analysis of related works, and provide key information on Prabhakar fractional calculus and bivariate Mittag-Leffler-type functions.

1.1. Motivation.

Fractional partial differential equations (FPDEs) have attracted special attention in recent decades due to their ability to model complex phenomena that involve memory effects and anomalous diffusion [19]. Among these, the time-fractional wave-diffusion equations represent a class of equations that interpolate between wave and diffusion, depending on the fractional order of the time derivative [6]. These equations are characterized by the presence of a fractional derivative in time, typically of the Caputo or Riemann-Liouville type, which introduces non-local behavior and captures the intrinsic memory of the system [20]. More general fractional derivatives also gain a certain interest due to their generalization behavior and the possibility of covering cases with fewer assumptions on process parameters [8]. In particular, the Prabhakar fractional derivative is a powerful mathematical tool that extends classical and fractional calculus by incorporating a generalized memory kernel based on the three-parameter Mittag-Leffler function. Its ability to model long-range memory effects, non-exponential

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relaxation, and complex transport phenomena makes it essential in Physics and Engineering (wave propagation, viscoelasticity) [24]; Dynamical Systems (control theory, oscillators, anomalous transport)[26]; Economics and Finance (market dynamics, economic growth models, memory-driven financial models) [27].

1.2. **Comparison analysis of related works.** First of all, we have to note the works [21] (Prabhakar integral), [18] (RL-Prabhakar derivative), [4] (Caputo-Prabhakar derivative), [7] (Hilfer-Prabhakar derivative), and [1] (psi-Hilfer-Prabhakar) in which Prabhakar fractional-order integral and integral-differential operators were introduced and investigated.

Cauchy problems for differential equations involving the Prabhakar fractional derivatives were investigated in [22], [12] (regularized Prabhakar), and [5] (Hilfer-Prabhakar).

The initial boundary problem involving the regularized Prabhakar derivative for the subdiffusion equation (see [17]), for the diffusion wave equation (see [29]), and for the mixed equation (see [28]) has been investigated. In [12] and [28], direct problems for a combination of wave and diffusion equations involving the regularized Prabhakar derivative were considered. The main differences of the present investigation are as follows:

- the correct formulation of the initial-boundary problem for the new combination of two waves and one diffusion equation (one needs to consider the initial and transmitting conditions carefully);
- the system of algebraic equations concerning unknown Fourier constants required a new solvability condition;
- more regularity conditions on the given data have been imposed;
- the convergence of infinite series has been proved similarly, but with more clarification on constants.

Inverse source problems for the subdiffusion equation involving the same derivative were considered in [11]. Nonlocal and inverse problems for generalized telegraph equations with the regularized Prabhakar derivative were investigated in [15], [16]. Leaving a huge amount of work devoted to studying the diffusion and the wave equation, we note recent publications such as [3] (anomalous diffusion laws on the hypersphere), [30] (multi-term subdiffusion).

In the present work, we study a direct problem for a combination of two-wave and one-subdiffusion equations involving the regularized Prabhakar derivative.

We also note works [9], [10], where bivariate Mittag-Leffler type functions were targeted. Since these functions play a crucial role in studying direct and inverse problems for PDEs involving the Prabhakar fractional derivatives, studying their properties remains very actual. In the present work, we use certain properties, studied in [17], [13], and [12].

1.3. **Prabhakar integral and integral-differential operators.** In this subsection, we will introduce only definitions.

Definition 1.1. The Prabhakar fractional integral operator ${}^{P}I_{0x}^{\alpha,\beta,\gamma,\delta}f(x)$ for $\alpha,\beta\in\mathbb{C}$ with positive real part and $\gamma,\delta\in\mathbb{C}$ arbitrary is defined as follows [21]:

$$ {}^{P}I_{0x}^{\alpha,\beta,\gamma,\delta}f(x)=\int_0^x (x-\xi)^{\beta-1}E_{\alpha,\beta}^{\gamma}(\delta(x-\xi)^{\alpha})f(\xi)d\xi,\ x>0, \tag{1.1} $$

where f is any function such that this integral converges and $E_{\alpha,\beta}^{\gamma}(z)$ is the 3-parameter Mittag-Leffler function, which is defined as follows:

$$ E_{\alpha,\beta}^{\gamma}(z)=\sum_{i=0}^{\infty}\frac{(\gamma)_i}{\Gamma(\alpha i+\beta)}\frac{z^i}{i!}. $$

Here $\alpha\in\mathbb{C}$ with positive real part and $\beta,\gamma\in\mathbb{C}$, $(\gamma)_i$ is the Pochhammer symbol

$$ (\gamma)_i=\gamma(\gamma+1)(\gamma+2)\dots(\gamma+i-1)=\frac{\Gamma(\gamma+i)}{\Gamma(\gamma)}, $$

$$ (-\gamma)_i=(-1)^i(\gamma-i+1)_i=(-1)^i\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)}. $$

Definition 1.2. The Prabhakar fractional derivative operators of Riemann-Liouville type and Caputo type, ${}^{PC}D_{0x}^{\alpha,\beta,\gamma,\delta}$ and ${}^{PR}D_{0x}^{\alpha,\beta,\gamma,\delta}$ are defined as follows [4], [18]:

$$ {}^{PC}D_{0x}^{\alpha,\beta,\gamma,\delta}f(x)={}^{P}I_{0x}^{\alpha,n-\beta,-\gamma,\delta}\frac{d^n}{dx^n}f(x), $$

$$ {}^{PR}D_{0x}^{\alpha,\beta,\gamma,\delta}f(x)=\frac{d^n}{dx^n}{}^{P}I_{0x}^{\alpha,n-\beta,-\gamma,\delta}f(x), $$

where $\alpha\in\mathbb{C}$ with positive real part and $\beta\in\mathbb{C}$ with non-negative real part and $n-1\le \mathrm{Re}\,(\beta)<n$, $n\in\mathbb{N}$ and $\gamma,\delta\in\mathbb{C}$.

1.4. **Bivariate Mittag-Leffler type function.** We recall bivariate Mittag-Leffler type function $E_2(x,y)$ presenting its double series form as follows:

$$ E_2\left(\begin{array}{c|c}\gamma_1,\alpha_1,\beta_1,\gamma_2,\alpha_2 & x\\ \delta_1,\alpha_3,\beta_2,\delta_2,\alpha_4,\delta_3,\beta_3 & y\end{array}\right) $$
$$ =\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma_1+\alpha_1 m+\beta_1 n)\,\Gamma(\gamma_2+\alpha_2 m)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\delta_1+\alpha_3 m+\beta_2 n)}\cdot\frac{x^m}{\Gamma(\delta_2+\alpha_4 m)}\cdot\frac{y^n}{\Gamma(\delta_3+\beta_3 n)} $$

is a bivariate Mittag-Leffler type function (see for details [17] or [12]).

For this function, the following statements are true:

Lemma 1.1. [12] *Let $\Re(\delta_1)>\Re(\gamma_1)>0$. If $\alpha_3=\alpha_1$ and $\beta_2=\beta_1$, then the following integral representation holds true:*

$$ E_2\left(\begin{array}{cc}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2 & |x\\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3 & |y\end{array}\right)= $$
$$ =\frac{1}{\Gamma(\gamma_1)\Gamma(\delta_1-\gamma_1)}\int_0^1\xi^{\gamma_1-1}(1-\xi)^{\delta_1-\gamma_1-1}E_{\alpha_4,\delta_2}^{\gamma_2,\alpha_2}(x\xi^{\alpha_1})\,E_{\beta_3,\delta_3}(y\xi^{\beta_1})\,d\xi. \tag{1.2} $$

Moreover, if $x<0$, $y\leq 0$, then for any $k>0$

$$\left|E_2\left(\begin{matrix}\gamma,\gamma,1;1,0 & |\lambda x^{\beta}\\ \beta+k+1,\beta,\alpha;\gamma,\gamma;1,1 & |\delta y^{\alpha}\end{matrix}\right)\right|\leq c$$

holds.

Here c is any positive real number and $E_{\beta_3,\delta_3}(z)$ is the two-parameter, and

$$E_{\alpha_4,\delta_2}^{\gamma_2,\alpha_2}(z)=\sum_{m=0}^{\infty}\frac{(\gamma_2)_{\alpha_2 m}z^m}{\Gamma(\alpha_4 m+\delta_2)}$$

is the four-parameter Mittag-Leffler function [19].

2. Formulation of a problem

We consider the following mixed equation:

$$0=\begin{cases} {}^{PC}D_{0t}^{\alpha,\beta_1,\gamma,\delta}u\left(t,x\right)-u_{xx}\left(t,x\right),\ (t,x)\in\Omega_1,\\ {}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}u\left(t,x\right)-u_{xx}\left(t,x\right),\ (t,x)\in\Omega_2,\\ {}^{PC}D_{T_2t}^{\alpha,\beta_3,\gamma,\delta}u\left(t,x\right)-u_{xx}\left(t,x\right),\ (t,x)\in\Omega_3\end{cases} \tag{2.1}$$

in a rectangular domain $\Omega=\Omega_1\cup\Omega_2\cup\Omega_3\cup J_1\cup J_2$, where
$J_1=\{(t,x):t=T_1,0<x<1\}$, $J_2=\{(t,x):t=T_2,0<x<1\}$,
$\Omega_1=\{(t,x):0<x<1,0<t<T_1\}$, $\Omega_2=\{(t,x):0<x<1,T_1<t<T_2\}$,
$\Omega_3=\{(t,x):0<x<1,T_2<t<T\}$, $T_1,T_2,T,\alpha,\gamma,\delta,\beta_1,\beta_2,\beta_3\in\mathbb{R}$ such that
$0<T_1<T_2<T$, $1<\beta_1\leq 2$, $0<\beta_2\leq 1$, $1<\beta_3\leq 2$.

Problem. To find a *regular solution* of equation (2.1) in Ω, satisfying the following initial, boundary and transmitting conditions:

$$u\left(0,x\right)=\varphi\left(x\right),\ 0\leq x\leq 1, \tag{2.2}$$

$$u\left(t,0\right)=u\left(t,1\right)=0,\ 0\leq t\leq T, \tag{2.3}$$

$$\lim_{t\to T_1+0}{}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}u\left(t,x\right)=\lim_{t\to T_1-0}u_t\left(t,x\right),\ 0<x<1, \tag{2.4}$$

$$\lim_{t\to T_2+0}u_t\left(t,x\right)=\lim_{t\to T_2-0}{}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}u\left(t,x\right),\ 0<x<1. \tag{2.5}$$

Definition 2.1. Function $u(t,x)$ is called a *regular solution* of (2.1) in Ω, if $u(t,x)\in C(\overline{\Omega})$, $u_{xx}(t,x)\in C(\Omega)$, ${}^{PC}D_{0t}^{\alpha,\beta_1,\gamma,\delta}u\left(t,x\right)\in C\left(\Omega_1\right)$, ${}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}u\left(t,x\right)\in C\left(\Omega_2\cup J_1\cup J_2\right)$, ${}^{PC}D_{T_2t}^{\alpha,\beta_3,\gamma,\delta}u\left(t,x\right)\in C\left(\Omega_3\right)$ and satisfies (2.1) in Ω.

Remark 2.1. In space variable, one can consider a more general operator rather than $Lu\equiv\frac{\partial^2}{\partial x^2}$. The only condition to such an operator is that it should be a positively defined elliptic operator with a discrete spectrum. The tool that we have used in this research will be valid for that case, as well. Consequently, one needs to change the definition of a regular solution. We refer the reader to the works [23]-[14], which consider more general operators in space variables.

Motivation related to the gas dynamics. It is well known that gas flow in a channel can be described by the wave equation [25]. In some cases, after a certain period, damage to the channel may cause the process to transition to diffusion before returning to its previous state. In such situations, a combination of wave-diffusion-wave equations can be used. In our formulation, one of the initial conditions is known (see (2.2)), and the transmission conditions are standard when the process changes its behavior (see (2.4) and (2.5)).

3. Formal solution

Since the equation is linear and the considered domain allows us to use the method of separation of variables, we look for a formal solution in a classical way. Namely, we search for a solution to the formulated problem as follows:

$$u\left(t,x\right)=\sum_{k=1}^{\infty}U_k\left(t\right)\sin k\pi x,\quad \left(t,x\right)\in\Omega_1, \tag{3.1}$$

$$u\left(t,x\right)=\sum_{k=1}^{\infty}V_k\left(t\right)\sin k\pi x,\quad \left(t,x\right)\in\Omega_2, \tag{3.2}$$

$$u\left(t,x\right)=\sum_{k=1}^{\infty}W_k\left(t\right)\sin k\pi x,\quad \left(t,x\right)\in\Omega_3. \tag{3.3}$$

Here $U_k(t)$, $V_k(t)$ and $W_k(t)$ are unknown functions to be found.

We note that the representation above is linked to the Sturm-Liouville problem, corresponding to space variables, once a method of separation of variables is used. According to the boundary conditions of the formulated problem, we will get eigenfunctions as $\{\sin k\pi x\}_{k=1}^{\infty}$, which form an orthonormal basis in L_2 space. Therefore, any smooth function, including $u(t,x)$ can be represented via this system.

We substitute (3.1), (3.2) and (3.3) into (2.1) and will get the following fractional differential equations:

$${}^{PC}D_{0t}^{\alpha,\beta_1,\gamma,\delta}U_k\left(t\right)+\left(k\pi\right)^2U_k\left(t\right)=0,\ \ 0<t<T_1, \tag{3.4}$$

$${}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}V_k\left(t\right)+\left(k\pi\right)^2V_k\left(t\right)=0,\ \ T_1\le t\le T_2, \tag{3.5}$$

$${}^{PC}D_{T_2t}^{\alpha,\beta_3,\gamma,\delta}W_k\left(t\right)+\left(k\pi\right)^2W_k\left(t\right)=0,\ \ T_2<t<T. \tag{3.6}$$

General solutions of (3.4),(3.5) and (3.6) can be written as follows (see [12]):

$$\begin{aligned}
&U_k\left(t\right)={}_1A_k+{}_2A_kt-{}_1A_k\left(k\pi\right)^2t^{\beta_1}\Gamma\left(\gamma\right)\\
&\times E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -\left(k\pi\right)^2t^{\beta_1}\\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta t^{\alpha}\end{array}\right)-{}_2A_k\left(k\pi\right)^2\\
&\times t^{\beta_1+1}\Gamma\left(\gamma\right)\cdot E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -\left(k\pi\right)^2t^{\beta_1}\\ \beta_1+2,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta t^{\alpha}\end{array}\right),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&V_k\left(t\right)={}_1B_k-{}_1B_k\cdot\left(k\pi\right)^2\cdot\left(t-T_1\right)^{\beta_2}\Gamma\left(\gamma\right)\\
&\times E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -\left(k\pi\right)^2\left(t-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha,\gamma,\gamma,1,1 & \delta\left(t-T_1\right)^{\alpha}\end{array}\right),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}&W_k(t) = {}_1F_k + {}_2F_k(t-T_2) - {}_1F_k(k\pi)^2(t-T_2)^{\beta_3}\Gamma(\gamma)\\&\times E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -(k\pi)^2(t-T_2)^{\beta_3}\\ \beta_3+1,\beta_3,\alpha,\gamma,\gamma,1,1 & \delta(t-T_2)^{\alpha}\end{array}\right) - {}_2F_k(k\pi)^2\\&\times(t-T_2)^{\beta_3+1}\Gamma(\gamma)\cdot E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -(k\pi)^2(t-T_2)^{\beta_3}\\ \beta_3+2,\beta_3,\alpha,\gamma,\gamma,1,1 & \delta(t-T_2)^{\alpha}\end{array}\right).\end{aligned} \tag{3.9}$$

Here ${}_1A_k, {}_2A_k,\ {}_1B_k, {}_1F_k,\ {}_2F_k$ are unknown constants to be found. To find them, we will first use

$$\lim_{t\to T_1-0} U_k(t) = \lim_{t\to T_1+0} V_k(t)$$

(this follows from the continuity of the function $u(t,x)$ on J_1) and due to (3.7),(3.8), we will obtain:

$$\begin{aligned}&{}_1A_k + {}_2A_kT_1 - {}_1A_k(k\pi)^2T_1^{\beta_1}\Gamma(\gamma)E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -(k\pi)^2{T_1}^{\beta_1}\\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta{T_1}^{\alpha}\end{array}\right)\\&-{}_2A_k(k\pi)^2{T_1}^{\beta_1+1}\Gamma(\gamma)E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -(k\pi)^2{T_1}^{\beta_1}\\ \beta_1+2,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta{T_1}^{\alpha}\end{array}\right) = {}_1B_k.\end{aligned} \tag{3.10}$$

Now we use the condition (2.2) which can be written as $U_k(0) = \varphi_k$, where $\varphi_k = 2\int_0^1 \varphi(x)\sin k\pi x dx$. This will give us ${}_1A_k = \varphi_k$. In order to use condition (2.4) we need to evaluate $U_k'(t)$. After certain evaluations, we obtain (see for details [12]):

$$\begin{aligned}&U_k'(t) = {}_2A_k - {}_1A_k(k\pi)^2t^{\beta_1-1}\Gamma(\gamma)E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -(k\pi)^2t^{\beta_1}\\ \beta_1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta t^{\alpha}\end{array}\right)\\&-{}_2A_k(k\pi)^2t^{\beta_1}\Gamma(\gamma)E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -(k\pi)^2t^{\beta_1}\\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta t^{\alpha}\end{array}\right).\end{aligned} \tag{3.11}$$

Now we evaluate $\lim\limits_{t\to T_1+0} {}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}u(t,x)$. For this aim, we pass to the limit as $t\to T_1+$ and from the equation ${}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}u(t,x) - u_{xx}(t,x) = 0$, considering $V_k(T_1) = {}_1B_k$, we will get

$$\lim_{t\to T_1+0} {}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}V_k(t) = -(k\pi)^2\,{}_1B_k.$$

Hence, the transmitting condition (2.4) or $\lim\limits_{t\to T_1+0} {}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}V_k(t) = \lim\limits_{t\to T_1-0} U_k'(t)$ gives us

$$\begin{aligned}&-(k\pi)^2\,{}_1B_k = {}_2A_k - {}_1A_k(k\pi)^2{T_1}^{\beta_1-1}\Gamma(\gamma)\\&\times E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -(k\pi)^2{T_1}^{\beta_1}\\ \beta_1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta{T_1}^{\alpha}\end{array}\right) - {}_2A_k(k\pi)^2{T_1}^{\beta_1}\Gamma(\gamma)\\&\times E_2\left(\begin{array}{c|c}\gamma,\gamma,1,1,0 & -(k\pi)^2{T_1}^{\beta_1}\\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta{T_1}^{\alpha}\end{array}\right)\end{aligned} \tag{3.12}$$

or considering the representation of ${}_1B_k$, we have

$$-{}_1A_k(k\pi)^2 - (k\pi)^2\,{}_2A_kT_1 + {}_1A_k(k\pi)^4{T_1}^{\beta_1}\Gamma(\gamma)$$

$$\times E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right) + {}_2A_k (k\pi)^4 {T_1}^{\beta_1+1}\Gamma(\gamma)$$

$$\times E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1+2,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right) = {}_2A_k - {}_1A_k (k\pi)^2 {T_1}^{\beta_1-1}\Gamma(\gamma)$$

$$\times E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right) - {}_2A_k (k\pi)^2 {T_1}^{\beta_1}\Gamma(\gamma)$$

$$\times E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right).$$

Rearranging similar terms, we obtain

$$-(k\pi)^2 {}_2A_k T_1 + {}_2A_k (k\pi)^4 {T_1}^{\beta_1+1}\Gamma(\gamma) E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1+2,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right) -$$

$$-{}_2A_k + {}_2A_k (k\pi)^2 {T_1}^{\beta_1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right)$$

$$= {}_1A_k (k\pi)^2 - (k\pi)^4 {}_1A_k T_1^{\beta_1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right)$$

$$-{}_1A_k (k\pi)^2 {T_1}^{\beta_1-1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right).$$

We will find ${}_2A_k$ from the above-given expression as

$${}_2A_k \Big(-(k\pi)^2 T_1 + (k\pi)^4 {T_1}^{\beta_1+1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1+2,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right)$$

$$-1 + (k\pi)^2 {T_1}^{\beta_1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right)\Big)$$

$$= {}_1A_k (k\pi)^2 - (k\pi)^4 {}_1A_k T_1^{\beta_1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right)$$

$$-{}_1A_k (k\pi)^2 {T_1}^{\beta_1-1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 {T_1}^{\beta_1} \\ \beta_1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta {T_1}^{\alpha} \end{array}\right).$$

Since ${}_1A_k = \varphi_k$, if $\Delta_k \neq 0$, we could find ${}_2A_k$ as follows:

$${}_2A_k = \tfrac{\varphi_k}{\Delta_k}\left((k\pi)^2 T_1^{\beta_1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta T_1^{\alpha} \end{array}\right) - 1\right.$$

$$\left. + T_1^{\beta_1-1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta T_1^{\alpha} \end{array}\right)\right). \tag{3.13}$$

Here

$$\Delta_k = T_1 - (k\pi)^2 T_1^{\beta_1+1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 T_1^{\beta_1} \\ \beta_1+2,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta T_1^{\alpha} \end{array}\right)$$

$$+\frac{1}{(k\pi)^2} - T_1^{\beta_1}\Gamma(\gamma)\ E_2\left(\begin{array}{c|c} \gamma,\gamma,1,1,0 & -(k\pi)^2 T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1 & \delta T_1^{\alpha} \end{array}\right). \tag{3.14}$$

Since the parameters of the bivariate Mittag-Leffler function $E_2(\cdot)$ involved Δ_k satisfy all conditions of Lemma 2.3 presented in [13], we can use the statement of

Lemma 3.1 (see [13]). That is, in the integral representation of E_2, two Wight-type functions appear. Using their asymptotics, we can obtain asymptotics of E_2, consequently stating that $\lim_{k\to\infty}\Delta_k = T_1 > 0$ is true. We refer the reader to the proof of Lemma 3.1, presented in [13], for details.

Now we start our evaluations to find unknown constants ${}_1F_k$ and ${}_2F_k$. For this aim, we will use $u(t,x) = C\left(\bar{\Omega}\right)$, which gives us $\lim_{t\to T_2-0} V_k(t) = \lim_{t\to T_2+0} W_k(t)$:

$$
\begin{gathered}
{}_1B_k - {}_1B_k\,(k\pi)^2\,(T_2-T_1)^{\beta_2}\,\Gamma(\gamma) \\
\times E_2\left(\begin{array}{l|c} \gamma,\gamma,1,1,0 & -(k\pi)^2\,(T_2-T_1)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha,\gamma,\gamma,1,1 & \delta\,(T_2-T_1)^{\alpha} \end{array}\right) = {}_1F_k.
\end{gathered}
$$

In order to use transmitting condition (2.5), we need to evaluate limit of ${}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}V_k(t)$ as $t\to T_2-0$. Therefore, we pass to the limit as $t\to T_2-$ and from the equation (3.5) we obtain:

$$
\begin{aligned}
\lim_{t\to T_2-0} {}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}V_k(t) &= -(k\pi)^2 \lim_{t\to T_2-0} V_k(t) = -(k\pi)^2\,V_k(T_2) \\
&= -(k\pi)^2\Bigg({}_1B_k - {}_1B_k\,(k\pi)^2\,(T_2-T_1)^{\beta_2}\,\Gamma(\gamma) \\
&\quad\times E_2\left(\begin{array}{l|c} \gamma,\gamma,1,1,0 & -(k\pi)^2\,(T_2-T_1)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha,\gamma,\gamma,1,1 & \delta\,(T_2-T_1)^{\alpha} \end{array}\right)\Bigg).
\end{aligned}
$$

Now let us evaluate $W_k'(t)$:

$$
\begin{gathered}
{W_k}'(t) = {}_2F_k - {}_1F_k\,(k\pi)^2\,(t-T_2)^{\beta_3-1}\,\Gamma(\gamma) \\
\times E_2\left(\begin{array}{l|c} \gamma,\gamma,1,1,0 & -(k\pi)^2\,(t-T_2)^{\beta_3} \\ \beta_3,\beta_3,\alpha,\gamma,\gamma,1,1 & \delta\,(t-T_2)^{\alpha} \end{array}\right) - {}_2F_k\,(k\pi)^2 \\
\times (t-T_2)^{\beta_3}\,\Gamma(\gamma)\;E_2\left(\begin{array}{l|c} \gamma,\gamma,1,1,0 & -(k\pi)^2\,(t-T_2)^{\beta_3} \\ \beta_3+1,\beta_3,\alpha,\gamma,\gamma,1,1 & \delta\,(t-T_2)^{\alpha} \end{array}\right).
\end{gathered}
$$

Hence, the transmitting condition (2.5) gives us

$$
\begin{gathered}
-(k\pi)^2\Bigg({}_1B_k - {}_1B_k\,(k\pi)^2\,(T_2-T_1)^{\beta_2}\,\Gamma(\gamma) \\
\times E_2\left(\begin{array}{l|c} \gamma,\gamma,1,1,0 & -(k\pi)^2\,(T_2-T_1)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha,\gamma,\gamma,1,1 & \delta\,(T_2-T_1)^{\alpha} \end{array}\right)\Bigg) = {}_2F_k
\end{gathered}
$$

or

$$
{}_2F_k = -(k\pi)^2\,{}_1F_k. \tag{3.15}
$$

Based on Lemma 2.3, given in [13], we write the following estimates:

$$\left|E_2\left(\begin{array}{c}\gamma,\gamma,1,1,0\\ \beta_3+1,\beta_3,\alpha,\gamma,\gamma,1,1\end{array}\middle|\begin{array}{c}-(k\pi)^2(t-T_2)^{\beta_3}\\ \delta(t-T_2)^{\alpha}\end{array}\right)\right|\leq$$
$$\frac{C}{(k\pi)^{2\theta}(t-T_2)^{\beta_3\theta+\alpha\theta}(-\delta)^{\theta}}\frac{\Gamma\left(\frac{\beta_3+1}{2}-\beta_3\theta\right)\Gamma\left(\frac{\beta_3+1}{2}-\alpha\theta\right)\Gamma(\gamma-\gamma\theta-\theta)}{\Gamma(\beta_3+1-\theta(\alpha+\beta_3))},$$
$$\left|E_2\left(\begin{array}{c}\gamma,\gamma,1,1,0\\ \beta_3+2,\beta_3,\alpha,\gamma,\gamma,1,1\end{array}\middle|\begin{array}{c}-(k\pi)^2(t-T_2)^{\beta_3}\\ \delta(t-T_2)^{\alpha}\end{array}\right)\right|\leq$$
$$\frac{C}{(k\pi)^{2\theta}(t-T_2)^{\beta_3\theta+\alpha\theta}(-\delta)^{\theta}}\frac{\Gamma\left(\frac{\beta_3+2}{2}-\beta_3\theta\right)\Gamma\left(\frac{\beta_3+2}{2}-\alpha\theta\right)\Gamma(\gamma-\gamma\theta-\theta)}{\Gamma(\beta_3+2-\theta(\alpha+\beta_3))}. \quad (3.16)$$

Estimates and other representations of the function $E_2(x,y)$ involved in the solution are discussed in detail in [13].

4. Convergence of infinite series

To prove a uniform convergence of infinite series (3.1) let us start with the following estimation:

$$|u(t,x)|=\left|\sum_{k=1}^{\infty}U_k(t)\sin k\pi x\right|\leq\sum_{k=1}^{\infty}|U_k(t)|,\ 0\leq t\leq T_1.$$

To complete this estimation, we need to estimate $|U_k(t)|$. For this aim, first using (3.7) we get

$$|U_k(t)|\leq|{}_1A_k|+|{}_2A_k|\,t+|{}_1A_k|(k\pi)^2t^{\beta_1}\Gamma(\gamma)$$
$$\times\left|E_2\left(\begin{array}{c}\gamma,\gamma,1,1,0\\ \beta_1+1,\beta_1,\alpha,\gamma,\gamma,1,1\end{array}\middle|\begin{array}{c}-(k\pi)^2t^{\beta_1}\\ \delta t^{\alpha}\end{array}\right)\right|+|{}_2A_k|(k\pi)^2t^{\beta_1+1}\Gamma(\gamma)$$
$$\times\left|E_2\left(\begin{array}{c}\gamma,\gamma,1,1,0\\ \beta_1+2,\beta_1,\alpha,\gamma,\gamma,1,1\end{array}\middle|\begin{array}{c}-(k\pi)^2t^{\beta_1}\\ \delta t^{\alpha}\end{array}\right)\right|.$$

Now, using the upper bound of the function E_2, we will get

$$|U_k(t)|\leq|\varphi_k|+|{}_2A_k|\,t+|\varphi_k|(k\pi)^{2-2\theta}t^{\beta_1-\beta_1\theta-\alpha\theta}H_1$$
$$+|{}_2A_k|(k\pi)^{2-2\theta}t^{\beta_1+1-\beta_1\theta-\alpha\theta}H_2, \quad (4.1)$$

$$|{}_2A_k|\leq\frac{1}{|\Delta_k|}\Bigg(|\varphi_k|(k\pi)^{2-2\theta}T_1^{\beta_1-\beta_1\theta-\alpha\theta}H_1+|\varphi_k|+$$
$$|\varphi_k|(k\pi)^{-2\theta}T_1^{\beta_1-\beta_1\theta-\alpha\theta-1}H_3\Bigg) \quad (4.2)$$

where,

$$H_1=\Gamma(\gamma)\frac{C}{(-\delta)^{\theta}}\frac{\Gamma\left(\frac{\beta_1+1}{2}-\beta_1\theta\right)\Gamma\left(\frac{\beta_1+1}{2}-\alpha\theta\right)\Gamma(\gamma-\gamma\theta-\theta)}{\Gamma(\beta_1+1-\theta(\alpha+\beta_1))},$$

$$H_2 = \Gamma(\gamma) \frac{C}{(-\delta)^\theta} \frac{\Gamma\left(\frac{\beta_1+2}{2} - \beta_1\theta\right)\Gamma\left(\frac{\beta_1+2}{2} - \alpha\theta\right)\Gamma(\gamma - \gamma\theta - \theta)}{\Gamma(\beta_1 + 2 - \theta(\alpha + \beta_1))},$$

$$H_3 = \Gamma(\gamma) \frac{C}{(-\delta)^\theta} \frac{\Gamma\left(\frac{\beta_1}{2} - \beta_1\theta\right)\Gamma\left(\frac{\beta_1}{2} - \alpha\theta\right)\Gamma(\gamma - \gamma\theta - \theta)}{\Gamma(\beta_1 - \theta(\alpha + \beta_1))}.$$

Now we consider the following estimates at $\theta \in \left[0\,;\frac{1}{2}\right)$;

Lemma 4.1 *If $\varphi \in C^m[0,1]$ such that $\varphi^{(2i)}(0) = \varphi^{(2i)}(1) = 0,\ i = 0, ..., [m/2]$ and $\varphi^{(m+1)}(x) \in L_2(0,1)$, then*

$$\sum_{k=1}^{\infty} |\varphi_k| (k\pi)^{m(1-\theta)} \le \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{2+2m\theta}} + \left\|\varphi^{(m+1)}(x)\right\|_2^2,$$

where m is a natural even number and it could be zero, as well.

Proof. Let the function $f(x)$ is continuous on $[0,1]$ and $f(x) = \sum\limits_{k=1}^{\infty} f_k \sin k\pi x$, where $f_k = 2\int\limits_0^1 f(x)\sin k\pi x dx$. Using integration by parts and assuming that $f(0) = f(1) = 0$, one can easily get $f_k = -\frac{2}{k\pi}\int\limits_0^1 f'(x)\cos k\pi x dx$. If one uses the well-known inequality $2ab \le a^2 + b^2$, the following inequality will be obvious:

$$|f_k| \le \frac{1}{(k\pi)^2} + \left(\int_0^1 f'(x)\cos k\pi x dx\right)^2.$$

Consequently, easy to deduce that

$$\sum_{k=1}^{\infty} |f_k| \le \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \sum_{k=1}^{\infty}\left(\int_0^1 f'(x)\cos k\pi x dx\right)^2.$$

Based on Parseval's equality, i.e. $\sum\limits_{k=1}^{\infty} |f_k|^2 = \|f(x)\|_2^2$, assuming that $f'(x) \in L(0,1)$, we obtain

$$\sum_{k=1}^{\infty} |f_k| \le \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \|f'(x)\|_2^2.$$

This proves Lemma 4.1 for the case $m = 0$. Other cases can be proved similarly. We note that we have chosen a specific power $m(1-\theta)$ for further usage, and a restriction on m, namely, it should be only even, is linked with writing the statement in a compact form, making it useful for usage in further evaluations. □

Considering (4.1), (4.2) and $\frac{1}{|\Delta_k|} \le \epsilon > 0$, we obtain

$$\sum_{k=1}^{\infty}|U_k(t)| \le \sum_{k=1}^{\infty}|\varphi_k|(1+t\epsilon)+\sum_{k=1}^{\infty}t\epsilon|\varphi_k|(k\pi)^{-2\theta}T_1^{\beta_1-\beta_1\theta-\alpha\theta-1}H_3+$$
$$\sum_{k=1}^{\infty}|\varphi_k|(k\pi)^{2-2\theta}\left(T_1^{\beta_1-\beta_1\theta-\alpha\theta}H_1t\epsilon+t^{\beta_1-\beta_1\theta-\alpha\theta}(H_1+t\epsilon H_2)\right)+$$
$$\sum_{k=1}^{\infty}\epsilon|\varphi_k|(k\pi)^{2-4\theta}T_1^{\beta_1-\beta_1\theta-\alpha\theta-1}H_2H_3+$$
$$\sum_{k=1}^{\infty}\epsilon|\varphi_k|(k\pi)^{4-4\theta}T_1^{\beta_1-\beta_1\theta-\alpha\theta}t^{\beta_1-\beta_1\theta-\alpha\theta+1}H_1H_2. \quad (4.3)$$

Using Lemma 4.1, in this particular case, we obtain

$$\sum_{k=1}^{\infty}|U_k(t)| \le \left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}+\|\varphi'(x)\|_2^2\right)(1+t\epsilon)+\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{2+4\theta}}+\|\varphi'(x)\|_2^2\right)\times$$
$$T_1^{\beta_1-\beta_1\theta-\alpha\theta-1}tH_3\epsilon+\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{4\theta+2}}+\left\|\varphi'''(x)\right\|_2^2\right)\times$$
$$\left(T_1^{\beta_1-\beta_1\theta-\alpha\theta}H_1t\epsilon+t^{\beta_1-\beta_1\theta-\alpha\theta}(H_1+t\epsilon H_2)\right)+\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{4\theta+2}}+\left\|\varphi^{V}(x)\right\|_2^2\right)\times$$
$$T_1^{\beta_1-\beta_1\theta-\alpha\theta}t^{\beta_1-\beta_1\theta-\alpha\theta+1}H_1H_2\epsilon+\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{8\theta+2}}+\left\|\varphi'''(x)\right\|_2^2\right)T_1^{\beta_1-\beta_1\theta-\alpha\theta-1}H_2H_3\epsilon.$$

Now let us look at the estimate of $V_k(t)$:

$$|u(t,x)|=\left|\sum_{k=1}^{\infty}V_k(t)\sin k\pi x\right| \le \sum_{k=1}^{\infty}|V_k(t)|\ ,\ T_1\le t\le T_2;$$

According to (3.8) we obtain:

$$\sum_{k=1}^{\infty}|V_k(t)| \le \sum_{k=1}^{\infty}|{}_1B_k|+\sum_{k=1}^{\infty}|{}_1B_k|(k\pi)^{2-2\theta}(t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}\Gamma(\gamma)\frac{C}{(-\delta)^{\theta}}\times$$
$$\frac{\Gamma\left(\frac{\beta_2+1}{2}-\beta_2\theta\right)\Gamma\left(\frac{\beta_2+1}{2}-\alpha\theta\right)\Gamma(\gamma-\gamma\theta-\theta)}{\Gamma(\beta_2+1-\theta(\alpha+\beta_2))}.$$

Introducing the following notations:

$$S_1=\frac{C}{(-\delta)^{\theta}}\frac{\Gamma\left(\frac{\beta_2+1}{2}-\beta_2\theta\right)\Gamma\left(\frac{\beta_2+1}{2}-\alpha\theta\right)\Gamma(\gamma-\gamma\theta-\theta)}{\Gamma(\beta_2+1-\theta(\alpha+\beta_2))}\Gamma(\gamma),$$

we rewrite the above representation $V_k(t)$ as follows:

$$\sum_{k=1}^{\infty}|V_k(t)| \le \sum_{k=1}^{\infty}|{}_1B_k|+\sum_{k=1}^{\infty}|{}_1B_k|(k\pi)^{2-2\theta}(t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1$$

Now we consider the estimate of ${}_1B_k$ using its representation:

$$|{}_1B_k| \le |\varphi_k| + |{}_2A_k| T_1 + |\varphi_k| (k\pi)^{2-2\theta} T_1^{\beta_1-\beta\theta-\alpha\theta} H_1 + |{}_2A_k| (k\pi)^{2-2\theta} T_1^{\beta_1-\beta\theta-\alpha\theta+1} H_2.$$

Therefore, we can rewrite the above-presented estimate as follows:

$$\sum_{k=1}^{\infty} |V_k(t)| \le \sum_{k=1}^{\infty} |\varphi_k| (1+T_1\epsilon) + \sum_{k=1}^{\infty} \epsilon |\varphi_k| (k\pi)^{-2\theta} {T_1}^{\beta_1-\beta_1\theta-\alpha\theta} H_3 +$$
$$\sum_{k=1}^{\infty} |\varphi_k| (k\pi)^{2-2\theta} \Big({T_1}^{\beta_1-\beta_1\theta-\alpha\theta+1} H_1\epsilon + {T_1}^{\beta_1-\beta_1\theta-\alpha\theta} H_1 +$$
$${T_1}^{\beta_1-\beta_1\theta-\alpha\theta+1} H_2\epsilon + (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 + T_1\epsilon (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 \Big) +$$
$$\sum_{k=1}^{\infty} |\varphi_k| (k\pi)^{2-4\theta} {T_1}^{\beta_1-\beta_1\theta-\alpha\theta} \Big({T_1}^{\beta_1-\beta_1\theta-\alpha\theta} H_2H_3 + (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} H_3S_1 \Big)\epsilon +$$
$$\sum_{k=1}^{\infty} |\varphi_k| (k\pi)^{4-4\theta} \Big(T_1^{2\beta_1-2\beta_1\theta-2\alpha\theta+1} H_1H_2\epsilon + T_1^{\beta_1-\beta_1\theta-\alpha\theta+1} (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} H_1S_1\epsilon +$$
$$T_1^{\beta_1-\beta_1\theta-\alpha\theta} (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} H_1S_1 + T_1^{\beta_1-\beta_1\theta-\alpha\theta+1} (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} H_2S_1\epsilon \Big) +$$
$$\sum_{k=1}^{\infty} |\varphi_k| (k\pi)^{4-6\theta} {T_1}^{2\beta_1-2\beta_1\theta-2\alpha\theta} (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1H_2H_3 +$$
$$\sum_{k=1}^{\infty} |\varphi_k| \epsilon (k\pi)^{6-6\theta} (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} T_1^{2\beta_1-2\beta_1\theta-2\alpha\theta+1} S_1H_1H_2.$$

Using the estimate in Lemma 4.1, we get the following:

$$\sum_{k=1}^{\infty} |V_k(t)| \le \left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \|\varphi'(x)\|_2^2 \right) (1+T_1\epsilon) +$$
$$\left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^{2+4\theta}} + \|\varphi'(x)\|_2^2 \right) {T_1}^{\beta_1-\beta_1\theta-\alpha\theta} H_3\epsilon + \left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^{4\theta+2}} + \left\|\varphi'''(x)\right\|_2^2 \right)$$
$$\Big({T_1}^{\beta_1-\beta_1\theta-\alpha\theta+1} H_1\epsilon + {T_1}^{\beta_1-\beta_1\theta-\alpha\theta} H_1 + {T_1}^{\beta_1-\beta_1\theta-\alpha\theta+1} H_2\epsilon + (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 +$$
$$T_1\epsilon (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 \Big) + \left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^{8\theta+2}} + \left\|\varphi'''(x)\right\|_2^2 \right) \times$$
$$\epsilon \Big({T_1}^{2\beta_1-2\beta_1\theta-2\alpha\theta} H_2H_3 + {T_1}^{\beta_1-\beta_1\theta-\alpha\theta} (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} H_3S_1 \Big) +$$
$$\left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^{8\theta+2}} + \left\|\varphi^{V}(x)\right\|_2^2 \right) \times$$
$$\Big(T_1^{2\beta_1-2\beta_1\theta-2\alpha\theta+1} H_1H_2\epsilon + T_1^{\beta_1-\beta_1\theta-\alpha\theta+1} (t-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} H_1S_1\epsilon +$$

$$T_1^{\beta_1-\beta_1\theta-\alpha\theta}\left(t-T_1\right)^{\beta_2-\beta_2\theta-\alpha\theta}H_1S_1+T_1^{\beta_1-\beta_1\theta-\alpha\theta+1}\left(t-T_1\right)^{\beta_2-\beta_2\theta-\alpha\theta}H_2S_1\epsilon\Big)+$$
$$\left(\sum_{k=1}^{\infty}\frac{1}{\left(k\pi\right)^{12\theta+2}}+\left\|\varphi^{V}\left(x\right)\right\|_2^2\right)T_1{}^{2\beta_1-2\beta_1\theta-2\alpha\theta}\left(t-T_1\right)^{\beta_2-\beta_2\theta-\alpha\theta}S_1H_2H_3+$$
$$\left(\sum_{k=1}^{\infty}\frac{1}{\left(k\pi\right)^{12\theta+2}}+\left\|\varphi^{VII}\left(x\right)\right\|_2^2\right)\left(t-T_1\right)^{\beta_2-\beta_2\theta-\alpha\theta}T_1^{2\beta_1-2\beta_1\theta-2\alpha\theta+1}S_1H_1H_2\epsilon.$$

Now we make a step to prove uniform convergence of the solution in Ω_3, namely

$$\left|u\left(t,x\right)\right|=\left|\sum_{k=1}^{\infty}W_k\left(t\right)\sin k\pi x\right|\leq\sum_{k=1}^{\infty}\left|W_k\left(t\right)\right|,\ T_2\leq t\leq T.$$

For this, considering (3), we need to look at the estimation of the $W_k\left(t\right)$:

$$\left|W_k\left(t\right)\right|\leq\left|{}_1F_k\right|+\left|{}_2F_k\right|\left(t-T_2\right)+\left|{}_1F_k\right|\left(k\pi\right)^2\left(t-T_2\right)^{\beta_3}\Gamma(\gamma)\times$$
$$\left|E_2\left(\begin{array}{c}\gamma,\gamma,1,1,0\\\beta_3+1,\beta_3,\alpha,\gamma,\gamma,1,1\end{array}\left|\begin{array}{c}-\left(k\pi\right)^2\left(t-T_2\right)^{\beta_3}\\\delta\left(t-T_2\right)^{\alpha}\end{array}\right.\right)\right|+\left|{}_2F_k\right|\left(k\pi\right)^2\left(t-T_2\right)^{\beta_3+1}\times$$
$$\Gamma(\gamma)\left|E_2\left(\begin{array}{c}\gamma,\gamma,1,1,0\\\beta_3+2,\beta_3,\alpha,\gamma,\gamma,1,1\end{array}\left|\begin{array}{c}-\left(k\pi\right)^2\left(t-T_2\right)^{\beta_3}\\\delta\left(t-T_2\right)^{\alpha}\end{array}\right.\right)\right|.$$

According to (3), we obtain:

$$\left|W_k\left(t\right)\right|\leq\left|{}_1F_k\right|+\left|{}_2F_k\right|\left(t-T_2\right)+\left|{}_1F_k\right|\left(k\pi\right)^{2-2\theta}\left(t-T_2\right)^{\beta_3-\beta_3\theta-\alpha\theta}\Gamma\left(\gamma\right)\times$$
$$\frac{C}{\left(-\delta\right)^{\theta}}\frac{\Gamma\left(\frac{\beta_3+1}{2}-\beta_3\theta\right)\Gamma\left(\frac{\beta_3+1}{2}-\alpha\theta\right)\Gamma\left(\gamma-\gamma\theta-\theta\right)}{\Gamma\left(\beta_3+1-\theta\left(\alpha+\beta_3\right)\right)}+\left|{}_2F_k\right|\left(k\pi\right)^{2-2\theta}\times$$
$$\left(t-T_2\right)^{\beta_3-\beta_3\theta-\alpha\theta+1}\Gamma\left(\gamma\right)\frac{C}{\left(-\delta\right)^{\theta}}\frac{\Gamma\left(\frac{\beta_3+2}{2}-\beta_3\theta\right)\Gamma\left(\frac{\beta_3+2}{2}-\alpha\theta\right)\Gamma\left(\gamma-\gamma\theta-\theta\right)}{\Gamma\left(\beta_3+2-\theta\left(\alpha+\beta_3\right)\right)}.$$

Introducing the following notations:

$$S_2=\Gamma\left(\gamma\right)\frac{C}{\left(-\delta\right)^{\theta}}\frac{\Gamma\left(\frac{\beta_3+1}{2}-\beta_3\theta\right)\Gamma\left(\frac{\beta_3+1}{2}-\alpha\theta\right)\Gamma\left(\gamma-\gamma\theta-\theta\right)}{\Gamma\left(\beta_3+1-\theta\left(\alpha+\beta_3\right)\right)},$$

$$S_3=\Gamma\left(\gamma\right)\frac{C}{\left(-\delta\right)^{\theta}}\frac{\Gamma\left(\frac{\beta_3+2}{2}-\beta_3\theta\right)\cdot\Gamma\left(\frac{\beta_3+2}{2}-\alpha\theta\right)\Gamma\left(\gamma-\gamma\theta-\theta\right)}{\Gamma\left(\beta_3+2-\theta\left(\alpha+\beta_3\right)\right)},$$

we rewrite the above representation of $\left|W_k\left(t\right)\right|$ as follows:

$$\left|W_k\left(t\right)\right|\leq\left|{}_1F_k\right|+\left|{}_2F\right|_k\left(t-T_2\right)+\left|{}_1F_k\right|\cdot\left(k\pi\right)^{2-2\theta}\cdot\left(t-T_2\right)^{\beta_3-\beta_3\theta-\alpha\theta}\cdot S_2+$$
$$+\left|{}_2F_k\right|\cdot\left(k\pi\right)^{2-2\theta}\cdot\left(t-T_2\right)^{\beta_3-\beta_3\theta-\alpha\theta+1}\cdot S_3;$$

To get the estimation of $W_k\left(t\right)$ via the given function, we first need to estimate ${}_1F_k$ and ${}_2F_k$.

Let us estimate ${}_1F_k$:

$$\begin{aligned}\left|{}_1F_k\right| \le \left|\varphi_k\right| + \left|{}_2A_k\right| T_1 + \left|\varphi_k\right| (k\pi)^{2-2\theta} T_1^{\beta_1-\beta_1\theta-\alpha\theta} H_1 + \\ + \left|{}_2A_k\right| (k\pi)^{2-2\theta} T_1^{\beta_1-\beta_1\theta-\alpha\theta+1} H_2 + \left|\varphi_k\right| (k\pi)^{2-2\theta} (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 + \\ + \left|{}_2A_k\right| (k\pi)^{2-2\theta} (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 T_1 + \left|\varphi_k\right| (k\pi)^{4-4\theta} T_1^{\beta_1-\beta_1\theta-\alpha\theta} \times \\ (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 H_1 + \left|{}_2A_k\right| (k\pi)^{4-4\theta} T_1^{\beta_1-\beta_1\theta-\alpha\theta+1} \times \\ (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 H_2.\end{aligned}$$

Now, let us look at the estimate of ${}_2F_k$. For this, we use the equation (3.15)

$$\begin{aligned}\left|{}_2F_k\right| \le \left|\varphi_k\right| (k\pi)^2 + (k\pi)^2 \left|{}_2A_k\right| T_1 + \left|\varphi_k\right| (k\pi)^{4-2\theta} T_1^{\beta_1-\beta_1\theta-\alpha\theta} H_1 + \\ \left|{}_2A_k\right| (k\pi)^{4-2\theta} T_1^{\beta_1-\beta_1\theta-\alpha\theta+1} H_2 + \left|\varphi_k\right| (k\pi)^{4-2\theta} (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 + \\ \left|{}_2A_k\right| (k\pi)^{4-2\theta} (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 T_1 + \left|\varphi_k\right| (k\pi)^{6-4\theta} T_1^{\beta_1-\beta_1\theta-\alpha\theta} \times \\ (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 H_1 + \left|{}_2A_k\right| (k\pi)^{6-4\theta} T_1^{\beta_1-\beta_1\theta-\alpha\theta+1} (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} \times \\ S_1 H_2.\end{aligned}$$

Now, using the above estimates and Lemma 4.1, after certain evaluations, we obtain the following estimate for $\sum_{k=1}^{\infty} \left|W_k(t)\right|$:

$$\begin{aligned}\sum_{k=1}^{\infty} \left|W_k(t)\right| \le \left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \left\|\varphi'(x)\right\|_2^2\right)(1+T_1\epsilon) + \left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \left\|\varphi'''(x)\right\|_2^2\right) \times \\ ((t-T_2)(1+T_1\epsilon)) + \left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^{2+4\theta}} + \left\|\varphi'(x)\right\|_2^2\right) T_1^{\beta_1-\beta_1\theta-\alpha\theta} H_3\epsilon + \\ \left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^{4\theta+2}} + \left\|\varphi'''(x)\right\|_2^2\right) \left(T_1^{\beta_1-\beta_1\theta-\alpha\theta} \Bigg(H_1(1+T_1\epsilon) + \right. \\ \left. T_1 H_2\epsilon + (t-T_2) H_3\epsilon \Bigg) + (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 (1+T_1\epsilon) + \right.\end{aligned}$$

$$\begin{aligned}\left.(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta} S_2 (1+T_1\epsilon)\right) + \left(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^{8\theta+2}} + \left\|\varphi'''(x)\right\|_2^2\right) T_1^{\beta_1-\beta_1\theta-\alpha\theta} \times \\ \epsilon \left(T_1^{\beta_1-\beta_1\theta-\alpha\theta} H_2 H_3 + (T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta} S_1 + (t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta} S_2 H_3\right) +\end{aligned}$$

$$\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{4\theta+2}}+\left\|\varphi^{V}(x)\right\|_{2}^{2}\right)\left(T_1^{\beta_1-\beta_1\theta-\alpha\theta}\left(T_1(t-T_2)\,\epsilon\,(H_1+H_2)+\right.\right.$$

$$(t-T_2)\,H_1+(1+T_1\epsilon)(t-T_2)\left((T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1+\right.$$

$$\left.(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta}S_3\right)+\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{8\theta+2}}+\left\|\varphi^{V}(x)\right\|_{2}^{2}\right)\times$$

$$\left(T_1^{\beta_1-\beta_1\theta-\alpha\theta}\left(T_1H_1\epsilon\left(T_1^{\beta_1-\beta_1\theta-\alpha\theta}H_2+(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1\right)+\right.\right.$$

$$(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1\left(H_1+(H_2T_1+H_3)\epsilon\right)+T_1^{\beta_1-\beta_1\theta-\alpha\theta}(t-T_2)\,H_2H_3\epsilon+$$

$$(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta}\left(H_1S_2(1+T_1\epsilon)+\epsilon\left(\frac{T_1^2H_2S_2+(t-T_2)\,H_3S_3}{T_1}\right)\right)+$$

$$\left.(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta}(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1S_2(1+T_1\epsilon)\right)+$$

$$\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{12\theta+2}}+\left\|\varphi^{V}(x)\right\|_{2}^{2}\right)T_1^{\beta_1-\beta_1\theta-\alpha\theta}H_3\left(T_1^{\beta_1-\beta_1\theta-\alpha\theta}S_1H_2\epsilon+\right.$$

$$\left.(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta}S_2\epsilon\left(T_1H_2+(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1\right)\right)+$$

$$\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{8\theta+2}}+\left\|\varphi^{VII}(x)\right\|_2^2\right)\left(T_1^{\beta_1-\beta_1\theta-\alpha\theta}\left(H_1(t-T_2)\left(T_1^{\beta_1-\beta_1\theta-\alpha\theta+1}H_2\epsilon+\right.\right.\right.$$

$$\left.(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}T_1S_1\epsilon+(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1\right)+(t-T_2)(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}\times$$

$$\left.S_1(H_1+H_2T_1\epsilon)\right)+(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta+1}S_3\times$$

$$\left.\left(H_1(1+T_1\epsilon)+T_1H_2\epsilon+(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1(1+T_1\epsilon)\right)\right)+$$

$$\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{12\theta+2}}+\left\|\varphi^{VII}(x)\right\|_2^2\right)\times$$

$$\left(T_1^{2\beta_1-2\beta_1\theta-2\alpha\theta}H_2\epsilon\left((T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1(T_1H_1+(t-T_2)H_3)+\right.\right.$$

$$\left.(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta}T_1S_2H_1\right)+T_1^{\beta_1-\beta_1\theta-\alpha\theta}(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta}\times$$

$$\left((T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1S_2(H_1+T_1\epsilon(H_1+H_2))+\right.$$

$$\left.\left.T_1^{\beta_1-\beta_1\theta-\alpha\theta}(t-T_2)H_2H_3S_3\epsilon\right)+\frac{(t-T_2)(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1T_1H_3S_3\epsilon}{T_1}\right)+$$

$$\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{16\theta+2}}+\left\|\varphi^{VII}(x)\right\|_2^2\right)T_1^{2\beta_1-2\beta_1\theta-2\alpha\theta}(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}\epsilon\times$$

$$(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta}S_1S_2H_2H_3+\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{12\theta+2}}+\left\|\varphi^{IX}(x)\right\|_2^2\right)\times$$

$$\left(T_1^{2\beta_1-2\beta_1\theta-2\alpha\theta+1}H_1H_2\epsilon\left((T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}(t-T_2)S_1+(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta+1}S_3\right)+\right.$$

$$\left.T_1^{\beta_1-\beta_1\theta-\alpha\theta}(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta+1}S_1H_1S_3(1+2T_1\epsilon)\right)+$$

$$\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{16\theta+2}}+\left\|\varphi^{IX}(x)\right\|_2^2\right)T_1^{2\beta_1-2\beta_1\theta-2\alpha\theta}(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}S_1H_1\times$$

$$(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta}\epsilon(T_1H_2S_2+(t-T_2)H_3S_3)+\left(\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{16\theta+2}}+\left\|\varphi^{XI}(x)\right\|_2^2\right)\times$$

$$T_1^{2\beta_1-2\beta_1\theta-2\alpha\theta+1}(T_2-T_1)^{\beta_2-\beta_2\theta-\alpha\theta}(t-T_2)^{\beta_3-\beta_3\theta-\alpha\theta+1}S_1S_3H_1^2.$$

We should also show that u_{xx} also converges uniformly. Because our equation also contains u_{xx}.

$$|u_{xx}| = \left|\sum_{k=1}^{\infty}(k\pi)^2 \cdot G_k(t) \cdot \sin k\pi x\right| \le \sum_{k=1}^{\infty}(k\pi)^2 \cdot |G_k(t)|,$$

where via $G_k(t)$ we denote $U_k(t)$, $V_k(t)$ and $W_k(t)$.

We will omit similar evaluations, stating that due to the multiplied term $(k\pi)^2$, we need to require two more derivatives from the given function $\varphi(x)$.

The uniform convergence of infinite series, corresponding to ${}^{PC}D_{0t}^{\alpha,\beta_1,\gamma,\delta}u(t,x)$, ${}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}u(t,x)$, ${}^{PC}D_{T_2t}^{\alpha,\beta_3,\gamma,\delta}u(t,x)$ can be obtained from the equation (2.1). Namely, based on (3.1), (3.2) and (3.3), one can get

$${}^{PC}D_{0t}^{\alpha,\beta_1,\gamma,\delta}u(t,x) = \sum_{k=1}^{\infty} {}^{PC}D_{0t}^{\alpha,\beta_1,\gamma,\delta}U_k(t)\sin k\pi x,$$

$${}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}u(t,x) = \sum_{k=1}^{\infty} {}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}V_k(t)\sin k\pi x,$$

$${}^{PC}D_{T_2t}^{\alpha,\beta_3,\gamma,\delta}u(t,x) = \sum_{k=1}^{\infty} {}^{PC}D_{T_2t}^{\alpha,\beta_3,\gamma,\delta}W_k(t)\sin k\pi x.$$

Considering (3.4), (3.5) and (3.6), we deduce

$$\left|{}^{PC}D_{0t}^{\alpha,\beta_1,\gamma,\delta}u(t,x)\right| \le \sum_{k=1}^{\infty}(k\pi)^2|U_k(t)|,$$

$$\left|{}^{PC}D_{T_1t}^{\alpha,\beta_2,\gamma,\delta}u(t,x)\right| \le \sum_{k=1}^{\infty}(k\pi)^2|V_k(t)|,$$

$$\left|{}^{PC}D_{T_2t}^{\alpha,\beta_3,\gamma,\delta}u(t,x)\right| \le \sum_{k=1}^{\infty}(k\pi)^2|W_k(t)|.$$

Since the latter series corresponds to the function $u_{xx}(t,x)$, the remaining proof will go similarly.

We note that the uniqueness of the solution to the problem can be proved based on the completeness of the sine Fourier series in $L_2(0,1)$.

Theorem 4.1 . If $\Delta_k \ne 0$, $0 \le \theta < \frac{1}{2}$, $\delta < 0$, $\gamma > \theta/(1-\theta)$, $2\alpha\theta < \lambda$ ($\lambda = \beta_1,\ \beta_2+1,\ \beta_3$), $\varphi(x) \in C^{12}[0,1]$ such that $\varphi^{(2i)}(0) = \varphi^{(2i)}(1) = 0$ $(i = \overline{0,6})$ and $\varphi^{XIII}(x) \in L_2(0,1)$, then the problem has a unique solution, represented by infinite series (3.1),(3.2) and (3.3).

5. Conclusion

We have investigated the unique solvability of the initial boundary value problem for the time-fractional wave-diffusion-wave equation. To solve this problem, we mainly employed the method of separating variables. Several conditions were established for the derivatives of the given function $\varphi(x)$ to ensure solvability. The constraints $0 \le \theta < \frac{1}{2}$, $\delta < 0$, $2\alpha\theta < \lambda$, arising from the estimate used for the function $E_2(x, y)$ caused difficulties to prove the convergence of infinite series, and require additional conditions on the derivatives of $\varphi(x)$. Although these conditions became more restrictive, we demonstrated that the multiple series involved in the solution converge uniformly by applying Lemma 4.1. Consequently, we have proved the existence of a unique solution to the problem, imposing stronger regularity on the given function. However, note that we are solving the problem with the minimum available data (one initial condition, two boundary condition, and standard transmitting conditions).

6. Data-availability and Conflict of Interests Declarations

The Authors declare that they do not have any conflicts of interest. Data sharing does not apply to this article as no new data were created or analyzed in this study.

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