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A GRAPH ASSOCIATED WITH TRI-POTENT ELEMENTS OF A COMMUTATIVE RING

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Abstract. In this paper, we introduce the tri-potent graph of a commutative ring R, denoted by $TP(R)$, where two distinct vertices x and y in R are adjacent if and only if $(x+y)^3 = x+y$. We conduct a comprehensive investigation of the graphical structural properties of tri-potent graph of a commutative ring R, including its diameter, connectedness, and size. It is shown that the tri-potent graph of a commutative ring R contains cycles with girth 3 and has no end vertices. Furthermore, we describe a significant spanning subgraph of the tri-potent graph of a commutative ring R and analyze the degree of each vertex in detail. Finally, we establish that, for a specific local ring, tri-potent graph of a commutative ring R forms a 3-partite graph, discuss its planarity, and determine its independence domination number.

1. Introduction

The interplay between finite commutative rings and graph theory has been extensively studied in recent decades. Several works, such as [2, 8, 9, 11], have investigated the intricate connections between these fields. Anderson and Badawi [1] initiated this exploration by introducing the total graph, wherein the vertices represent elements of a ring R and adjacency occurs precisely when the sum of two elements is a zero-divisor. This was followed by Ashrafi et al. [3], who introduced the unit graph, establishing adjacency whenever the sum of two distinct vertices is a unit. An extension of this line of research was made by Razzaghi and Sahebi [10], who introduced the idempotent graph, which denotes by $G_{\mathrm{Id}}(R)$, wherein two distinct elements are adjacent if and only if their sum is idempotent. In this paper, we define the tri-potent graph of a commutative ring R, denoted by $TP(R)$, and examine its fundamental graphical properties, including connectedness and diameter. We also compute the degree of each vertex in $TP(R)$, demonstrating that the graph contains an odd cycle, has no end vertices, and satisfies $\mathrm{gir}(TP(R)) = \omega(TP(R)) = 3$. Moreover, we determine the size of the tri-potent graph where R is a local ring and establish its relation to $|Tri(R)|$. In addition, we characterize $TP(R)$ as a 3-partite graph when $R \cong \mathbb{Z}_r$ with $r = p^\alpha$, where $p \geq 3$ is a prime, $\alpha \in \mathbb{Z}^+$, and we obtain its independence domination

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number. Finally, we investigate the planarity of the tri-potent graph in the case when $|Tri(R)| = 3$.

2. Preliminaries

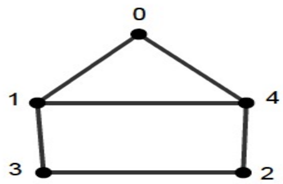
A graph G consists of a set of vertices $V(G)$ and a set of edges $E(G)$ connecting pairs of vertices. The order (size) of G refers to the number of vertices (edges), denoted by n and m, respectivley. The degree of a vertex x, denoted $\deg(x)$, is the number of edges incident to x, and its neighborhood, $N_G(x)$, is the set of vertices adjacent to x. A path in G is a sequence of distinct vertices where consecutive vertices are adjacent. A cycle is a closed path in which each vertex has degree two. The distance between vertices x and y, denoted $d_G(x, y)$, is the length of the shortest path connecting them. A graph is connected if every pair of vertices is joined by a path; otherwise, it is disconnected. The diameter of a connected graph G, denoted $diam(G)$, is the maximum distance between any two vertices, while the girth, $gir(G)$, is the length of the shortest cycle, taken to be infinite if G has no cycle. The clique number $\omega(G)$ is the largest number of vertices in a complete subgraph of G. A graph where every two distinct vertices are adjacent is called a complete graph, denoted K_n. A graph is bipartite if its vertex set can be partitioned into two independent sets. A spanning subgraph is a subgraph containing all the vertices of G [5]. An independent dominating set in G, denoted $\gamma_{\text{ind}}(G)$, is an independent set that also dominates the graph, that is; every vertex not in the set is adjacent to at least one vertex in it [6]. A graph is planar if it can be embedded in the plane without edge crossings; by Kuratowski's Theorem, this occurs if and only if G does not contain any subdivision of K_5 or $K_{3,3}$.

Throughout this paper, let R denote a commutative ring with identity. An element $a \in R$ is called idempotent if $a^2 = a$, and the set of all idempotent elements is denoted $Id(R)$. An element $s \in R$ is called tri-potent if $s^3 = s$, and the set of all tri-potent elements is denoted $Tri(R)$. In $\mathbb{Z}$, the tri-potent elements are precisely $\{0, 1, -1\}$, and every idempotent is necessarily tri-potent. A ring R is local if it has a unique maximal ideal; notably, $\mathbb{Z}_{p^\alpha}$ is local for every $\alpha \in \mathbb{Z}^+$ [4].

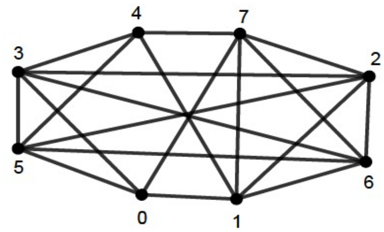
3. Main Results

In this section, we introduce the concept of the tri-potent graph associated with a ring R and provide several illustrative examples. We examine the adjacency criteria between pairs of vertices in the tri-potent graph and establish that it necessarily contains at least one odd cycle, and it forms a complete graph if and only if $R = Tri(R)$. We further compute the degree of each vertex and determine the connectivity and diameter of the tri-potent graph of R. Moreover, we demonstrate that the tri-potent graph has no end vertices and that the idempotent graph forms a spanning subgraph of the tri-potent graph.

Definition 3.1. Let R be a ring; define the tri-potent graph as a graph with the set of vertices as the element of R, and for two distinct elements t_1 and t_2 in R, they are adjacent if $t_1 + t_2 = s$, where $s \in Tri\,(R)$, and denote by $TP\,(R)$. In Figure 1, we provide some examples of tri-potent graphs of commutative rings.



(1.1) $TP(Z_5)$



(1.2) $TP(Z_8)$

FIGURE 1. Tri-potent Graph of Commutative Rings

Remark 3.2. Consider the Table 1, which presents the properties of $TP(R)$.

TABLE 1. Properties of TP(R)

$V(TP(R))$	**Type of a ring** R	$TP(R)$
2	Z_2	K_2
3	Z_3	C_3
4	Z_4 or $Z_2[x]/(x^2)$	$K_1 + C_3$
5	Z_5	Figure 1.1
6	Z_6	K_6
8	Z_8	Figure 1.2
p^α where $p \geq 3, \alpha \in \mathbb{Z}^+$	Z_p^α	Figure 3

Theorem 3.3. *Let R be a ring, then $TP(R)$ has an odd cycle.*

Proof. Let R be a ring; then for any $s \in Tri(R)$, it implies that $s^3 = s$, so $s(s^2 - 1) = 0$. Thus, the tri-potent elements of R are $\{0, 1, -1\}$; that is, for every $t_1, t_2 \in V(TP(R))$, then $t_1 + t_2 \in Tri(R)$, we have a path $0 - 1 - (-1) - 0$ which is of length 3. Hence $TP(R)$ has at least one odd cycle of length 3. $\square$

Corollary 3.4. *Let R be a ring, then $gir(TP(R)) = \omega(TP(R)) = 3$.*

Proof. It's direct, by Theorem 3.3. $\square$

Lemma 3.5. *Let x be a vertex of $TP(R)$, then the degree of x is either $|Tri(R)|$ or $|Tri(R)| - 1$.*

Proof. Let t_1 and t_2 be two distinct vertices of $TP(R)$. Then $t_1 + t_2 = s$ for some $s \in Tri(R)$. If $2 \cdot t_1 \in Tri(R)$, then t_1 is adjacent to $s - t_1$ for any $s \in Tri(R) \setminus \{2 \cdot x\}$. Hence, the degree of t_1 is $|Tri(R)| - 1$. Next, if $2 \cdot t_1 \notin Tri(R)$, then t_1 is adjacent to $s - t_1$ for any $s \in Tri(R)$. Hence the degree of t_1 is $|Tri(R)|$. $\square$

Proposition 3.6. *Let R be a ring, then a graph $TP(R)$ is complete if and only if $Tri(R) = R$.*

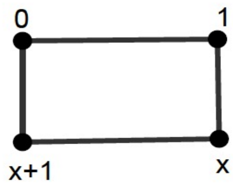
Proof. Let $TP\left(R\right)$ be a complete graph, and for $t_1 \in TP\left(R\right)$, then t_1 is adjacent to every other vertex of $TP\left(R\right)$, that is, $deg\left(t_1\right) = \left|V\left(TP\left(R\right)\right)\right| - 1$, and the set of vertices of $TP\left(R\right)$ equals the element of R, then $deg\left(t_1\right) = |R| - 1$, since by Lemma 3.5, we obtain $deg\left(t_1\right) = \left|Tri\left(R\right)\right| - 1$, which obtains $R \subset Tri(R)$. On the other hand, consistently $Tri(R) \subset R$. Thus $Tri(R) = R$.
Conversely, let R be a ring such that $Tri(R) = R$, and let $t_1 \in R$; then $(t_1)^3 = t_1$. Now, for two distinct vertices $t_1, t_2 \in TP\left(R\right)$, then $t_1 + t_2 \in Tri\left(R\right)$, that is, $d\left(t_1, t_2\right) = 1$, so every vertex of $TP\left(R\right)$ is adjacent to other vertices of $TP\left(R\right)$, which means $deg\left(t_1\right) = |R| - 1$. Thus, $TP\left(R\right)$ is a complete graph. □

Theorem 3.7. *Let R be a ring, then $TP\left(R\right)$ is connected and $diam\left(TP\left(R\right)\right) = \left\{\frac{2^\alpha}{4}, \frac{p^\alpha - 1}{2}, \infty\right\}$, where $p \geq 3$ and $\alpha \in \mathbb{Z}^+$.*

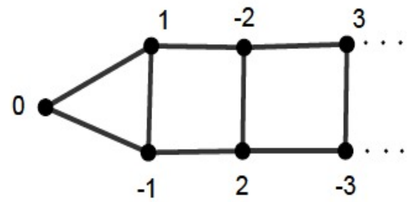
Proof. For two distinct vertices t_1 and t_2 in $TP(R)$, suppose that $t_1 + t_2$ is equal to only the trivial tri-potent element of R; then $TP(R)$ is disconnected since there is at least one component of $TP(R)$ that 0 is adjacent to 1, as illustrated in Figure 2.1. On the other hand, if $t_1 + t_2 \in Tri\left(R\right)$, then there is at least a path from 0 to t_1, such that $0 - y_1 - ... - y_n - t_1$, for some y_i, which is of even length such that $0 + t_1 \in Tri\left(R\right)$, and explained in Figure 2.2, thus $TP\left(R\right)$ is connected.
Now, we discuss the following cases to find the maximum distance between every two vertices in $TP\left(R\right)$:
Case 1. Let t_1 and $t_2 \in R$; if $t_1 + t_2 \in Tri\left(R\right)$, then there is a path $t_1 - t_2$ and $d\left(t_1, t_2\right) = 1$, so $diam\left(TP\left(R\right)\right) = 1$.
Case 2. Let t_1 and $t_2 \in R$; if $t_1 + t_2 \notin Tri\left(R\right)$, then there exists some $a_1, a_2, \ldots, a_n \in R$, such that there exists a path like $t_1 - a_1 - t_2$, has $diam\left(TP\left(R\right)\right) = 2$. Moreover, if $t_1 + a_1 \notin Tri\left(R\right)$, then there is another $a_i \in R$, such that $t_1 - a_1 - a_2 - .. - a_n - t_2$. So, we examine the following cases:

- If $R \cong Z_r$, where $r = 2^\alpha$, with $\alpha \geq 3$, then $d\left(0, 2^{\alpha-2}\right) = d\left(0, r - 2^{\alpha-2}\right) = 2^{\alpha-2}$, that is $diam\left(TP\left(R\right)\right) = \frac{2^\alpha}{4} = \frac{r}{4}$.
- If $R \cong Z_r$, for $r = p^\alpha$, with $p \geq 3$ and $\alpha \in \mathbb{Z}^+$, then $d\left(0, \frac{p^\alpha - 1}{2}\right) = d\left(0, \frac{p^\alpha + 1}{2}\right) = \frac{p^\alpha - 1}{2}$. Thus $diam\left(TP\left(R\right)\right) = \frac{p^\alpha - 1}{2} = \frac{r-1}{2}$.
- If $R \cong \mathbb{Z}$, then $diam\left(TP\left(R\right)\right) = \infty$.

□



(2.1) $TP(Z_2[x]/(x^2))$



(2.2) $TP(\mathbb{Z})$

FIGURE 2. Tri-potent Graph of Commutative Rings

Proposition 3.8. *Let R be a ring, then $TP\,(R)$ has no an end-vertex.*

Proof. For a ring R; suppose $TP\,(R)$ has an end-vertex. Now, let $t_1 \in V(TP(R))$, then there exists some $t_2 \in V(TP(R))$ such that $t_1 + t_2 \in \text{Tri}(R)$.
However $d(0,1) = d(0,-1) = 1$, then $N_{TP(R)}\,(0) = \{1,-1\}$. Consider that, if $2 \cdot t_1 \in Tri\,(R)$, then $N_{TP(R)}\,(t_1) = \{b-1, b+1 : b \in R - \{2 \cdot t_1\}\}$, so the degree of t_1 in $TP\,(R)$ is at least 2. Also, if $2 \cdot t_1 \notin Tri\,(R)$, take $d(1,0) = d(1,-1) = d(1,b) = 1$, where $b = 2 \cdot t_1$, then $N_{TP(R)}\,(t_1) = \{b, b-1, b+1 : b \in R\}$, so the degree of t_1 in $TP\,(R)$ is equal to 3. As a result, $deg\,(t_1) = 2$ or 3, and it's a contradiction. Hence, $TP\,(R)$ has no end–vertex. □

Proposition 3.9. *Let R be a ring, then $G_{Id}\,(R)$ is a spanning subgraph of $TP\,(R)$.*

Proof. For any ring R, by assumption of $G_{Id}\,(R)$, the element of R refers to the set of vertices of $G_{Id}\,(R)$ and also of $TP\,(R)$. Now, for two distinct vertices $t_1, t_2 \in G_{Id}\,(R)$, then $t_1 + t_2 = e \in Id\,(R)$. Since for every idempotent element $e^2 = e$ is also tri-potent, then $e^2 = e^3 = e$, that is, $t_1 + t_2 = e \in Tri\,(R)$. Hence $G_{Id}\,(R)$ is a spanning subgraph of $TP\,(R)$. □

4. Structure of Tri-potent graph

In this section, we aim to compute the size of $TP(R)$, where R is a local ring. In particular, the ring $\mathbb{Z}_{p^\alpha}$ is local for any $\alpha \in \mathbb{Z}^+$. We begin this section with the following lemma, which characterizes the properties of $TP(R)$ for a local ring.

Lemma 4.1. *Let R be a ring, and $R \cong Z_r$, where $r = p^\alpha$, for $\alpha \in \mathbb{Z}^+$. Then define $\mathcal{M}(R) = \{x : 2x \in Tri\,(R)\}$ such that:*

1. *If p is even, and $\alpha \geq 3$, then $|Tri\,(R)| = 5$ and $|\mathcal{M}(R)| = 2$.*
2. *If p is odd, and $\alpha \in \mathbb{Z}^+$, $|Tri\,(R)| = 3$ and $|\mathcal{M}(R)| = 2$.*

Proof. let $R \cong Z_r$, where $r = p^\alpha$, and $\alpha \in \mathbb{Z}^+$, to find $|Tri\,(R)|$ and $|\mathcal{M}(R)|$, we discuss the following cases:

(1) If p is even, then $Tri\,(R) = \{0, 1, 2^{\alpha-1}-1, 2^{\alpha-1}+1, 2^{\alpha}-1\}$, by [12, Proposition 1.7], so $|Tri\,(R)| = 5$. Now to find $|\mathcal{M}(R)|$, suppose that $s \in Tri\,(R)$, then $s^3 \equiv s \pmod{2^\alpha}$. That is, for any $x \in R$, if $2 \cdot x \in Tri\,(R)$, then $2x \equiv s \pmod{2^\alpha}$; if $s \in Tri\,(R) \setminus \{0\}$, then x doesn't exist, and if $s = 0$, then $2x \equiv 0 \pmod{2^\alpha}$, implying that $x \in \{0, 2^{\alpha-1}\}$. Thus $|\mathcal{M}(R)| = 2$.

(2) If p is odd, then $Tri\,(R) = \{0, 1, p^\alpha - 1\}$, by [12, Lemma 1.4]. So $|Tri\,(R)| = 3$. Now, we seek to find $|\mathcal{M}(R)|$, since by definition of $\mathcal{M}(R)$, then $2 \cdot x \in Tri\,(R)$, implying that $(2x)^3 \equiv 2x \pmod{p^\alpha}$, so $2x(2x-1)(2x+1) \equiv \pmod{p^\alpha}$. Thus we have $2x \equiv 0 \pmod{p^\alpha}$, which means $x = 0$. Also $(2x-1) \equiv \pmod{p^\alpha}$, so $x = \frac{p^\alpha+1}{2}$. At last, $(2x+1) \equiv \pmod{p^\alpha}$, then $x = \frac{p^\alpha-1}{2}$. Thus $|\mathcal{M}(R)| = 3$.

□

Proposition 4.2. *Let R be a local ring, then* $m\,(TP\,(R)) = \left\{\frac{5r-2}{2}, \frac{3(r-1)}{2}\right\}$

Proof. Let $TP(R)$ be a tri-potent graph; since every edge is incident to exactly two vertices, then $m\,(TP\,(R)) = \sum_{i=1}^{r} deg(v_i)/2$, and by Lemma 3.5, we have $deg\,(v_i) = |Tri\,(R)| - 1$, or $deg\,(v_i) = |Tri\,(R)|$, that is;

$$m\,(TP\,(R)) = \left(\sum_{i=1}^{k} deg(v_i) \cdot (|Tri\,(R)| - 1) + \sum_{i=k+1}^{|R|} deg(v_i) \cdot (|Tri\,(R)|)\right)/2$$

Now put $k = |\mathcal{M}(R)|$, which defines in Lemma 4.1, we have two following cases:

(1) For $|Tri\,(R)| = 5$ and $|R| = r$, so we have

$$\begin{aligned} m\,(TP\,(R)) &= \left(\sum_{i=1}^{2} deg(v_i) \cdot (5-1) + \sum_{i=3}^{r} deg(v_i) \cdot (5)\right)/2 \\ &= \left(\sum_{i=1}^{2} deg(v_i) \cdot 5 - \sum_{i=1}^{2} deg(v_i) + \sum_{i=3}^{r} deg(v_i) \cdot 5\right)/2 \\ &= \left(\sum_{i=1}^{r} deg(v_i) \cdot 5 - \sum_{i=1}^{2} deg(v_i)\right)/2 \end{aligned}$$

Thus $m\,(TP\,(R)) = \frac{5r-2}{2}$.

(2) If $k = 3$, then $|Tri\,(R)| = 3$, so

$$\begin{aligned} m\,(TP\,(R)) &= \left(\sum_{i=1}^{3} deg(v_i) \cdot (3-1) + \sum_{i=4}^{r} deg(v_i) \cdot (3)\right)/2 \\ &= \left(\sum_{i=1}^{3} deg(v_i) \cdot 3 - \sum_{i=1}^{3} deg(v_i) + \sum_{i=4}^{r} deg(v_i) \cdot 3\right)/2 \\ &= \left(\sum_{i=1}^{r} deg(v_i) \cdot 3 - \sum_{i=1}^{3} deg(v_i)\right)/2 \end{aligned}$$

Thus $m\,(TP\,(R)) = \frac{3(r-1)}{2}$.

□

Theorem 4.3. *Let R be a ring, then $TP\,(R)$ is 3-partite graph, if $R \cong Z_r$, for $r = p^{\alpha}$, with $p \geq 3$, and $\alpha \in \mathbb{Z}^{+}$.*

Proof. Since $R \cong \mathbb{Z}_r$, where $r = p^{\alpha}$ with $p \geq 3$ and $\alpha \in \mathbb{Z}^{+}$, then we have

$$V\,(TP(R)) = \{0, 1, 2, \ldots, r-1\}.$$

Consider the element 0 is adjacent only to 1 and $r-1$. Then, we define the first partite set as:

$$V_1 = \{0\}.$$

Now, let $x \in V(TP(R))$. Then there exists some $y \in V(TP(R))$ such that $x + y \in \mathrm{Tri}(R)$, implies that $d(x, y) = 1$, which explains in Figure 3 and we get

$$x \in \left\{1, 2, \ldots, \frac{r-1}{2}\right\} = V_2, \quad \text{and} \quad y \in \left\{\frac{r+1}{2}, \ldots, r-1\right\} = V_3.$$

where the distance between the vertices of V_1 and V_2 equals to 1. Thus, we obtain three disjoint and independent sets as V_1, V_2, and V_3, which form a 3-partite graph corresponding to the element of a ring R. □

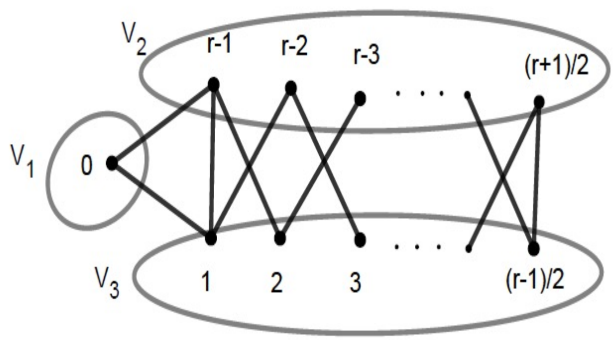


FIGURE 3. 3-partite graph of R

Theorem 4.4. *Let R be a ring, then the independence domination number of $TP\,(R)$ is given by $\gamma_{ind}\,(TP\,(R)) = \frac{r-1}{2}$.*

Proof. Let R be a ring, by Theorem 4.3, consider the set $S = \left\{1, 2, \ldots, \frac{r-1}{2}\right\}$ as a subset of the set of vertices $V(TP(R))$. Our claim is to prove, every vertices of $TP(R)$ is adjacent to S. Now, let $t_2 \in V(TP(R)) \setminus S$, then there exists at least $t_1 \in S$ such that $t_1 + t_2 = 0 \in Tri(R)$, which implies that t_1 and t_2 are adjacent in $TP(R)$. Note that the vertex 0 is adjacent only to 1, since $0+1 = 1 \in Tri(R)$. On the other hand, for two distinct vertices $t_1, t_2 \in S$, we have $t_1 + t_2 \notin Tri(R)$, so no two vertices in S are adjacent. Thus, S forms an independent set.
Furthermore, S is also a minimal dominating set, since there is no non-empty proper subset $\emptyset \neq J \subset S$ such that every vertex in $V(TP(R)) \setminus S$ is adjacent to

some vertex in J. Hence, S is a minimal independent dominating set. Therefore, the independence domination number of $TP(R)$ represents as $|S| = \frac{r-1}{2}$ where $r = p^{\alpha}$ for some prime $p \geq 3$ and $\alpha \in \mathbb{Z}^{+}$. □

Theorem 4.5. *Let R be a ring, then $TP\,(R)$ is planar if and only if $|Tri\,(R)| = 3$.*

Proof. Let R be a ring, and for two distinct vertices t_1 and t_2 in $TP(R)$, we have $t_1 + t_2 \in Tri(R)$. Now, Assume that $TP(R)$ is planar. By Kuratowski's Theorem, this implies that $TP(R)$ does not contain a subdivision of either K_5 or $K_{3,3}$. Consider a vertex $t_1 \in TP(R)$. If $2 \cdot t_1 \in \mathrm{Tri}(R)$, then t_1 is adjacent to all other elements of $Tri(R)$ except itself, and hence $\deg(t_1) = |Tri(R)| - 1$. However, by Lemma 4.1, $\deg(t_1) = 2$, which implies $|Tri(R)| = 3$. On the other hand, if $2 \cdot t_1 \notin Tri(R)$, then t_1 is adjacent to all elements of $Tri(R)$, and thus $\deg(t_1) = |Tri(R)|$. Again, by the same reason, we get $\deg(t_1) = 3$, implies that $|Tri(R)| = 3$. Therefore, in both cases, we conclude that $|Tri(R)| = 3$. Conversely, suppose that $|Tri(R)| > 3$. Then there exists a vertex $t_1 \in TP(R)$ with $\deg(t_1) = 4$. Moreover, by Lemma 4.1, the supremum of the vertex degrees in $TP(R)$ is

$$\sup_{TP(R)} (\deg(t_1)) = 5,$$

which implies that $TP(R)$ contains a subdivision of K_5, which contradicts our assumption. Hence, $|Tri(R)| = 3$. □

Conclusion

In conclusion, the tri-potent graph associated with a commutative ring R exhibits several significant structural properties, including the connectivity, the absence of end vertices, and the presence of cycles with the smallest possible girth. A detailed examination of key properties such as vertex degrees, diameter, and the independence domination number, within the context of a specific local ring, highlights the potential of this graph as a valuable tool in algebraic graph theory. Furthermore, this study opens new avenues for future research into the interaction between ring-theoretic properties and graph structures.

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