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ANALYTICAL SOLUTIONS OF FRACTIONAL PDES VIA BANACH SPACE TENSOR PRODUCT THEORY

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Abstract. In this paper, we propose a novel class of atomic solutions for fractional partial differential equations involving three independent variables, constructed through the framework of conformable derivatives. By leveraging the structural properties of Banach spaces and the theory of tensor product decomposition, we develop a rigorous analytical foundation for the formulation and validation of these solutions. The conformable derivative is employed for its compatibility with standard calculus rules, enabling efficient separation of variables and simplification of the solution process. Through analytical derivation and case analysis, we demonstrate the effectiveness, generality, and applicability of the proposed method in addressing high-dimensional fractional models.

1. Introduction

R. Khalil [14] proposed an innovative method for solving both ordinary and fractional differential equations. This new approach is grounded in the tensor product theory of Banach spaces and can be employed to derive what are referred to as "atomic solutions" for the differential equations being investigated [5, 21, 4, 19, 18].

Often, partial differential equations can be solved by separating variables under certain structural assumptions. However, a significant limitation of the separation of variables method is its inapplicability to nonlinear equations or even to certain linear equations for which variable separation is not straightforward. A clear example of this challenge is illustrated by the following equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x \partial y \partial z} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = u. \tag{1.1}$$

Here, u denotes a function that is continuously differentiable within the space $C(\mathbb{R}^3, \mathbb{R})$, so that it is difficult to find a solution to this equation using separation of variables.

Equations of this type may arise in models involving anomalous diffusion or viscoelastic wave propagation, where the fractional order captures memory or hereditary properties of the medium. Therefore, providing analytical solutions for such

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PDEs is of both theoretical and applied interest.
We generalize PDE_s(1.1) to its fractional form by introducing the conformable derivative as follows:

$$D_x^{(\alpha)}D_x^{(\alpha)}u + D_x^{(\alpha)}D_y^{(\alpha)}D_z^{(\alpha)}u + D_y^{(\alpha)}D_y^{(\alpha)}u + D_z^{(\alpha)}D_z^{(\alpha)}u = u, \quad \alpha \in (0,1] \tag{1.2}$$

The choice to expand the equation into a fractional partial differential equation arises from recognizing that fractional calculus offers a broader framework, better suited to capturing the complex dynamics of these systems [10, 17, 20, 8, 9].
We adopt the conformable derivative due to its analytical tractability and its adherence to standard product and chain rules. These properties facilitate the manipulation of equations in Banach space frameworks. In contrast to Caputo or Riemann–Liouville derivatives, the conformable derivative supports more straightforward variable separation and function composition, making it particularly well-suited for the construction of tensor-based atomic solutions.For further insights and applications involving the conformable derivative, the reader may consult the referenced studies.[6, 7]

2. Preliminaries

The conformable fractional derivative was first introduced in [13] as a new approach to fractional differentiation.

Definition 2.1. [13] Let $h : [0, \infty) \to \mathbb{R}$ be a real valued function. Then the conformable fractional derivative of h of order $\alpha \in (0, 1]$, at $t > 0$ is defined by

$$D^{\alpha}(h)(t) = \lim_{s \to 0} \frac{h(t + st^{1-\alpha}) - h(t)}{s}.$$

When the limit exists, we say that h is α-differentiable at t.

If h is α-differentiable in some $(0, a]$, $a > 0$, and $\lim_{t \to 0^+} D^{\alpha}(h)(t)$ exists, then we define $D^{\alpha}(h)(0) = \lim_{t \to 0^+} D^{\alpha}(h)(t)$. We will sometimes, write $h^{(\alpha)}(t)$ for $D^{\alpha}(h)(t)$ to denote the conformable fractional derivative of h of order α.

Theorem 2.2. [13] *For $0 < \alpha \leq 1$ and u, v be α-differentiable at a point $t > 0$ we have the following properties:*

(1) $D^{\alpha}(au + bv) = aD^{\alpha}(u) + bD^{\alpha}(v)$, *for all* $a, b \in R$.
(2) $D^{\alpha}(t^p) = pt^{p-\alpha}$, *for all* $p \in R$.
(3) $D^{\alpha}(uv) = uD^{\alpha}(v) + vD^{\alpha}(u)$.
(4) $D^{\alpha}(\frac{u}{v}) = \dfrac{vD^{\alpha}(u) - uD^{\alpha}(v)}{v^2}$.
(5) $D^{\alpha}(\lambda) = 0$, *λ is a constant function.*
(6) *If u is differentiable, then* $D^{\alpha}(u)(t) = t^{1-\alpha}\dfrac{du}{dt}(t)$.

The most important case for the range of α is $(0, 1)$. When $\alpha \in (n, n + 1]$ the definition would be as follows:

Definition 2.3. [13] Let $\alpha \in (n, n+1]$, and h be an n-differentiable at t, where $t > 0$. Then the conformable fractional derivative of u of order α is defined as

$$D^{\alpha}(h)(t) = \lim_{s \to 0} \frac{h^{(\lceil \alpha \rceil - 1)}(t + st^{\lceil \alpha \rceil - \alpha}) - h^{(\lceil \alpha \rceil - 1)}}{s},$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α. When the limit exists, we say that h is α-differentiable at t. If h is α-differentiable in some $(0, a], a > 0$ and $\lim_{t \to 0^+} D^{\alpha}(h)(t)$ exists, then we define $h^{\alpha}(0) = \lim_{t \to 0^+} h^{\alpha}(t)$ here, $h^{\alpha}(t)$ denotes $D^{\alpha}(h)(t)$, and $h^{2\alpha}(t)$ denotes $D^{\alpha}D^{\alpha}(h)(t)$.

The corresponding definition of the integral operator, analogous to the derivative operator, is provided as follows.

Definition 2.4. [13] Let $0 < \alpha \leq 1$, and $h : [0, \infty) \to \mathbb{R}$ be a function. For $a \geq 0$ the conformable fractional integral of h of order α is defined by

$$I_a^{\alpha}(h)(t) = I_a^1(t^{1-\alpha}h(t)) = \int_a^t \frac{h(x)}{x^{1-\alpha}}\, dx,$$

where the integral is the usual Riemann improper integral.

For further insights into the conformable fractional derivative, additional references can be found in [1, 2, 3] and [13, 16].
Let E and F be two Banach spaces, with E^* representing the dual of E. Assume $x \in E, y \in F$. The operator $\Psi : E^* \to F$ is defined as

$$\Psi(x^*) = x^*(x)y = \langle x^*, x \rangle y.$$

The expression "$x \otimes y$" denotes the bounded one-rank linear operator Ψ. Such operators are referred to as atoms, which are an important component of Banach space tensor product theory. The theorem cited in [11] carries importance for our paper and is essential for our analysis.

Theorem 2.5. *If $x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3$, then either $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$.*

Lemma 2.6. *If $x_1 \otimes y_1 = x_2 \otimes y_2$, then $x_1 = x_2$ and $y_1 = y_2$, where x_1, x_2, and x_3 are elements of a Banach space.*

For further insights into tensor products of Banach spaces, see references [11] and [15].

3. Atomic solution

In this section, we'll explore several potential atomic solutions for fractional differential equation (1.2). An atomic solution to the fractional partial differential equation involving variables x, y, and z takes the following form:

$$u(x, y, z) = \rho(x)\varphi(y)\psi(z).$$

Consider the fractional partial differential equation below:

$$D_x^{(\alpha)}D_x^{(\alpha)}u + D_x^{(\alpha)}D_y^{(\alpha)}D_z^{(\alpha)}u + D_y^{(\alpha)}D_y^{(\alpha)}u + D_z^{(\alpha)}D_z^{(\alpha)}u = u, \quad \alpha \in (0, 1] \tag{3.1}$$

where

$$D_x^{(\alpha)} D_x^{(\alpha)} u = \frac{\partial^\alpha}{\partial x}\left(\frac{\partial^\alpha u(x,y,z)}{\partial x}\right), \quad D_x^{(\alpha)} D_y^{(\alpha)} D_z^{(\alpha)} u = \frac{\partial^\alpha}{\partial x}\left(\frac{\partial^\alpha}{\partial y}\left(\frac{\partial^\alpha u(x,y,z)}{\partial z}\right)\right),$$

$$D_y^{(\alpha)} D_y^{(\alpha)} u = \frac{\partial^\alpha}{\partial y}\left(\frac{\partial^\alpha u(x,y,z)}{\partial y}\right), \quad D_z^{(\alpha)} D_z^{(\alpha)} u = \frac{\partial^\alpha}{\partial z}\left(\frac{\partial^\alpha u(x,y,z)}{\partial z}\right).$$

Although this is a linear fractional partial differential equation, it is very difficult to employ the separation of variables technique for its solution. Therefore, we are compelled to resort to atomic solutions.

Let

$$u(x,y,z) = \rho(x)\varphi(y)\psi(z) = \rho \otimes \varphi \otimes \psi \tag{3.2}$$

where $u(x,y,z)$ represents an atomic solution of equation (3.2), and $\rho(x)$, $\varphi(y)$, and $\psi(z)$ are non-constant functions. Additionally, let's make the following assumptions.

$$\begin{cases} \rho(0) = \rho^\alpha(0) = 1 \ldots (i) \\ \varphi(0) = \varphi^\alpha(0) = 1 \ldots (ii) \\ \psi(0) = \psi^\alpha(0) = 1 \ldots (iii) \end{cases}$$

Now, substitute equation (3.2) into equation (3.1) to get:

$$\rho^{2\alpha} \otimes \varphi \otimes \psi + \rho^\alpha \otimes \varphi^\alpha \otimes \psi^\alpha + \rho \otimes \varphi^{2\alpha} \otimes \psi + \rho \otimes \varphi \otimes \psi^{2\alpha} = \rho \otimes \varphi \otimes \psi \tag{3.3}$$

Hence, equation (3.3) is written in the following form

$$\rho^{2\alpha} \otimes \varphi \otimes \psi + \rho^\alpha \otimes \varphi^\alpha \otimes \psi^\alpha = \rho \otimes \left[-\varphi^{2\alpha} \otimes \psi - \varphi \otimes \psi^{2\alpha} + \varphi \otimes \psi\right] \tag{3.4}$$

We formulate equation (3.4) as follows :

$$\rho^{2\alpha} \otimes H_1 + \rho^\alpha \otimes H_2 = \rho \otimes H_3 \tag{3.5}$$

where

$$\varphi \otimes \psi = H_1, \quad \varphi^\alpha \otimes \psi^\alpha = H_2, \quad \text{and } \left[-\varphi^{2\alpha} \otimes \psi - \varphi \otimes \psi^{2\alpha} + \varphi \otimes \psi\right] = H_3$$

Hence, applying Theorem 2.5 to equation (3.5) results in two possible cases:

- **Case 1:** $\rho^{2\alpha} = \rho^\alpha = \rho$
- **Case 2:** $H_1 = H_2 = H_3$

3.1. **Case 1:** $\rho^{2\alpha} = \rho^\alpha = \rho$**.** In this case I have the following situations:

(1) $\rho^{2\alpha} = \rho$
(2) $\rho^{2\alpha} = \rho^\alpha$
(3) $\rho^\alpha = \rho$

Situation 01:

$$\rho^{2\alpha} = \rho$$

Referring to the findings in [12], the corresponding characteristic equation is given by $r^2 = 1$. Hence ($r = 1$ or $r = -1$)

Therefore

$$\rho(x) = c_1 e^{\frac{1}{\alpha}x^\alpha} + c_2 e^{-\frac{1}{\alpha}x^\alpha}$$

Using the initial conditions (i) gives us

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 1 \end{cases} \quad \text{Hence, } c_1 = 1 \text{ and } c_2 = 0 \text{ then } \quad \rho(x) = e^{\frac{1}{\alpha}x^\alpha}$$

Situation 02:

$$\rho^{2\alpha} = \rho^{\alpha}$$

Once again, employing the outcomes from [12], the associated characteristic equation can be expressed as $r^2 = r$ as a result ($r = 0$, or $r = 1$)
As a consequence

$$\rho(x) = c_1 + c_2 e^{\frac{1}{\alpha}x^{\alpha}}$$

We utilize condition (i) to obtain

$$\begin{cases} c_1 + c_2 = 1 \\ c_2 = 1 \end{cases} \quad \text{Thus,} \quad c_1 = 0 \quad \text{So} \quad \rho(x) = e^{\frac{1}{\alpha}x^{\alpha}}$$

Situation o3:

$$\rho^{\alpha} = \rho$$

Applying the result of [12], the associated characteristic equation is $r = 1$:

$$\rho(x) = c e^{\frac{1}{\alpha}x^{\alpha}}$$

So, according to condition (i), then $c = 1$, Thus

$$\rho(x) = e^{\frac{1}{\alpha}x^{\alpha}}$$

Thus $\rho(x) = e^{\frac{1}{\alpha}x^{\alpha}}$ satisfies all situations of the first case. Now, we substute $\rho(x)$ in equation (3.5)

$$e^{\frac{x^{\alpha}}{\alpha}} \otimes H_1 + e^{\frac{x^{\alpha}}{\alpha}} \otimes H_2 = e^{\frac{x^{\alpha}}{\alpha}} \otimes H_3.$$

This gives us

$$H_1 + H_2 = H_3.$$

That is

$$\varphi^{\alpha} \otimes \psi^{\alpha} + \varphi^{2\alpha} \otimes \psi = \varphi \otimes -\psi^{2\alpha}. \tag{3.6}$$

From Theorem 2.5, we have two possibilities:

$$\begin{cases} (1) \quad \varphi^{\alpha} = \varphi^{2\alpha} = \varphi \\ \qquad\quad \text{or} \\ (2) \quad \psi = \psi^{\alpha} = -\psi^{2\alpha} \end{cases}$$

Possibility 1

$$\varphi^{\alpha} = \varphi^{2\alpha} = \varphi$$

In this possibility, we have the following situations:

(1) $\varphi^{2\alpha} = \varphi^{\alpha}$
(2) $\varphi^{2\alpha} = \varphi$
(3) $\varphi^{\alpha} = \varphi$

Situation 01:

$$\varphi^{2\alpha} = \varphi^{\alpha}$$

Based on the findings in [12], the corresponding characteristic equation is as follows $r^2 = r$ so $r = 0$, or $r = 1$ as a result,

$$\varphi(y) = c_1 + c_2 e^{\frac{1}{\alpha}y^{\alpha}}$$

Applying the conditions outlined in (ii), we obtain:

$$\varphi(y) = e^{\frac{1}{\alpha}y^{\alpha}}$$

Situation 02:

$$\varphi^{2\alpha} = \varphi.$$

The characteristic equation is $r^2 = 1$. According to condition (ii), we have

$$\varphi(y) = e^{\frac{1}{\alpha}y^\alpha}.$$

Situation 03:

$$\varphi^{\alpha} = \varphi.$$

Continuing with the method used previously, we arrive at:

$$\varphi(y) = e^{\frac{1}{\alpha}y^\alpha}$$

Since situation 1, situation 2, and situation 3 gave us the same function $\varphi(y)$, then we substitute $\varphi(y)$ into equation (3.6):

$$e^{\frac{1}{\alpha}y^\alpha} \otimes \psi^\alpha + e^{\frac{1}{\alpha}y^\alpha} \otimes \psi = e^{\frac{1}{\alpha}y^\alpha} \otimes -\psi^{2\alpha}.$$

This gives us

$$\psi + \psi^\alpha = -\psi^{2\alpha} \tag{3.7}$$

According to the conclusions outlined in the referenced work [12] , the characteristic equation for equation (3.7) is given by $1 + r + r^2 = 0$. Hence $r_1 = \frac{-1+i\sqrt{3}}{2}$ and $r_1 = \frac{-1-i\sqrt{3}}{2}$ so $\psi(z) = e^{\frac{-z^\alpha}{2\alpha}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2\alpha}z^\alpha\right) + c_2 \sin\left(\frac{\sqrt{3}}{2\alpha}z^\alpha\right)\right]$

Using to condition (iii), we obtain:

$$\psi(z) = e^{\frac{-z^\alpha}{2\alpha}}\left[\cos\left(\frac{\sqrt{3}}{2\alpha}z^\alpha\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2\alpha}z^\alpha\right)\right]$$

The atomic solution for this possibility is:

$$\begin{aligned} u(x,y,z) &= e^{\frac{1}{\alpha}x^\alpha} \otimes e^{\frac{1}{\alpha}y^\alpha} \otimes \left[e^{\frac{-z^\alpha}{2\alpha}}\left[\cos\left(\frac{\sqrt{3}}{2\alpha}z^\alpha\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2\alpha}z^\alpha\right)\right]\right] \\ &= e^{\frac{1}{\alpha}x^\alpha}.e^{\frac{1}{\alpha}y^\alpha}.e^{\frac{-z^\alpha}{2\alpha}}\left[\cos\left(\frac{\sqrt{3}}{2\alpha}z^\alpha\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2\alpha}z^\alpha\right)\right] \end{aligned}$$

As shown in Figure 1, the atomic solution for $z = -2$ is illustrated, demonstrating the behavior of the system under this specific condition.

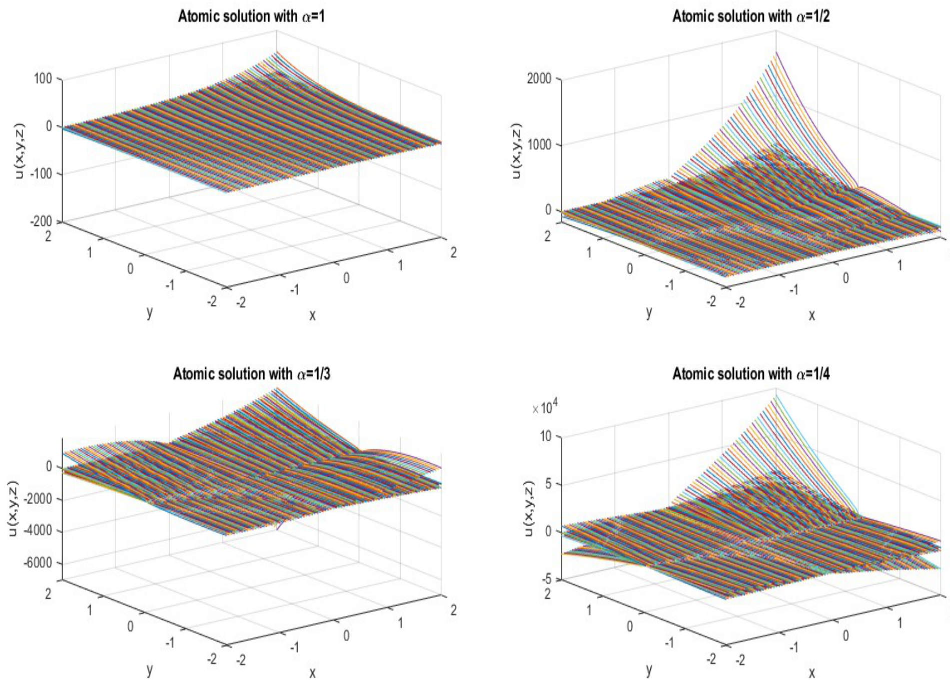

FIGURE 1. Atomic solution for z=-2 based on the proposed analytical method

Next, we study possibility 2. **Possibility 2**

$$\psi = \psi^{\alpha} = -\psi^{2\alpha} \tag{3.8}$$

According to equation (3.8) we obtain the following situations:

(1) $\psi = \psi^{\alpha}$
(2) $\psi = -\psi^{2\alpha}$
(3) $\psi^{\alpha} = -\psi^{2\alpha}$

Situation 01:

$$\psi^{\alpha} = \psi$$

Continuing from our previous method and taking into consideration condition (iii), we come to the following result:

$$\psi(z) = e^{\frac{1}{\alpha} z^{\alpha}}$$

Situation 02:

$$\psi = -\psi^{2\alpha}$$

Applying the method used earlier and considering condition (iii), we reach:

$$\psi(z) = \cos\left(\frac{1}{\alpha} z^{\alpha}\right) + \sin\left(\frac{1}{\alpha} z^{\alpha}\right)$$

Situation 03:

$$\psi^{\alpha} = -\psi^{2\alpha}$$

Applying the previously used method and considering condition (iii), we get:

$$\psi(z) = e^{\frac{-1}{\alpha} z^{\alpha}}$$

Since the expression ψ differs in situations 1, 2, and 3, then we deduce there is no atomic solution corresponding to possibility 2.

3.2. **Case 2.**

$$H_1 = H_2 = H_3.$$

This implies that

$$\varphi \otimes \psi = \varphi^{\alpha} \otimes \psi^{\alpha} = \left[-\varphi^{2\alpha} \otimes \psi - \varphi \otimes \psi^{2\alpha} + \varphi \otimes \psi\right]$$

The next situations for the second case are explained as follows .

(1) $\varphi \otimes \psi = \varphi^{\alpha} \otimes \psi^{\alpha}$
(2) $\varphi \otimes \psi = [-\varphi^{2\alpha} \otimes \psi - \varphi \otimes \psi^{2\alpha} + \varphi \otimes \psi]$
(3) $\varphi^{\alpha} \otimes \psi^{\alpha} = [-\varphi^{2\alpha} \otimes \psi - \varphi \otimes \psi^{2\alpha} + \varphi \otimes \psi]$

The situations can be expressed as follows

(1) $\varphi \otimes \psi = \varphi^{\alpha} \otimes \psi^{\alpha}$
(2) $\varphi^{2\alpha} \otimes \psi = \varphi \otimes -\psi^{2\alpha}$
(3) $\varphi^{\alpha} \otimes \psi^{\alpha} + \varphi^{2\alpha} \otimes \psi = \varphi \otimes [-\psi^{2\alpha} + \psi]$

Situation 1:

$$\varphi^{\alpha} \otimes \psi^{\alpha} = \varphi \otimes \psi$$

Applying Lemma 2.6 we have:

$$\begin{cases} \varphi^{\alpha} = \varphi \\ \text{and} \\ \psi^{\alpha} = \psi \end{cases}$$

In accordance with the initial conditions denoted as (ii) and (iii), we observe that $\varphi(y) = e^{\frac{1}{a} y^{\alpha}}$, and $\psi(z) = e^{\frac{1}{a} z^{\alpha}}$.

Situation 2:

$$\varphi^{2\alpha} \otimes \psi = \varphi \otimes (-\psi^{2\alpha})\cdot$$

According to Lemma 2.6, we have:

$$\begin{cases} \varphi^{2\alpha} = \varphi \\ \text{and} \\ -\psi^{2\alpha} = \psi \end{cases}$$

By applying the same method used previously and considering the initial conditions (ii) and (iii), we have $\psi(z) = \cos\left(\frac{1}{\alpha} z^{\alpha}\right) + \sin\left(\frac{1}{\alpha} z^{\alpha}\right)$ and $\varphi(y) = e^{\frac{1}{\alpha} y^{\alpha}}$. Comparing the expression for ψ in situation 1 with the expression for ψ in situation 2, it becomes evident that the second case cannot admit an atomic solution.

4. Conclusion

In this paper, we introduced an innovative analytical framework for solving linear fractional partial differential equations (PDEs) in three independent variables by constructing atomic solutions within the setting of Banach space tensor product theory. By employing conformable fractional derivatives, we formulated a generalized version of the given PDE and derived explicit solution structures using rank-one tensor decompositions. The use of atomic operators enabled the separation of variables in a rigorous functional-analytic context, resulting in a structured classification of cases and solution possibilities.

Our methodology not only provides exact solutions but also highlights the advantages of combining fractional calculus with operator-theoretic techniques. The detailed analysis demonstrates that this framework can successfully handle higher-dimensional and non-standard fractional systems, offering clear mathematical insight and computational tractability.

Future work will aim to extend this approach to nonlinear fractional PDEs, systems with boundary conditions, and equations involving different types of fractional derivatives such as Caputo or Riemann–Liouville. Moreover, numerical validation and stability analysis of these atomic solutions under various perturbations could offer practical relevance in modeling real-world phenomena governed by complex fractional dynamics.

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