

WEAKLY S -2-ABSORBING IDEALS IN NONCOMMUTATIVE RINGS

ALAA ABOUHALAKA^{1*} and NICO GROENEWALD²

ABSTRACT. This paper introduces and explores the notion of weakly S -2-absorbing ideals within the framework of noncommutative rings. Extending the concepts of weakly 2-absorbing ideals and weakly S -prime ideals, an ideal K of a ring R , which is disjoint from an m -system S , is defined as a (weakly) S -2-absorbing ideal if, for all $r_1, r_2, r_3 \in R$ with $r_1 R r_2 R r_3 \subseteq K$ (or $0 \neq r_1 R r_2 R r_3 \subseteq K$), there exists $s \in S$ such that $r_1 r_2 \langle s \rangle \subseteq K$, $r_1 r_3 \langle s \rangle \subseteq K$, or $r_2 r_3 \langle s \rangle \subseteq K$. We investigate key properties of weakly S -2-absorbing ideals, highlighting their differences from related structures such as weakly 2-absorbing ideals. Additionally, we analyze weakly S -2-absorbing ideals in various ring constructions.

1. INTRODUCTION

The concept of weakly 2-absorbing ideals in a commutative ring R , introduced in [6], serves as a generalization of 2-absorbing ideals, which were initially studied in [5]. Both notions have been extensively explored and generalized in various directions; see, for example, [7, 8]. Consider an R -module M and a multiplication closed subset S of R . The notion of S -2-absorbing submodules was originally presented in [13]. A submodule K of M is called S -2-absorbing if it satisfies the following condition: $(K :_R M) \cap S = \emptyset$, and there exists a fixed element $s \in S$ such that for all $a, b \in R$ and $m \in M$, the implication

$$abm \in K \Rightarrow sab \in (K :_R M) \quad \text{or} \quad sam \in K \quad \text{or} \quad sbm \in K,$$

holds. In particular, an S -2-absorbing ideal of a ring R can be viewed as a special case of an S -2-absorbing submodule when $M = R$. The class of commutative rings in which every ideal is S -2-absorbing was investigated in [10].

In the setting of noncommutative rings, the notion of weakly 2-absorbing ideals was first extended in [11]. As defined in [11], an ideal K of a noncommutative ring R is said to be a weakly 2-absorbing ideal if, for all $r_1, r_2, r_3 \in R$ satisfying $0 \neq r_1 R r_2 R r_3 \subseteq K$, at least one of the following conditions holds:

$$r_1 r_2 \in K, \quad r_1 r_3 \in K, \quad \text{or} \quad r_2 r_3 \in K.$$

Date: Received: Apr 2, 2025; Accepted: Apr 28, 2025.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary 16W99; Secondary 16N60.

Key words and phrases. Prime ideals; S -prime ideals; weakly S -prime ideals; S -2-absorbing ideals; weakly S -2-absorbing ideals.

Similarly, the notion of S -prime ideals was extended to noncommutative rings in [1] by considering S as an m -system, leading to fruitful results. In [1], an ideal K of R , disjoint from an m -system S , is defined as a right S -prime ideal if, for every pair of ideals I_1, I_2 of R satisfying $I_1 I_2 \subseteq K$, there exists some $s \in S$ such that

$$I_1 \langle s \rangle \subseteq K \quad \text{or} \quad I_2 \langle s \rangle \subseteq K.$$

This concept has been further generalized in several works [9, 2, 3].

In this paper, we introduce the concept of weakly S -2-absorbing ideals in noncommutative rings. Specifically, an ideal K of a ring R , disjoint from an m -system S , is called a weakly S -2-absorbing ideal if, for all $r_1, r_2, r_3 \in R$ satisfying

$$0 \neq r_1 R r_2 R r_3 \subseteq K,$$

there exists $s \in S$ such that

$$r_1 r_2 \langle s \rangle \subseteq K, \quad r_1 r_3 \langle s \rangle \subseteq K, \quad \text{or} \quad r_2 r_3 \langle s \rangle \subseteq K.$$

In Section 2, we investigate several fundamental properties of weakly S -2-absorbing ideals, analyzing their structural characteristics and exploring various equivalent formulations and applications. Several examples of weakly S -2-absorbing ideals which are not S -2-absorbing ideals are given in Example 2.3. Proposition 2.6 states that if the set of zero divisors does not intersect the m -system S , then the cube of any weakly S -2-absorbing ideal that is not S -2-absorbing must be zero (i.e., it must be contained in the prime radical $\text{Nil}(R)$).

We also investigate the behavior of weakly S -2-absorbing ideals in related rings, such as homomorphic images, trivial ring extensions, and product rings. The connection between (weakly) S -prime ideals and weakly S -2-absorbing ideals is established in Proposition 4.3. According to Proposition 4.3, in $R = R_1 \times R_2$, where R_1 and R_2 are rings with identities, let $S = S_1 \times S_2$, where S_1 and S_2 are m -systems of R_1 and R_2 , respectively. If I is a nonzero proper ideal of R_1 , then $I \times \{0\}$ is a weakly S -2-absorbing ideal of R with $I \cap S_1 = \emptyset$ and $\{0\} \cap S_2 = \emptyset$ if and only if I is a weakly S_1 -prime ideal of R_1 and $\{0\}$ is an S_2 -prime ideal of R_2 .

Weakly S -2-absorbing ideals naturally extend the framework of 2-absorbing ideals by incorporating an m -system S , allowing for a more flexible study of ideal-theoretic behavior. Beyond being a generalization, these ideals can play a meaningful role in understanding and analyzing rings in which every ideal is weakly S -2-absorbing, similar to how rings with all prime, weakly prime, or S -2-absorbing ideals have been studied.

Throughout the paper, although if not explicitly stated, R is understood to denote a noncommutative ring, and S an m -system of R .

2. PROPERTIES OF WEAKLY S -2-ABSORBING IDEALS

Weakly S -2-absorbing ideals are defined as follows:

Definition 2.1. A proper ideal I of R disjoint from S is called a weakly S -2-absorbing ideal if whenever $r_1, r_2, r_3 \in R$ satisfying $0 \neq r_1 R r_2 R r_3 \subseteq K$, there exists $s \in S$ such that $r_1 r_2 \langle s \rangle \subseteq K$, $r_1 r_3 \langle s \rangle \subseteq K$, or $r_2 r_3 \langle s \rangle \subseteq K$.

Clearly, every weakly 2-absorbing ideal, disjoint from S , is also a weakly S -2-absorbing ideal, and the two concepts coincide when $S \subseteq U(R)$. In addition, an ideal K of R that is disjoint from S is said to be a weakly right S -prime ideal if whenever A and B are ideals of R with $0 \neq AB \subseteq K$, implies either $A\langle s \rangle \subseteq K$ or $B\langle s \rangle \subseteq K$. In the next corollary we show that every weakly right S -prime ideal is a weakly S -2-absorbing ideal.

Corollary 2.2. *Let I and J be ideals of a ring R , and S an m -system of R .*

(1) *If I is a weakly S -2-absorbing ideal, and J meets S , then IJ is a weakly S -2-absorbing ideal of R .*

(2) *If I is a weakly S -2-absorbing ideal, and J meets S , then $I \cap J$ is a weakly S -2-absorbing ideal of R .*

(3) *If I is a weakly right S -prime ideal of R , then I is a weakly S -2-absorbing ideal of R .*

Proof. (1) Let $s \in S \cap J$. Assume $0 \neq aRbRc \subseteq IJ \subseteq I$ for some nonunit elements $a, b, c \in R$. Since I is weakly S -2-absorbing, there exists $t \in S$ such that $ab\langle t \rangle \subseteq I$, $ac\langle t \rangle \subseteq I$, or $bc\langle t \rangle \subseteq I$. For some $r \in R$, we have $s' = srt \in S$. Consequently, $ab\langle s' \rangle \subseteq ab\langle t \rangle \subseteq IJ$, $ac\langle s' \rangle \subseteq ac\langle t \rangle \subseteq IJ$, or $bc\langle s' \rangle \subseteq bc\langle t \rangle \subseteq IJ$. Therefore, IJ is a weakly S -2-absorbing ideal of R .

(2) Similar to (1).

(3) Suppose that I is a weakly right S -prime ideal of R . Let $a, b, c \in R$ with $0 \neq aRbRc \subseteq I$. Now $0 \neq \langle a \rangle \langle b \rangle \langle c \rangle \subseteq I$. Hence, either $\langle a \rangle \langle s \rangle \subseteq I$ or $\langle b \rangle \langle s \rangle \subseteq I$ or $\langle c \rangle \langle s \rangle \subseteq I$. Consequently, either $ab\langle s \rangle \subseteq a\langle s \rangle \subseteq \langle a \rangle \langle s \rangle \subseteq I$ or $ac\langle s \rangle \subseteq a\langle s \rangle \subseteq \langle a \rangle \langle s \rangle \subseteq I$ or $bc\langle s \rangle \subseteq b\langle s \rangle \subseteq \langle b \rangle \langle s \rangle \subseteq I$. Hence I is a weakly S -2-absorbing ideal. $\square$

Example 2.3. (1) The zero ideal consistently satisfies the properties of a weakly S -2-absorbing ideal, which may not be S -2-absorbing.

(2) Let $R = M_2(\mathbb{Z}[X])$, and $K = M_2(\langle 2X^2 \rangle)$. Let $S = \{s, s^2, s^4, s^8, \dots\}$, where $s = 4(e_{11} + e_{22})$. Then S is an m -system of R , and $K \cap S = \emptyset$. Notice that, K is not a weakly 2-absorbing ideal of R , since

$$0 \neq 2(e_{11} + e_{22})RM_2(X)RM_2(X) \subseteq K.$$

However,

$$2(e_{11} + e_{22})M_2(X) \not\subseteq K, \text{ and } M_2(X)M_2(X) \not\subseteq K.$$

On the other hand, if $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in R$, with $0 \neq \mathcal{M}_1R\mathcal{M}_2R\mathcal{M}_3 \subseteq K$, then either $\mathcal{M}_1\mathcal{M}_2 = X^2\mathcal{A}_1$ or $\mathcal{M}_1\mathcal{M}_3 = X^2\mathcal{B}_1$ or $\mathcal{M}_2\mathcal{M}_3 = X^2\mathcal{C}_1$ for some matrices $\mathcal{A}_1$, $\mathcal{B}_1$, and $\mathcal{C}_1$ of R , and hence either $\mathcal{M}_1\mathcal{M}_2\langle s \rangle \subseteq K$ or $\mathcal{M}_1\mathcal{M}_3\langle s \rangle \subseteq K$ or $\mathcal{M}_2\mathcal{M}_3\langle s \rangle \subseteq K$. It implies that K is weakly S -2-absorbing.

(3) Let $R = M_2(\mathbb{Z}_{210})$. For $s = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, let $S = \{2^k, k \in \mathbb{N} \cup \{0\}\}$. Then S is an m -system of R . The ideal $K = \{0\}$ is a weakly S -2-absorbing ideal. However,

it is not S -2-absorbing, because $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} R \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} R \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \subseteq K$, but,

$$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \langle s \rangle \not\subseteq K, \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \langle s \rangle \not\subseteq K, \text{ and} \\ \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \langle s \rangle \not\subseteq K.$$

(4) Let $R = M_2(\mathbb{Z}_{24})$, and let S be the m -system: $S = \{s^k, k \in \mathbb{N} \cup \{0\}\}$, for $s = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. For $\mathcal{M} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, take the ideal $K = \langle \mathcal{M} \rangle$. Since

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} R \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} R \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \subseteq K, \text{ but } \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \notin K,$$

then K is not a weakly 2-absorbing ideal. However, it is clear that K is a weakly S -2-absorbing ideal.

Definition 2.4. Let K be an ideal of a ring R such that $K \cap S = \emptyset$. We say that a triple $(x, y, z) \in R^3$ is called an S -triple-zero of K if the product $xRyRz = 0$, and for every $s \in S$, none of the sets $xy\langle s \rangle$, $yz\langle s \rangle$, or $xz\langle s \rangle$ are fully contained in K ; that is,

$$xy\langle s \rangle \not\subseteq K, \quad yz\langle s \rangle \not\subseteq K, \quad \text{and} \quad xz\langle s \rangle \not\subseteq K.$$

It is worth observing that if an ideal K of R is weakly S -2-absorbing but fails to be S -2-absorbing, then there exist elements $x, y, z \in R$ such that the triple (x, y, z) forms an S -triple-zero of K .

Proposition 2.5. Let K be a weakly S -2-absorbing ideal of a ring R , and suppose that (a, b, c) is an S -triple-zero of K for some $a, b, c \in R$. Then:

(1) $aRbK\langle s_1 \rangle = KbRc\langle s_1 \rangle = aRcK\langle s_1 \rangle = 0$ for some $s_1 \in S$.

(2) $aK^2\langle s_1 \rangle = KbK\langle s_1 \rangle = K^2c\langle s_1 \rangle = 0$ for some $s_1 \in S$.

Proof. (1) Suppose $ar_1bx\langle s_1 \rangle \neq \{0\}$ for some $x \in K$ and $r_1 \in R$. Then $0 \neq ar_1br_2(c+x)\langle s_1 \rangle = ar_1br_2c\langle s_1 \rangle + ar_1br_2x\langle s_1 \rangle \subseteq K$ for $r_2 \in R$. Hence $0 \neq aRbR(c+x)s_1 \subseteq K$. Since (a, b, c) is an S -triple-zero of K , we have $ab\langle s \rangle \not\subseteq K$ for all $s \in S$, it follows that either $a(c+x)s_1\langle s \rangle \subseteq K$ or $b(c+x)s_1\langle s \rangle \subseteq K$. This is not possible since $ac\langle s \rangle \not\subseteq K$ and $bc\langle s \rangle \not\subseteq K$ for all $s \in S$. Thus $aRbRK\langle s_1 \rangle = \{0\}$ for some $s_1 \in S$.

Similarly we can show that $aRcK\langle s_1 \rangle = 0$ for some $s_1 \in S$.

Suppose $KbRc\langle s_1 \rangle \neq \{0\}$. Hence $ibr_1c\langle s \rangle \neq \{0\}$ for some $i \in K$ and $r_1 \in R$. So we have $(a+i)RbRc\langle s \rangle \neq \{0\}$. Thus $(a+i)RbRct \neq \{0\}$ for some $t \in \langle s \rangle$. Since $bct\langle s \rangle \not\subseteq K$, we have $(a+i)b\langle s_2 \rangle \subseteq K$ or $(a+i)ct\langle s_2 \rangle \subseteq K$ for some $s_2 \in S$ since K is weakly S -2-absorbing. Hence $ab\langle s_2 \rangle \subseteq K$ or $act\langle s_2 \rangle \subseteq K$ which is not possible. Hence $KbRc\langle s_1 \rangle = \{0\}$ for some $s_1 \in S$.

(2) Suppose $aK^2\langle s_1 \rangle \neq \{0\}$. There exist $i, j \in K$ such that $aij\langle s_1 \rangle \neq \{0\}$. Hence $a(b+i)(c+j)\langle s_1 \rangle = abc\langle s_1 \rangle + abj\langle s_1 \rangle + aic\langle s_1 \rangle + aij\langle s_1 \rangle \neq \{0\}$. From (1), $a(b+i)(c+j)\langle s_1 \rangle = aij\langle s_1 \rangle \neq \{0\}$. Hence $aijt \neq 0$ for $t \in \langle s_1 \rangle$. We now have $\{0\} \neq (a+i)R(b+j)Rct \subseteq K$. Since K is weakly S -2-absorbing, $(a+i)(b+j)\langle s \rangle \subseteq K$ or $(a+i)ct\langle s \rangle \subseteq K$ or $(b+j)ct\langle s \rangle \subseteq K$. Hence $ab\langle s \rangle \subseteq K$

or $act\langle s \rangle \subseteq K$ or $bct\langle s \rangle \subseteq K$. This is a contradiction since (a, b, c) is an S -triple-zero of K . Hence $aK^2\langle s_1 \rangle \neq \{0\}$. Similarly we can show that $K^2c\langle s_1 \rangle = 0$ for some $s_1 \in S$.

Suppose $KbK\langle s_1 \rangle \neq \{0\}$. There exist $i, j \in K$ such that $ibj\langle s_1 \rangle \neq 0$. Now $(a+i)b(c+j)\langle s_1 \rangle = abc\langle s_1 \rangle + abj\langle s_1 \rangle + ibc\langle s_1 \rangle + ibj\langle s_1 \rangle = ibj\langle s_1 \rangle \neq \{0\}$. We now have $\{0\} \neq (a+i)RbR(c+j)t$ for some $t \in \langle s_1 \rangle$. Since K is weakly S -2-absorbing $(a+i)b\langle s \rangle \subseteq K$ or $(a+i)(c+j)t\langle s \rangle \subseteq K$ or $b(c+j)t\langle s \rangle \subseteq K$. Consequently $ab\langle s \rangle \subseteq K$ or $act\langle s \rangle \subseteq K$ or $bct\langle s \rangle \subseteq K$ which is not possible since (a, b, c) is an S -triple-zero of K . Thus $KbK\langle s_1 \rangle = \{0\}$ for some $s_1 \in S$. $\square$

Proposition 2.6. *Let K be a weakly S -2-absorbing ideal of R that is not an S -2-absorbing ideal. If $Z(R) \cap S = \emptyset$, then $K^3 = \{0\}$.*

Proof. Since K is not an S -2-absorbing ideal of R , there exists an S -triple-zero (a, b, c) in K for some $a, b, c \in R$. Suppose $xyz \neq 0$ for some $x, y, z \in K$. By Proposition 2.5, we have

$$(a+x)(b+y)(c+z)\langle s_1 \rangle = xyz\langle s_1 \rangle \subseteq K.$$

Since $Z(R) \cap S = \emptyset$, it follows that $(a+x)R(b+y)R(c+z)s_2 \neq \{0\}$ for some $s_2 \in \langle s_1 \rangle$. Hence, for some $s \in S$, at least one of the following holds:

$$(a+x)(b+y)\langle s \rangle \subseteq K, \quad (a+x)(c+z)s_2\langle s \rangle \subseteq K, \quad \text{or} \quad (b+y)(c+z)s_2\langle s \rangle \subseteq K.$$

Hence $ab\langle s \rangle \subseteq K$ or $acs_1\langle s \rangle \subseteq K$ or $bcs_1\langle s \rangle \subseteq K$. Let $s' \in s_1\langle s \rangle \subseteq S$. Then $ab\langle s' \rangle \subseteq K$ or $ac\langle s' \rangle \subseteq K$ or $bc\langle s' \rangle \subseteq K$. This is not possible since (a, b, c) is an S -triple-zero in K . Hence $K^3 = 0$. $\square$

Corollary 2.7. [11, Corollary 4.18] *Let I be a weakly 2-absorbing ideal of R that is not a 2-absorbing ideal. Then $I^3 = 0$.*

Proof. The proof follows by Proposition 2.6 by taking $S = \{1\}$. $\square$

Corollary 2.8. *Let I be a weakly S -2-absorbing ideal of a ring R . If I is not S -2-absorbing, then $I \subseteq \text{Nil}(R)$.*

Proof. This is clear since $I^3 = \{0\}$ and $\text{Nil}(R)$ is a semiprime ideal. $\square$

2.1. Homomorphic image. Let S be an m -system in a ring A . According to Lemma 2.1 in [1], if $\phi : A \rightarrow B$ is a surjective ring homomorphism, then the image $\phi(S)$ forms an m -system in B .

Theorem 2.9. *Let A and B be rings, and suppose $\phi : A \rightarrow B$ is an injective ring homomorphism with $\phi(1_A) = 1_B$. Assume that K is a weakly $\phi(S)$ -2-absorbing ideal in B , and that $\phi(a)$ is a non-invertible element in B for each nonunit $a \in A$. Then the preimage $\phi^{-1}(K)$ is a weakly S -2-absorbing ideal in A .*

Proof. Let $x, y, z \in A$ be non-invertible elements such that $0 \neq xAyAz \subseteq \phi^{-1}(K)$. Since ϕ is a homomorphism, we obtain:

$$\phi(xAyAz) = \phi(x)B\phi(y)B\phi(z) \subseteq K,$$

and because ϕ is injective, the image is nonzero. Given that K is weakly $\phi(S)$ -2-absorbing in B , there exists $s \in S$ such that:

$$\phi(x)\phi(y)\langle\phi(s)\rangle \subseteq K \quad \text{or} \quad \phi(x)\phi(z)\langle\phi(s)\rangle \subseteq K \quad \text{or} \quad \phi(y)\phi(z)\langle\phi(s)\rangle \subseteq K.$$

Applying the properties of ϕ , we find that:

$$xy\langle s \rangle \subseteq \phi^{-1}(K) \quad \text{or} \quad xz\langle s \rangle \subseteq \phi^{-1}(K) \quad \text{or} \quad yz\langle s \rangle \subseteq \phi^{-1}(K).$$

Moreover, since $\phi(S) \cap K = \emptyset$, it follows that $S \cap \phi^{-1}(K) = \emptyset$. Therefore, $\phi^{-1}(K)$ satisfies the definition of a weakly S -2-absorbing ideal in A . $\square$

Theorem 2.10. *Let A and B be rings, with S an m -system in A . Suppose $\phi : A \rightarrow B$ is a surjective homomorphism such that $\phi(1_A) = 1_B$. If K is a weakly S -2-absorbing ideal of A and $\ker(\phi) \subseteq T$, then $\phi(K)$ is a weakly $\phi(S)$ -2-absorbing ideal of B .*

Proof. Since $\phi(S) \cap \phi(K) = \emptyset$, we can proceed. Let $x, y, z \in B$ be nonunits such that $0 \neq xByBz \subseteq \phi(K)$. Then, there are elements $a, b, c \in A$ with $\phi(a) = x$, $\phi(b) = y$, and $\phi(c) = z$, which implies:

$$\phi(aAbAc) = xByBz \subseteq \phi(K).$$

Given that $\ker(\phi) \subseteq K$, it follows that $0 \neq aAbAc \subseteq K$. Since K is weakly S -2-absorbing, we have:

$$ab\langle s \rangle \subseteq K \quad \text{or} \quad ac\langle s \rangle \subseteq K \quad \text{or} \quad bc\langle s \rangle \subseteq K$$

for some $s \in S$. Applying ϕ , we obtain:

$$xy\langle \phi(s) \rangle \subseteq \phi(K) \quad \text{or} \quad xz\langle \phi(s) \rangle \subseteq \phi(K) \quad \text{or} \quad yz\langle \phi(s) \rangle \subseteq \phi(K),$$

demonstrating that $\phi(K)$ is indeed weakly $\phi(S)$ -2-absorbing in B . $\square$

Theorem 2.11. *Let R be a ring and S an m -system within R . Suppose $J \subsetneq Q$ are ideals in R with $Q \cap S = \emptyset$. If Q is a weakly S -2-absorbing ideal in R , then the quotient Q/J inherits the weakly S -2-absorbing property in the factor ring R/J . Conversely, if J is weakly S -2-absorbing and Q/J is also weakly S -2-absorbing in R/J , then Q must be weakly S -2-absorbing in R .*

Proof. Assume that Q is a weakly S -2-absorbing ideal in R . Let $\alpha, \beta, \gamma \in R$ be non-invertible elements such that their images $\alpha + J, \beta + J, \gamma + J$ in R/J satisfy:

$$0 \neq (\alpha + J)(R/J)(\beta + J)(R/J)(\gamma + J) \subseteq Q/J.$$

This implies $0 \neq \alpha R \beta R \gamma \subseteq Q$. Since Q is weakly S -2-absorbing, there exists an element $t \in S$ such that one of the following holds:

$$\alpha\beta\langle t \rangle \subseteq Q, \quad \alpha\gamma\langle t \rangle \subseteq Q, \quad \text{or} \quad \beta\gamma\langle t \rangle \subseteq Q.$$

Passing to the quotient, we conclude:

$$(\alpha + J)(\beta + J)\langle \bar{t} \rangle \subseteq Q/J, \quad (\alpha + J)(\gamma + J)\langle \bar{t} \rangle \subseteq Q/J, \quad \text{or} \quad (\beta + J)(\gamma + J)\langle \bar{t} \rangle \subseteq Q/J,$$

which shows that Q/J satisfies the definition in R/J .

Conversely, suppose both J and Q/J are weakly S -2-absorbing ideals, and consider elements $x, y, z \in R$ with $0 \neq xRyRz \subseteq Q$. If $xRyRz \subseteq J$, then the weakly S -2-absorbing property of J implies there exists $t \in S$ such that at least one of the following holds:

$$xy\langle t \rangle \subseteq J \subseteq Q, \quad xz\langle t \rangle \subseteq J \subseteq Q, \quad yz\langle t \rangle \subseteq J \subseteq Q.$$

If $xRyRz \not\subseteq J$, then $0 \neq (x+J)(R/J)(y+J)(R/J)(z+J) \subseteq Q/J$. Using the assumption on Q/J , there exists $\bar{t} = t + J \in S + J$ such that:

$$(x+J)(y+J)\langle\bar{t}\rangle \subseteq Q/J, \quad (x+J)(z+J)\langle\bar{t}\rangle \subseteq Q/J, \quad \text{or} \quad (y+J)(z+J)\langle\bar{t}\rangle \subseteq Q/J.$$

Translating back to R , this gives:

$$xy\langle t \rangle \subseteq Q, \quad xz\langle t \rangle \subseteq Q, \quad \text{or} \quad yz\langle t \rangle \subseteq Q.$$

Hence, Q is weakly S -2-absorbing in R . $\square$

3. IDEALIZATION

We now explore the construction of weakly S -2-absorbing ideals through the process of idealization. Let M be an R - R -bimodule. Understanding the ideal structure of $R \boxplus M$ is crucial for this discussion. For a more detailed study on the idealization of a ring, we refer the reader to [4, 14].

According to [14], the prime ideals of $R \boxplus M$ have the specific form $K \boxplus M$, where K is a prime ideal of R . Furthermore, if S is an m -system in R , then the set $S \boxplus M = \{(t, k) \mid t \in S, k \in M\}$ forms an m -system in $R \boxplus M$.

It is also important to observe that the condition $(S \boxplus M) \cap (K \boxplus M) = \emptyset$ holds if and only if $S \cap K = \emptyset$.

We aim to characterize when an ideal of the form $K \boxplus M$ in the trivial ring extension $R \boxplus M$ becomes a weakly $S \boxplus M$ -2-absorbing ideal. The following theorem provides a necessary and sufficient condition for this property, connecting it to the structure of K in R and its interaction with the module M .

Theorem 3.1. *Let R be a ring with a proper ideal K and let M be an R - R -bimodule. Suppose S is an m -system of R . Then $K \boxplus M$ is a weakly $S \boxplus M$ -2-absorbing ideal of the trivial ring extension $R \boxplus M$ if and only if K is a weakly S -2-absorbing ideal of R , and for any S -triple zero (a, b, c) of K , we have $aRbRM = MRbRc = aMc = \{0\}$.*

Proof. Assume first that there exist elements $a, b, c \in R$ such that $\{0\} \neq aRbRc \subseteq K$. We consider the elements $(a, 0), (b, 0), (c, 0) \in R \boxplus M$. Then,

$$\{(0, 0)\} \neq (a, 0)R \boxplus M(b, 0)R \boxplus M(c, 0) \subseteq K \boxplus M.$$

Since $K \boxplus M$ is assumed to be a weakly $S \boxplus M$ -2-absorbing ideal, there exists $(s, m) \in S \boxplus M$ such that at least one of the following conditions is satisfied:

$$\begin{aligned} (a, 0)(b, 0)\langle(s, m)\rangle &= (ab, 0)(RsR, RsM + RmR + MsR) \subseteq (ab\langle s \rangle, M) \subseteq K \boxplus M, \\ (a, 0)(c, 0)\langle(s, m)\rangle &= (ac, 0)(RsR, RsM + RmR + MsR) \subseteq (ac\langle s \rangle, M) \subseteq K \boxplus M, \\ (b, 0)(c, 0)\langle(s, m)\rangle &= (bc, 0)(RsR, RsM + RmR + MsR) \subseteq (bc\langle s \rangle, M) \subseteq K \boxplus M. \end{aligned}$$

In each case, the containment in $K \boxplus M$ implies that $ab\langle s \rangle \subseteq K$, $ac\langle s \rangle \subseteq K$, or $bc\langle s \rangle \subseteq K$. Hence, K is a weakly S -2-absorbing ideal of R .

Now suppose (a, b, c) is an S -triple zero of K . We claim that $aRbRM = MRbRc = aMc = 0$.

Assume, for contradiction, that $aRbRM \neq 0$. Then there exist $m \in M$ and $r_1, r_2 \in R$ such that $ar_1br_2m \neq 0$. Consider the following element:

$$(ar_1br_2c, ar_1br_2m) = (a, 0)(r_1, 0)(b, 0)(r_2, 0)(c, m) \in (a, 0)R \boxplus M(b, 0)R \boxplus M(c, m).$$

Since $aRbRc \subseteq K$ and $ar_1br_2m \neq 0$, it follows that:

$$(0, 0) \neq (ar_1br_2c, ar_1br_2m) \in K \boxplus M.$$

But this contradicts the assumption that $K \boxplus M$ is weakly $S \boxplus M$ -2-absorbing, given that the triple (a, b, c) is an S -triple zero of K . Therefore, $aRbRM = 0$. Similar arguments yield $MRbRc = 0$ and $aMc = 0$.

Conversely, assume that K is a weakly S -2-absorbing ideal of R and that for any S -triple zero (a, b, c) of K , we have $aRbRM = MRbRc = aMc = 0$. Let $(a, n), (b, m), (c, t) \in R \boxplus M$ satisfy:

$$\{(0, 0)\} \neq (a, n)R \boxplus M(b, m)R \boxplus M(c, t) \subseteq K \boxplus M.$$

This implies that $aRbRc \subseteq K$. We now consider two cases.

Case 1: $aRbRc \neq 0$. Since K is weakly S -2-absorbing, it follows that one of $ab\langle s \rangle$, $ac\langle s \rangle$, or $bc\langle s \rangle$ is contained in K . Then, the corresponding product in $R \boxplus M$ satisfies:

$$\begin{aligned} & (a, n)(b, m)\langle (s, t) \rangle \subseteq K \boxplus M, \\ \text{or} \quad & (a, n)(c, t)\langle (s, t) \rangle \subseteq K \boxplus M, \\ \text{or} \quad & (b, m)(c, t)\langle (s, t) \rangle \subseteq K \boxplus M. \end{aligned}$$

Hence, $K \boxplus M$ is weakly $S \boxplus M$ -2-absorbing.

Case 2: $aRbRc = 0$. Suppose, for contradiction, that none of $ab\langle s \rangle$, $ac\langle s \rangle$, or $bc\langle s \rangle$ are contained in K . Then (a, b, c) is an S -triple zero of K . By assumption, this forces $aRbRM = MRbRc = aMc = 0$, implying that the entire product

$$(a, n)R \boxplus M(b, m)R \boxplus M(c, t) \subseteq \{(0, 0)\},$$

contradicting the assumption that it is nonzero. Therefore, $K \boxplus M$ must be weakly $S \boxplus M$ -2-absorbing. $\square$

Corollary 3.2. *Let R be a semi-prime ring with identity which is not a prime ring, and let M be an R - R -bimodule. Then the unique weakly $S \boxplus M$ -2-absorbing ideal of the trivial ring extension $R \boxplus M$ that is not an $S \boxplus M$ -2-absorbing ideal and has the form $I \boxplus M$, where I is an ideal of R , is the ideal $\{0\} \boxplus M = 0 \boxplus M$.*

Proof. Let I be an ideal of R , and let $J = I \boxplus M$ such that J is a weakly $S \boxplus M$ -2-absorbing ideal of $R \boxplus M$, but not an $S \boxplus M$ -2-absorbing ideal. Then I is a weakly S -2-absorbing ideal of R which is not an S -2-absorbing ideal of R .

From Proposition 2.6, it follows that $I \subseteq \text{Nil}(R) = 0$. Hence, $J = 0 \boxplus M$.

The zero ideal $\{0\}$ is weakly S -2-absorbing in R . Let (a, b, c) be an S -triple zero of $\{0\}$, so that

$$aRbRc = 0, \quad \text{with} \quad ab\langle s \rangle \neq 0, \quad ac\langle s \rangle \neq 0, \quad \text{and} \quad bc\langle s \rangle \neq 0.$$

We claim that

$$aRbRM = MRbRc = aMc = 0.$$

Suppose, to the contrary, that $aRbRM \neq 0$. Then there exist $r_1, r_2 \in R$ and $n \in M$ such that

$$ar_1br_2n \neq 0.$$

Now consider:

$$(0, ar_1br_2n) = (a, 0)(r_1, 0)(b, 0)(r_2, 0)(c, n) \in (a, 0)(R \boxplus M)(b, 0)(R \boxplus M)(c, n).$$

Since $(a, 0)R \boxplus M(b, 0)R \boxplus M(c, n) = \{0\} \boxplus M = J$, and J is weakly $S \boxplus M$ -2-absorbing, we must have:

$$\begin{aligned} (a, 0)(b, 0)\langle(s, m)\rangle &\subseteq J, & \text{or} \\ (a, 0)(c, n)\langle(s, m)\rangle &\subseteq J, & \text{or} \\ (b, 0)(c, n)\langle(s, m)\rangle &\subseteq J. \end{aligned}$$

However, none of these conditions hold, which contradicts the assumption that J is weakly $S \boxplus M$ -2-absorbing. By Theorem 3.1, the assumptions $MRbRc \neq 0$ or $aMc \neq 0$ lead to a contradiction. Therefore, the claim is validated.

Conversely, according to Theorem 3.1 and the fact that $J^3 = 0$, it follows that J cannot be an $S \boxplus M$ -2-absorbing ideal of $R \boxplus M$. This completes the proof. $\square$

Corollary 3.3. *Let (R, M) be a local ring, S be an m -system of R and E be an R - R -bimodule such that $ME = EM = 0$. Then $I \boxplus E$ is a weakly $S \boxplus E$ -2-absorbing ideal of $R \boxplus E$ if and only if I is a weakly S -2-absorbing ideal of R .*

Proof. Let $a, b, c \in R$ with $aRbRc = 0$, but $ab\langle s \rangle \not\subseteq I$ and $ac\langle s \rangle \not\subseteq I$ and $bc\langle s \rangle \not\subseteq I$ for each $s \in S$. Suppose that $aRbR \not\subseteq M$ or $RbRc \not\subseteq M$ or $aMc \not\subseteq M$. Suppose $aRbR \not\subseteq M$ and $r_1, r_2 \in R$ such that $ar_1br_2 \notin M$. Then there exists $d \in R$ such that $ar_1br_2d = 1$ and hence $c = ar_1br_2dc \in aRbRc = \{0\}$. Hence $c = 0$. Since $ac\langle s \rangle \not\subseteq I$, there exists $t \in \langle s \rangle$ such that $act \notin I$. If now $c = 0$, we get $0 \notin I$, a contradiction. If $RbRc \not\subseteq M$, $l_1, l_2 \in R$ such that $l_1bl_2c \notin M$. So there is a $p \in R$ such that $l_1bl_2cp = 1$. Hence $a = al_1bl_2cp \in aRbRcp = \{0\}$, again this gives a contradiction. If $aMc \not\subseteq M$, there exists $l \in R$ such that $alc \notin M$ and we have a $q \in R$ such that $alcq = 1$ and $bc = alcq1b1c \in aRbRc = \{0\}$. Hence $0 = bc\langle s \rangle \not\subseteq I$ a contradiction. Therefore $aRbRE = ERbRc = aEc = \{0\}$. Apply Theorem 3.1 to complete the proof. $\square$

Remark 3.4. Consider a local ring $(R, \mathfrak{m})$ with maximal ideal $\mathfrak{m}$ satisfying $\mathfrak{m}^2 = 0$, and let S be an m -system in R . Observe that any proper ideal of R that is disjoint from S must be weakly S -prime, and consequently, weakly S -2-absorbing.

To see this, let K be a proper ideal of R such that $K \cap S = \emptyset$. Given elements $s \in S$ and $a, b \in R$ with $0 \neq aRb \subseteq K$, we analyze the implications. Since $\mathfrak{m}^2 = 0$, at least one of a or b cannot belong to $\mathfrak{m}$. If both were in $\mathfrak{m}$, then the containment $0 \neq aRb \subseteq \mathfrak{m}^2 = 0$ would be a contradiction.

Without loss of generality, assume $a \notin \mathfrak{m}$. Then, we can express b as $b = a^{-1}ab$, which ensures $b \in aRb \subseteq K$. Consequently, we conclude that $b\langle s \rangle$ is also contained in K , as required.

We now demonstrate how Corollary 3.3 can be utilized to generate more examples of weakly $S \boxplus E$ -2-absorbing ideals of the type $I \boxplus E$ within the trivial ring extension $R \boxplus E$.

Example 3.5. Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m}^2 = 0$, and E be an $R - R$ -bimodule such that $\mathfrak{m}E = E\mathfrak{m} = 0$. Let K be an ideal of R disjoint with S . Then $K \boxplus E$ is weakly $S \boxplus E$ -2-absorbing.

Proof. Follows immediately from (3) of Corollary 2.2, Remark 3.4 and Corollary 3.3. $\square$

4. PRODUCT OF RINGS

It is well known that if the rings R_1, R_2 are with identities, then any ideal of $R_1 \times R_2$ has the form $A \times B$, where A and B are ideals of R_1 and R_2 , respectively. Moreover, if S_1 is an m -system of R_1 and S_2 is an m -system of R_2 , then, $S_1 \times S_2$ is an m -system of the ring $R_1 \times R_2$, see Lemma 2.1 of [1]. It is easy to check that for an ideal A of R_1 , $(A \times R_2) \cap (S_1 \times S_2) = \emptyset$, if and only if $A \cap S_1 = \emptyset$. In this section, we assume that $Z(R) \cap S = \emptyset$.

The following two theorems demonstrate the nuanced behavior of weakly S -2-absorbing ideals across different ring constructions. In Theorem 3.1, set in the framework of a trivial ring extension $R \boxplus M$, it suffices for the base ideal K to be weakly S -2-absorbing in R , along with certain annihilation conditions involving the module M , to ensure that $K \boxplus M$ inherits the weakly $S \boxplus M$ -2-absorbing property. In contrast, Theorem 4.1 highlights that in the case of a product ring $R = A \times B$, the weakly S -2-absorbing nature of an ideal of the form $K \times B$ is equivalent to K being an S_1 -2-absorbing ideal in A .

Theorem 4.1. *Let $R = A \times B$, where A and B are rings with identities, and let $S = S_1 \times S_2$, where S_1 and S_2 are m -systems of A and B , respectively. The following statements are equivalent:*

- (1) $K \times B$ is a weakly S -2-absorbing ideal of R .
- (2) $K \times B$ is an S -2-absorbing ideal of R .
- (3) K is an S_1 -2-absorbing ideal of A .

Proof. (1) $\Rightarrow$ (2): Suppose $K \times B$ is a weakly S -2-absorbing ideal of R with $(K \times B) \cap (S_1 \times S_2) = \emptyset$. If $K \times B$ is not S -2-absorbing, then by Proposition 2.6, $K \times B \subseteq \text{Nil}(R) = \text{Nil}(A) \times \text{Nil}(B)$, which is a contradiction.

(2) $\Rightarrow$ (3): Suppose $K \times B$ is an S -2-absorbing ideal of R . Then $(K \times B) \cap S = \emptyset$, which implies $K \cap S_1 = \emptyset$. Let $a_1 A a_2 A a_3 \subseteq K$ for some $a_1, a_2, a_3 \in A$. Then, for any $b_1, b_2, b_3 \in B$, we have

$$(a_1, b_1)R(a_2, b_2)R(a_3, b_3) \subseteq K \times B.$$

Thus, either

$$(a_1, b_1)(a_2, b_2) \langle (s_1, s_2) \rangle = (a_1, b_1)(a_2, b_2) \langle \langle s_1 \rangle, \langle s_2 \rangle \rangle \subseteq K \times B,$$

or

$$(a_1, b_1)(a_3, b_3) \langle (s_1, s_2) \rangle = (a_1, b_1)(a_3, b_3) \langle \langle s_1 \rangle, \langle s_2 \rangle \rangle \subseteq K \times B,$$

or

$$(a_2, b_2)(a_3, b_3) \langle (s_1, s_2) \rangle = (a_2, b_2)(a_3, b_3) \langle \langle s_1 \rangle, \langle s_2 \rangle \rangle \subseteq K \times B.$$

Consequently, either $a_1 a_2 \langle s_1 \rangle \subseteq K$ or $a_1 a_3 \langle s_1 \rangle \subseteq K$ or $a_2 a_3 \langle s_1 \rangle \subseteq K$. Hence, K is an S_1 -2-absorbing ideal of A .

(3) $\Rightarrow$ (1): Suppose K is an S_1 -2-absorbing ideal of A . Then $K \cap S_1 = \emptyset$, which implies $(K \times B) \cap S = \emptyset$. Now let $(a_1, b_1)R(a_2, b_2)R(a_3, b_3) \subseteq K \times B$ for some $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$. Then $a_1 A a_2 A a_3 \subseteq K$, and hence either

$$a_1 a_2 \langle s_1 \rangle \subseteq K, \quad \text{or} \quad a_1 a_3 \langle s_1 \rangle \subseteq K, \quad \text{or} \quad a_2 a_3 \langle s_1 \rangle \subseteq K$$

for some $s_1 \in S_1$. Consequently,

$$(a_1, b_1)(a_2, b_2) \langle (s_1, s_2) \rangle = (a_1 a_2 \langle s_1 \rangle, b_1 b_2 \langle s_2 \rangle) \subseteq K \times B,$$

or similar for the other combinations. Hence, $K \times B$ is an S -2-absorbing ideal of R , and therefore a weakly S -2-absorbing ideal. $\square$

Theorem 4.2. *Let $R = A \times B$, where A and B are rings with identity elements. Let $S = S_1 \times S_2$, where S_1 and S_2 are m -systems of A and B , respectively. Suppose K is a nonzero ideal of A , and J is a nonzero ideal of B . Then, following statements are equivalent:*

- (1) $K \times J$ is a weakly 2-absorbing ideal of R , with $K \cap S_1 = \emptyset$ and $J \cap S_2 = \emptyset$.
- (2) Either $J = B$ and K is an S_1 -2-absorbing ideal of A , or J is an S_2 -prime ideal of B and K is an S_1 -prime ideal of A .
- (3) $K \times J$ is an S -2-absorbing ideal of R , with $K \cap S_1 = \emptyset$ and $J \cap S_2 = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Assume $K \times J$ is a weakly 2-absorbing ideal of R , where $K \cap S_1 = \emptyset$ and $J \cap S_2 = \emptyset$. This implies $(K \times J) \cap S = \emptyset$, so $K \times J$ is also a weakly S -2-absorbing ideal of R .

First, consider the case $J = B$. Then, by Theorem 4.1, it follows that K must be an S_1 -2-absorbing ideal of A .

Next, suppose $J \neq B$. We aim to show that under this assumption, J is an S_2 -prime ideal of B , and simultaneously, K is an S_1 -prime ideal of A .

Let $a, b \in B$ satisfy $aBb \subseteq J$, and let $0 \neq k \in K$. Then,

$$(k, 1)R(1, a)R(1, b) \subseteq K \times BaBb \subseteq (K \times J) \setminus \{(0, 0)\}.$$

Since

$$(1, a)(1, b) \langle (s_1, s_2) \rangle = (1, a)(1, b) \langle \langle s_1 \rangle, \langle s_2 \rangle \rangle = (\langle s_1 \rangle, ab \langle s_2 \rangle) \not\subseteq K \times J$$

due to $K \cap S_1 = \emptyset$, we conclude that either

$$(k, 1)(1, a) \langle \langle s_1 \rangle, \langle s_2 \rangle \rangle = (k \langle s_1 \rangle, a \langle s_2 \rangle) \subseteq K \times J,$$

or

$$(k, 1)(1, b) \langle \langle s_1 \rangle, \langle s_2 \rangle \rangle = (k \langle s_1 \rangle, b \langle s_2 \rangle) \subseteq K \times J.$$

Thus, either $a \langle s_2 \rangle \subseteq J$ or $b \langle s_2 \rangle \subseteq J$, implying J is an S_2 -prime ideal of B .

A similar argument works for K : take $c, d \in A$ such that $cAd \subseteq K$, and $0 \neq j \in J$. Then

$$(c, 1)R(d, 1)R(1, j) \subseteq cAd \times J \subseteq (K \times J) \setminus \{(0, 0)\}.$$

Now,

$$(c, 1)(d, 1) \langle \langle s_1 \rangle, \langle s_2 \rangle \rangle = (cd \langle s_1 \rangle, \langle s_2 \rangle) \not\subseteq K \times J$$

since $J \cap S_2 = \emptyset$. Therefore, either

$$(c, 1)(1, j) \langle \langle s_1 \rangle, \langle s_2 \rangle \rangle = (c \langle s_1 \rangle, j \langle s_2 \rangle) \subseteq K \times J$$

or

$$(d, 1)(1, j)(\langle s_1 \rangle, \langle s_2 \rangle) = (d\langle s_1 \rangle, j\langle s_2 \rangle) \subseteq K \times J.$$

Hence, either $c\langle s_1 \rangle \subseteq K$ or $d\langle s_1 \rangle \subseteq K$, showing that K is an S_1 -prime ideal.

(2) $\Rightarrow$ (3): If $J = B$ and K is an S_1 -2-absorbing ideal of A , then by Theorem 4.1, $K \times B$ is an S -2-absorbing ideal of R .

Alternatively, if both K and J are S_1 - and S_2 -prime ideals respectively, and $K \cap S_1 = \emptyset$, $J \cap S_2 = \emptyset$, then $K \times J \cap S = \emptyset$. For any $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in R$ such that

$$(a_1, b_1)R(a_2, b_2)R(a_3, b_3) \subseteq K \times J,$$

we deduce that at least one among the a_i 's lies in K and one among the b_j 's lies in J . So for some $(s_1, s_2) \in S$, we have

$$(a_1, b_1)(a_2, b_2)(\langle s_1 \rangle, \langle s_2 \rangle) \subseteq K \times J.$$

Therefore, $K \times J$ is an S -2-absorbing ideal.

(3) $\Rightarrow$ (1): This follows immediately from the definitions, as every S -2-absorbing ideal is, in particular, weakly S -2-absorbing. $\square$

Proposition 4.3. *Let $R = A \times B$, where A and B are rings with identities, and let $S = S_1 \times S_2$, where S_1 and S_2 are m -systems of A and B , respectively. Let K be a nonzero proper ideal of A . Then $K \times \{0\}$ is a weakly S -2-absorbing ideal of $R = A \times B$ with $K \cap S_1 = \emptyset$ and $\{0\} \cap S_2 = \emptyset$ if and only if K is a weakly S_1 -prime ideal of A and $\{0\}$ is an S_2 -prime ideal of B .*

Proof. Assume that $K \times \{0\}$ is a weakly S -2-absorbing ideal in the direct product ring $A \times B$, where $S = S_1 \times S_2$ is an m -system of $A \times B$, such that $K \cap S_1 = \emptyset$ and $\{0\} \cap S_2 = \emptyset$. Consequently, $(K \times \{0\}) \cap S = \emptyset$.

We first establish that K is a weakly S_1 -prime ideal in A . Take any $x, y \in A$ with $0 \neq xAy \subseteq K$. Consider the elements $(x, 1), (y, 1), (1, 0) \in A \times B$. Then,

$$(x, 1)(y, 1)(1, 0) = (xy, 0) \in K \times \{0\},$$

and since $xy \neq 0$, we get $(xy, 0) \neq (0, 0)$. As $K \times \{0\}$ is weakly S -2-absorbing, there exist elements $s_1 \in S_1$, $s_2 \in S_2$ such that one of the following inclusions holds:

$$(x, 1)(y, 1)\langle (s_1, s_2) \rangle \subseteq K \times \{0\}, \quad (x, 1)(1, 0)\langle (s_1, s_2) \rangle \subseteq K \times \{0\}, \quad \text{or} \quad (y, 1)(1, 0)\langle (s_1, s_2) \rangle \subseteq K \times \{0\}.$$

Now, $(x, 1)(y, 1)\langle (s_1, s_2) \rangle = (xy\langle s_1 \rangle, \langle s_2 \rangle)$, and since $\langle s_2 \rangle \neq 0$ (as $0 \notin S_2$), we conclude that this product is not in $K \times \{0\}$. Thus, either:

$$(x\langle s_1 \rangle, 0) \subseteq K \times \{0\} \quad \text{or} \quad (y\langle s_1 \rangle, 0) \subseteq K \times \{0\},$$

which implies $x\langle s_1 \rangle \subseteq K$ or $y\langle s_1 \rangle \subseteq K$. Therefore, K is weakly S_1 -prime in A .

Next, we show that $\{0\}$ is S_2 -prime in B . Let $u, v \in B$ with $uBv = \{0\}$ and choose $a \in K$ such that $a \neq 0$. Then:

$$(a, uv) = (a, 1)(1, u)(1, v) \in K \times \{0\}.$$

Since $uv = 0$, we have $(a, 0) \in K \times \{0\}$ and $(a, uv) \neq (0, 0)$. Again, because $K \cap S_1 = \emptyset$, the product:

$$(1, u)(1, v)\langle (s_1, s_2) \rangle = (\langle s_1 \rangle, uv\langle s_2 \rangle) \not\subseteq K \times \{0\}.$$

Thus, applying the weakly S -2-absorbing property to $(a, 1)(1, u)(1, v)$ implies:

$$(a\langle s_1 \rangle, u\langle s_2 \rangle) \subseteq K \times \{0\}, \quad \text{or} \quad (a\langle s_1 \rangle, v\langle s_2 \rangle) \subseteq K \times \{0\},$$

so either $u\langle s_2 \rangle = 0$ or $v\langle s_2 \rangle = 0$. Hence, $\{0\}$ is S_2 -prime in B .

Conversely, suppose K is weakly S_1 -prime in A , and $\{0\}$ is S_2 -prime in B , with $K \cap S_1 = \emptyset$ and $\{0\} \cap S_2 = \emptyset$. Then $(K \times \{0\}) \cap S = \emptyset$. We now prove that $K \times \{0\}$ is weakly S -2-absorbing in $A \times B$.

Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in A \times B$ be such that:

$$(0, 0) \neq (a_1, b_1)(a_2, b_2)(a_3, b_3) = (a_1 a_2 a_3, b_1 b_2 b_3) \in K \times \{0\}.$$

This means $a_1 a_2 a_3 \in K$ and $b_1 b_2 b_3 = 0$. Since K is weakly S_1 -prime, at least one of a_1, a_2 , or a_3 (say a_1) satisfies $a_1\langle s_1 \rangle \subseteq K$ for some $s_1 \in S_1$. Similarly, as $\{0\}$ is S_2 -prime and $b_1 b_2 b_3 = 0$, at least one of b_1, b_2 , or b_3 (say b_2) satisfies $b_2\langle s_2 \rangle = 0$ for some $s_2 \in S_2$.

Therefore,

$$(a_1, b_1)(a_2, b_2)\langle (s_1, s_2) \rangle = (a_1 a_2 \langle s_1 \rangle, b_1 b_2 \langle s_2 \rangle) \subseteq K \times \{0\}.$$

Thus, $K \times \{0\}$ satisfies the condition of being a weakly S -2-absorbing ideal in $A \times B$. $\square$

Proposition 4.4. *Let $R = A \times B$, where A and B are rings with identities, and let $S = S_1 \times S_2$, where S_1 and S_2 are m -systems of A and B , respectively. Let K_1 be a nonzero proper ideal of A and K_2 a proper ideal of B . Then $K_1 \times K_2$ is a weakly S -2-absorbing ideal of $A \times B$ that is not an S -2-absorbing ideal, with $K_1 \cap S_1 = \emptyset$ and $K_2 \cap S_2 = \emptyset$, if and only if $K_2 = \{0\}$ is an S_2 -prime ideal of B and K_1 is a weakly S_1 -prime ideal of A that is not an S_1 -prime ideal.*

Proof. Assume that $K_1 \times K_2$ is a weakly S -2-absorbing ideal of $A \times B$ but not an S -2-absorbing ideal, with $K_1 \cap S_1 = \emptyset$ and $K_2 \cap S_2 = \emptyset$. Suppose, for contradiction, that $K_2 \neq \{0\}$. Then by Theorem 4.2, $K_1 \times K_2$ must be an S -2-absorbing ideal of $A \times B$, a contradiction. Hence, $K_2 = \{0\}$.

According to Proposition 4.3, $K_2 = \{0\}$ is an S_2 -prime ideal of B and K_1 is a weakly S_1 -prime ideal of A . If K_1 were an S_1 -prime ideal, then again by Theorem 4.2, $K_1 \times K_2$ would be an S -2-absorbing ideal, contradicting the assumption. Thus, K_1 is not an S_1 -prime ideal.

Conversely, suppose K_1 is a weakly S_1 -prime ideal of A that is not S_1 -prime, and $K_2 = \{0\}$ is an S_2 -prime ideal of B . Then clearly $K_1 \cap S_1 = \emptyset$ and $K_2 \cap S_2 = \emptyset$. By Proposition 4.3, $K_1 \times \{0\}$ is a weakly S -2-absorbing ideal of $A \times B$.

We now demonstrate that $K_1 \times \{0\}$ is not an S -2-absorbing ideal. Since K_1 is not S_1 -prime, there exist $x, y \in A$ such that $xAy = \{0\} \subseteq K_1$, but for all $s \in S_1$, neither $x\langle s \rangle \subseteq K_1$ nor $y\langle s \rangle \subseteq K_1$. Consider:

$$(x, 1)(A \times B)(y, 1)(A \times B)(1, 0) = (xAyA, 0) \subseteq K_1 \times \{0\}.$$

However, for any $t \in S_2$ and all $s \in S_1$:

$$\begin{aligned}(x, 1)(y, 1)\langle(s, t)\rangle &= (xy\langle s\rangle, \langle t\rangle) \not\subseteq K_1 \times \{0\}, \\(x, 1)(1, 0)\langle(s, t)\rangle &= (x\langle s\rangle, 0) \not\subseteq K_1 \times \{0\}, \\(y, 1)(1, 0)\langle(s, t)\rangle &= (y\langle s\rangle, 0) \not\subseteq K_1 \times \{0\}.\end{aligned}$$

Thus, $K_1 \times \{0\}$ is not an S -2-absorbing ideal of $A \times B$. $\square$

CONCLUSION

In summary, we have introduced and explored the notion of weakly S -2-absorbing ideals in the context of noncommutative rings. This new class of ideals generalizes the concept of weakly 2-absorbing ideals by using an m -system. We have examined their core properties, presented illustrative examples distinguishing them from stronger counterparts, and analyzed their behavior under various ring constructions.

Acknowledgement. The authors thank the anonymous referees for their helpful comments and suggestions that improved the paper.

REFERENCES

1. Abouhalaka, A. (2024). S -prime ideals, S -Noetherian noncommutative rings, and the S -Cohen's theorem. *Mediterranean Journal of Mathematics*, 21, 43. <https://doi.org/10.1007/s00009-023-02578-w>
2. Abouhalaka, A., Çay, H., & Ersoy, B. A. (2024). $S\mathcal{J}$ -ideals: A study in commutative and noncommutative rings. *Journal of Mathematics*, Article ID 1707271, 10 pages. <https://doi.org/10.1155/2024/1707271>
3. Abouhalaka, A., Çay, H., & Groenewald, N. (2024). On weakly $S\rho$ -ideals in noncommutative rings. *Gulf Journal of Mathematics*, 18(2), 1–15. <https://doi.org/10.56947/gjom.v18i2.2221>
4. Anderson, D. D., & Winders, M. (2009). Idealization of a module. *Journal of Commutative Algebra*, 1(1), 3–56. <https://doi.org/10.1216/JCA-2009-1-1-3>
5. Badawi, A. (2007). On 2-absorbing ideals of commutative rings. *Bulletin of the Australian Mathematical Society*, 75(3), 417–429. <https://doi.org/10.1017/S0004972700039344>
6. Badawi, A., & Darani, A. Y. (2013). On weakly 2-absorbing ideals of commutative rings. *Houston Journal of Mathematics*, 39(2), 441–452.
7. Badawi, A., Tekir, U., & Yetkin, E. (2014). On 2-absorbing primary ideals in commutative rings. *Bulletin of the Korean Mathematical Society*, 51(4), 1163–1173. <https://doi.org/10.4134/BKMS.2014.51.4.1163>
8. Badawi, A., & Fahid, B. (2020). On weakly 2-absorbing δ -primary ideals of commutative rings. *Georgian Mathematical Journal*, 27(4), 503–516. <https://doi.org/10.1515/gmj-2018-0070>
9. Çay, H., Abouhalaka, A., & Ersoy, B. A. (2024). Generalization of ρ -ideals associated with an m -system and a special radical class. *Ricerche di Matematica*. <https://doi.org/10.1007/s11587-024-00912-2>
10. El Khair, I., Kim, H., & Mahdou, N. (2025). Commutative rings in which every ideal is S -2-absorbing. *Rendiconti del Circolo Matematico di Palermo, Serie II*, 74, 16. <https://doi.org/10.1007/s12215-024-01159-4>
11. Groenewald, N. (2021). On weakly 2-absorbing ideals of noncommutative rings. *African Mathematics*, 32, 1669–1683. <https://doi.org/10.1007/s13370-021-00927-0>

12. Mahdou, N., Moutui, M., & Zahir, Y. (2022). On S -weakly prime ideals of commutative rings. *Georgian Mathematical Journal*, 29(3), 397–405. <https://doi.org/10.1515/gmj-2022-2141>
13. Ulucak, G., Tekir, Ü., & Koç, S. (2020). On S -2-absorbing submodules and vn-regular modules. *Analele Științifice ale Universității "Ovidius" Constanța. Seria Matematică*, 28(2), 239–257. <https://doi.org/10.2478/auom-2020-0030>
14. Veldsman, S. (2008). A note on the radicals of idealizations. *Southeast Asian Bulletin of Mathematics*, 32, 545–551.

¹ DEPARTMENT OF MATHEMATICS, ÇUKUROVA UNIVERSITY, ADANA, TURKEY.
Email address: alaa1aclids@gmail.com

² DEPARTMENT OF STATISTICS, NELSON MANDELA UNIVERSITY, GQEBERHA, SOUTH AFRICA
Email address: nico.groenewald@mandela.ac.za