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LOWER BOUNDS ON THE EIGENVALUE RATIO WITH ROBIN BOUNDARY CONDITIONS

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Abstract. We study the problem of minimizing the ratio of the first eigenvalues of vibrating string equations subject to the Robin boundary conditions for the class of concave weights. We show that, unlike the Dirichlet, Neumann and mixed boundary conditions, the constant weight is not minimizing for the class of concave weights. In addition, we prove a relation between the eigenvalues and real roots of the first Airy functions and their derivatives.

1. Introduction

The eigenvalues and eigenfunctions draw motivation from mathematical physics and offer an interesting insight into their different applications that range far and wide, such as quantum mechanics, acoustics, and structural engineering. A number of authors have given due regard to the behavior of vibrating strings on account of their practical implication in several fields. The present paper builds upon the foundational work of Huang [18] and the subsequent expansion by Horváth [15], focusing on the problem of vibrating strings under Robin boundary conditions.

At the heart of the current work research is the vibrating string problem, a classical model in physics and engineering that describes the vibration of string fixed at two ends. The mathematical representation of this problem involves differential equations subject to specific boundary conditions. The Robin boundary conditions, a focus of this study, are a mix of Dirichlet and Neumann conditions that allow for a more generalized and versatile model of physical phenomena. The above conditions are particularly relevant in scenarios where the behavior of a system is influenced by both the values of a function and its derivative at the boundary, such as heat transfer problems and electromagnetic fields.

We consider here the low-lying eigenvalues λ of a self-adjoint vibrating string problem on a finite interval, scaled without loss of generality to have length π,

$$Hu := -\frac{d^2u}{dx^2} = \lambda w(x)u, \ x \in [0, \pi], \tag{1.1}$$

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where w is a positive concave function and represents the mass density.
The subject of a lower optimal bound for the eigenvalues ratios $\frac{\lambda_2}{\lambda_1}$ of vibrating string equations has gained considerable attention during the last decades cf.[5, 7, 10, 14, 16, 18, 15, 17, 19] and references therein. For the problem (1.1) with Dirichlet boundary conditions Huang [18] proved that for the class of symmetric single-well density w the eigenvalues ratios satisfy $\frac{\lambda_2}{\lambda_1} \leq 4$ and for the class of concave or symmetric single-barrier density w the eigenvalues ratios satisfy $\frac{\lambda_2}{\lambda_1} \geq 4$ and equality holds if and only if w is constant. In [15], Horváth expanded on Huang's results by exploring single-barrier functions without relying on symmetry assumptions, thus broadening the scope of the research. Recently, Gu and Sun [13] investigated the lower bound for the ratio of the first two eigenvalues for vibrating string with a single-barrier density and transition point at 0 with a mixed boundary condition. These developments have laid the groundwork for a deeper understanding of the dynamics of vibrating strings and the mathematical structures governing them.
Throughout the present paper, we consider the following problem subject to Robin boundary conditions.

$$\begin{cases} -u'' = \lambda w u & x \in (0,\pi) \\ u'(0) = -\alpha u(0), \\ u'(\pi) = \beta u(\pi). \end{cases} \tag{1.2}$$

α and β are two constants, there have a physical reason to writing with opposite signs (see [22]). It is well known that the spectrum of vibrating string problem with Robin boundary conditions consists of infinitely many simple eigenvalues and is bounded from below [24]. If the eigenvalues are arranged such that

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_n \to \infty,$$

then the n^{th} eigenfunction u_n corresponding to the eigenvalue λ_n has exactly $n-1$ zeros in $(0,\pi)$.

The eigenvalues of the differential operator governing the vibrating string problem are directly related to the frequencies of vibration. Understanding the distribution and ratios of these eigenvalues is crucial for predicting the behavior of the system under various conditions. The problem of determining lower bounds on the eigenvalue ratio is particularly significant as it provides fundamental insights into the stability and resonance properties of vibrating systems. In the context of Robin boundary conditions, this problem becomes even more complex and intriguing due to the mixed nature of the conditions, offering a rich field for mathematical exploration.

The significance of this topic is that it might go a long way in aiding toward more insight into vibrating systems, which happen to be very basic in most technological and scientific applications. Such insights from the study of the eigenvalues and eigenfunctions of vibrating strings can be extremely powerful in leading to very important new developments that span the gamut from musical instrument design to advanced materials and structures. In this paper, the analysis is extended to Robin boundary conditions, further filling one of the lacunae in the existing literature and thus presenting new perspectives on a classical problem.

One of the innovative aspects of this paper is the derivation of simple properties of eigenvalue ratio under Robin boundary conditions, a topic that has not been extensively explored in previous studies. Additionally, the establishment of a link between the eigenvalues of the problem and the real roots of Airy functions and their derivatives represents a novel approach to understanding the underlying mathematical structures. This connection not only enriches the theoretical framework but also opens up new avenues for research, particularly in the study of eigenvalue ratios in the context of vibrating strings problem.

This paper is arranged as follows: in section §2, we derive some simple properties of eigenvalue ratio. In section §3, we will study the optimal concave weight for vibrating string problem subject to Robin boundary conditions. In section §4, we prove a relation between the eigenvalues of problem (1.2) and the real roots of Airy functions and their derivatives Ai′ and Bi′, where Ai and Bi Airy functions on the bounded solution of the ordinary differential equation $u''(x) = xu(x)$.

2. Preliminaries and Basics

In this section, we will give some basics and preliminaries of problem (1.2) with Robin boundary conditions.

Definition 2.1. The eigenvalue ratio which is denoted by $\Lambda[w]$, is defined by the first two eigenvalues

$$\Lambda[w] = \frac{\lambda_2[w]}{\lambda_1[w]}.$$

It is clear that the eigenvalue ratio depends on the weight w and the length of the underlying interval.

Lemma 2.2. *The eigenvalue ratio of vibrating strings problem subject to Robin boundary conditions on an interval* $[0, \pi]$ *is not affected by multiplying the density by a constant, that is*

$$\Lambda[w] = \Lambda[cw]$$

for any constant c *different of zero.*

Proof. If $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$ denote the first two eigenvalues of problem (1.2) for w, then the first two eigenvalues $\lambda_1^{(2)}, \lambda_2^{(2)}$ of problem (1.2) for cw satisfy $\lambda_1^{(2)} = \dfrac{\lambda_1^{(1)}}{c}$ and $\lambda_2^{(2)} = \dfrac{\lambda_2^{(1)}}{c}$, as is easily seen. Hence,

$$\Lambda[w] = \Lambda[cw].$$

$\square$

Now, we give the Feynman-Hellmann formula for the variation of eigenvalues with respect to a family of weight of the problem (1.2).

Lemma 2.3. *Suppose that $w(.,t)$ is a one-parameter family of real-valued, locally L^1 function with $\frac{\partial w}{\partial t}(x,t) \in L^1(0,\pi)$. Then*

$$\frac{d\lambda_n(t)}{dt} = -\lambda_n \int_0^\pi \frac{\partial w}{\partial t}(x,t) u_n^2(x,t) dx.$$

Proof. Is a standard argument based on Kato-type perturbation theory for forms; however, since in the standard references [20] the functions are usually taken analytic in t, for completeness, we provide a sketch of the proof.
Denote $\dot{u} = \frac{du}{dt}$ the function w depends integrably on x and differentiably on t.
We define $R_t : H^1(0,\pi) \setminus \{0\} \to \mathbb{R}$, the Rayleigh quotient by

$$R_t(u) = \frac{\displaystyle\int_0^\pi (u')^2 dx + \alpha u^2(0) + \beta u^2(\pi)}{\displaystyle\int_0^\pi w u^2 dx}$$

Then

$$\frac{d\lambda_n(t)}{dt} = \frac{1}{\displaystyle\int_0^\pi w u_n^2 dx} \left(\int_0^\pi 2 u_n' \dot{u_n'} dx + 2\alpha u_n(0) \dot{u}_n(0) + 2\beta u_n(\pi) \dot{u_n}(\pi) \right)$$

$$- \frac{1}{\left(\displaystyle\int_0^\pi w u_n^2 dx \right)^2} \left(\int_0^\pi (u_n')^2 dx + \alpha u_n^2(0) + \beta u_n^2(\pi) \right) \left(\int_0^\pi 2 w u_n \dot{u}_n dx + \int_0^\pi \dot{w} u_n^2 dx \right).$$

Integrating by parts twice, we obtain

$$\int_0^\pi u_n' \dot{u'}_n dx = \int_0^\pi \lambda_n w u_n \dot{u_n} dx - \alpha u_n(0) \dot{u}_n(0) - \beta u_n(\pi) \dot{u}_n(\pi),$$

and the fact that $\displaystyle\int_0^\pi w u_n^2 dx = 1$, we deduce that

$$\frac{d\lambda_n(t)}{dt} = -\lambda_n(t) \int_0^\pi \frac{\partial w}{\partial t}(x,t) u_n^2(x,t) dx.$$

□

Remark 2.4. We have

$$\frac{d\lambda_n(t)}{dt} = -\lambda_n(t) \int_0^\pi \frac{\partial w}{\partial t}(x,t) u_n^2(x,t) dx.$$

Then

$$\frac{d\Lambda[w]}{dt} = \frac{d}{dt}\left(\frac{\lambda_2(t)}{\lambda_1(t)}\right) = \frac{\dot{\lambda}_2(t)\lambda_1(t) - \dot{\lambda}_1(t)\lambda_2(t)}{\lambda_1(t)^2} = -\frac{\lambda_2(t)}{\lambda_1(t)} \int_0^\pi \frac{\partial w}{\partial t}(x,t) \left(u_2^2(x,t) - u_1^2(x,t) \right) dx.$$

Next Following [3, 6], we have the technical results:

Lemma 2.5. *Consider the problem (1.2) with $w \in L^1(0,\pi)$. If u_1, u_2 are respectively the first and second eigenfunctions then $\frac{u_2}{u_1}$ is decreasing on $(0,\pi)$.*

Proof. u_1 has no zero and is positive on $(0,\pi)$ and u_2 has a unique zero x_0, $0 < x_0 < \pi$. We assume that

$$\begin{cases} u_2(x) > 0 & \text{on} \quad (0, x_0), \\ u_2(x) < 0 & \text{on} \quad (x_0, \pi). \end{cases} \tag{2.1}$$

We only need to show that $\left(\frac{u_2}{u_1}\right)' < 0$ on $(0, x_0)$.
The Wronskian is defined as

$$W(x) = u_1(x)u_2'(x) - u_2(x)u_1'(x).$$

Then

$$\left(\frac{u_2}{u_1}\right)' = \frac{1}{u_1^2(x)} \int_0^x (\lambda_1 - \lambda_2) w(t) u_1(t) u_2(t) dt < 0, \tag{2.2}$$

since $\lambda_1 < \lambda_2$ and $u_1, u_2 > 0$ on $(0, x_0)$.
In the same way, we can prove that $\frac{u_2}{u_1}$ is decreasing on (x_0, π). This ends the proof.

□

Lemma 2.6. *The equation $|u_2(x)| = |u_1(x)|$ has at most two solutions on $(0,\pi)$.*

Proof. Suppose that there exist two points distinct $\zeta_i \in (0,\pi)$ such that

$$u_2(\zeta_i) = u_1(\zeta_i), \ \ i = 1, 2.$$

Define the function

$$z(x) = \frac{u_2(x)}{u_1(x)}.$$

Then $z(\zeta_1) = z(\zeta_2)$, by Rolle's theorem there exists $\beta \in (0, x_0)$ such that $z'(\beta) = 0$, but this contradicts (2.2). A similar argument applies to the interval (x_0, π). This proves Lemma 2.6.

□

Using the above Lemma, we deduce the following result:

Lemma 2.7. *Consider the problem (1.2) with Robin boundary conditions. There exists two points x_- and x_+ satisfying $0 \le x_- < x_+ \le \pi$ and*

$$\begin{cases} u_2^2(x) \geqslant u_1^2(x) & \text{on} \quad (0, x_-) \cup (x_+, \pi), \\ u_1^2(x) > u_2^2(x) & \text{on} \quad (x_-, x_+). \end{cases} \tag{2.3}$$

3. Characterization of optimizers

The purpose of this section is to illustrate that, in general, the constant weight does not minimize the eigenvalue ratio $\Lambda[w]$ among concave weights.

Conjecture 3.1. *We consider the problem (1.2) under Robin boundary conditions, then the constant weight minimizes the eigenvalue ratio $\Lambda[w]$ among concave weights w.*

Remark 3.2. In the cases of Dirichlet boundary conditions ($\alpha = \beta = \infty$) or Neumann boundary conditions ($\alpha = \beta = 0$), the constant weight minimizes the eigenvalue ratio $\Lambda[w]$ in classes of concave or single-barrier weights [7, 10].

In the following we will show that unlike the previous two cases, the constant weight is not minimized for any α and β.

Theorem 3.3. *Consider the problem (1.2) , then for every concave and not affine weight w, there exists an affine weight $w_a = ax + b$ such that*

$$\Lambda[w] \geqslant \Lambda[w_a].$$

Proof. Let w be a concave, not linear weight, and $w_a = ax+b$ be the affine weight such that
$w(x_\pm) = w_a(x_\pm)$.
From Lemma 2.5, there exist $0 \le x_- < x_+ \le \pi$ and

$$\begin{cases} u_2^2(x) > u_1^2(x) & \text{on} \quad (0, x_-) \cup (x_+, \pi), \\ u_1^2(x) > u_2^2(x) & \text{on} \quad (x_-, x_+). \end{cases}$$

By concavity of w

$$\begin{cases} w - w_a \leqslant 0 & \text{on} \quad (0, x_-) \cup (x_+, \pi), \\ w - w_a \geqslant 0 & \text{on} \quad (x_-, x_+). \end{cases}$$

Therefore

$$\int_0^\pi (w - w_a)(u_2^2 - u_1^2) dx < 0.$$

Let

$$L_w(t) = tw + (1-t)w_a, \ t \in (0, 1).$$

Consequently

$$\dot{L_w}(t) = w - w_a.$$

Then

$$\frac{d\Lambda(L_w(t))}{dt} = -\frac{\lambda_2}{\lambda_1} \int_0^\pi (w - w_a)(u_2^2 - u_1^2) dx > 0.$$

Integrating this inequality with respect to t over $(0, 1)$, we find

$$\int_0^1 \frac{d\Lambda(L_w(t))}{dt} = (\Lambda(w) - \Lambda(w_a)) \geqslant 0.$$

Then

$$\Lambda(w) \geqslant \Lambda(w_a).$$

□

We will give a simple corollary of Lemma 2.3 for showing non-optimality of a given weight w within class $\mathcal{C} \subset L^\infty(0, \pi)$.

Corollary 3.4. *Given a weight $w \in \mathcal{C}$, suppose that w_0 is a weight such that $w_0 + tw \in \mathcal{C}$ for each $t \geq 0$, respectively $t \leq 0$, sufficiently small. If there exist normalized eigenfunctions u_1 associated with $\lambda_1(w_0)$ and u_2 associated with $\lambda_2(w_0)$ such that*

$$\int_0^\pi w(x)[u_2^2(x) - u_1^2(x)]dx < 0, \text{ respectively, } > 0, \tag{3.1}$$

then w does not minimize the eigenvalue ratio in $\mathcal{C}$.

Example 3.5. We consider the problem (1.2) subject to Robin boundary conditions.

$$\begin{cases} -u'' = \lambda w u & x \in (0, \pi) \\ u'(0) = -\alpha u(0), \\ u'(\pi) = \beta u(\pi). \end{cases}$$

where $\alpha = 0$ and $\beta = \infty$, we claim that in this case the constant weight $w = 1$ is not optimal. We observe that the L^2-normalized eigenfunctions are $u_1(x) = C\sin(\pi(x + \frac{\pi}{2})/2\pi)$ and $u_2(x) = C\sin(3\pi(x + \frac{\pi}{2})/2\pi)$ where $C^2 = \frac{2}{\pi}$.

Then

$$\int_0^\pi w(x)[u_2^2(x) - u_1^2(x)]dx < 0, \tag{3.2}$$

By Corollary 3.4, we expect that the minimizing weight is non-constant.

The above example suggests that for $\alpha = 0$ and $\beta = \infty$ the minimizing weight is non-constant, more generally for the problem (1.2) with Robin boundary conditions, we do not expect the constant weight to be optimal; indeed, the optimal must be of the form $w(x) = ax + b$, where $a \neq 0$.

4. Estimates on the eigenvalue ratio for linear weight

In this section, we prove a relation between the eigenvalue ratio and the real roots of the first Airy functions Ai and Bi. Throughout, we consider only the case of linear weight $w(x) = ax$ with $a > 0$ and $\alpha = \beta$.

Proposition 4.1. *The eigenvalues of problem (1.2) with Robin boundary conditions correspond to the real roots of the transcendental equation*

$$\begin{aligned} &\left[(\lambda a)^{\frac{1}{3}}\mathrm{Ai}'(0) + \alpha\mathrm{Ai}(0)\right]\left[(\lambda a)^{\frac{1}{3}}\mathrm{Bi}'(-(\lambda a)^{\frac{1}{3}}\pi) - \alpha\mathrm{Bi}(-(\lambda a)^{\frac{1}{3}}\pi)\right] \\ &= \left[(\lambda a)^{\frac{1}{3}}\mathrm{Ai}'(-(\lambda a)^{\frac{1}{3}}\pi) - \alpha\mathrm{Ai}(-(\lambda a)^{\frac{1}{3}}\pi)\right]\left[(\lambda a)^{\frac{1}{3}}\mathrm{Bi}'(0) + \alpha\mathrm{Bi}(0)\right] \end{aligned} \tag{4.1}$$

Proof. Consider the vibrating string equation

$$-u''(x) = \lambda w(x)u(x).$$

Using the change of variable $z = (\lambda a)^{\frac{1}{3}}x$. We get the new equation $u''(z) = -zu(z)$.

The eigenfunctions of the vibrating string equation with linear weight have the form

$$y(z) = C_1\mathrm{Ai}(-z) + C_2\mathrm{Bi}(-z)$$

with $C_1, C_2 \in \mathbb{R}$. Then

$$u(x) = C_1\mathrm{Ai}(-(\lambda a)^{\frac{1}{3}}x) + C_2\mathrm{Bi}(-(\lambda a)^{\frac{1}{3}}x). \tag{4.2}$$

The Robin boundary conditions

$$\begin{aligned} u'(0) &= -\alpha u(0), \\ u'(\pi) &= \alpha u(\pi), \end{aligned} \tag{4.3}$$

give the transcendental equation (4.1). This ends the proof. □

Example 4.2. The goal of this example is to show how the eigenvalue ratio depend on the boundary conditions. By varying α in the equation (4.1), according to the Lemma 2.2 it is enough to take $a = 1$, and using Maple, obtaining the results in the following table

α	λ_1	λ_2	Λ
0.5	0.1515621382	1.182135538	7.799675777
0.8	0.2121429590	1.329717011	6.268023305
1	0.2448740432	1.412220037	5.767128351
10^3	0.6105615046	2.637686454	4.320099505
10^8	10.99670749	34.27314172	3.116673036
10^{10}	17.32312947	44.89673729	2.591722089

4.1. **Dirichlet boundary conditions.** We take the Robin boundary conditions and sending $\alpha \longrightarrow \infty$ so, we recover the Dirichlet boundary conditions . Let α_n and β_n the zeros of the first Airy functions Ai and Bi.

Proposition 4.3. *The eigenvalues of problem (1.2) with Dirichlet boundary conditions correspond to the real roots of the transcendental equation*

$$\mathrm{Bi}(-(\lambda a)^{\frac{1}{3}}\pi) = \sqrt{3}\mathrm{Ai}(-(\lambda a)^{\frac{1}{3}}\pi). \tag{4.4}$$

Proof. The proposition is an immediate consequence of the proposition 4.1. □

Example 4.4. Using Maple, we can approximate the nonnegative real roots of the transcendental equation (4.4). For $a = 1$, we obtain:

$$\lambda_1 \approx 0.6113686524 \textit{ and } \lambda_2 \approx 2.640967963, \textit{ then } \Lambda_D[w] \approx 4.319763456$$

4.2. **Neumann boundary conditions.** We take the Robin boundary conditions and sending $\alpha \longrightarrow 0$ so, we recover the Neumann boundary conditions. Let α'_n and β'_n the zeros of the derivatives of the first Airy functions Ai′ and Bi′.

Proposition 4.5. *The eigenvalues of problem (1.2) with Neumann boundary conditions correspond to the real roots of the transcendental equation*

$$\mathrm{Bi}'(-(\lambda a)^{\frac{1}{3}}\pi) = -\sqrt{3}\mathrm{Ai}'(-(\lambda a)^{\frac{1}{3}}\pi). \tag{4.5}$$

Proof. The proposition is an immediate consequence of the proposition 4.1. □

Example 4.6. Using Maple, we can approximate the nonnegative real roots of the transcendental equation (4.5). For $a = 1$, we obtain

$$\lambda_1 \approx 0.8268706895 \textit{ and } \lambda_2 \approx 3.094520082 \textit{ then } \Lambda_N[w] \approx 3.742447424$$

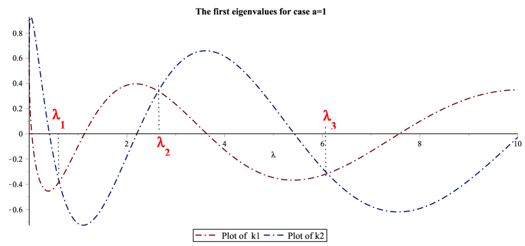

Figure 1: The eigenvalues with D.B.C

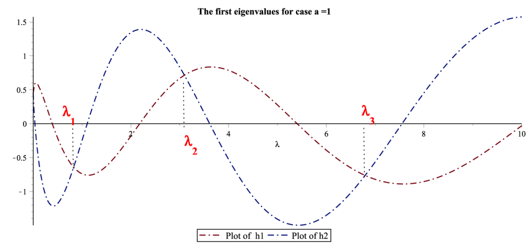

Figure 2: The eigenvalues with N.B.C

5. Conclusion

In our work, we proved that the minimizing concave weight for the eigenvalue ratio of the vibrating strings problem with Robin boundary conditions are affine weight, and using the real roots of Airy functions and their derivatives Ai′ and Bi′, it can be seen from the above examples that

$$\Lambda_N[w] < \Lambda_D[w] < \Lambda_R[w].$$

Different boundary conditions (e.g., Neumann, Dirichlet, or mixed conditions) impose different constraints on the ends of the vibrating string. These conditions can significantly affect the eigenvalues of the system, which in turn influence the vibrational modes and frequencies of the string. By exploring various boundary conditions, researchers can gain a more holistic understanding of the dynamics of vibrating strings in different physical and engineering contexts, such as acoustics, structural engineering, and musical instrument design [9, 21], where the behavior of vibrating strings is of significant importance. The outlined future work aims to broaden the scope of research in vibrating strings problems by exploring different boundary conditions and weight functions. Each of these areas holds the potential to significantly advance our understanding of vibrating systems, with wide-ranging applications in physics, engineering, and beyond.

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