

CLASSIFICATION OF POLYNOMIALS AND AN ORTHONORMAL BASIS CONSTRUCTION ON THE ASSOCIATED BASIN OF ATTRACTION

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ABSTRACT. We study a construction that associates certain Fatou subsets with reproducing kernel Hilbert spaces and a method for constructing an orthonormal basis for said Hilbert space, which depends on the map of the given Fatou set. We provide a classification of those polynomials the method applies to, and we extend a previous result concerning the dynamics of these orthonormal bases. Essential to the proof of this classification are the Newton–Girard identities and Vieta’s formulas.

1. INTRODUCTION

We examine an orthonormal basis construction developed by Alpay, Jorgensen, Lewkowicz, and Martziano [1] for reproducing kernel Hilbert spaces arising from iterated function systems. We start with a positive definite function $k(z, w)$ on a topological space $\mathcal{U}$ and let $\{e_j\}_{j \in J}$ be an orthonormal basis for the reproducing kernel Hilbert space $\mathcal{H}_k$ associated to $k(z, w)$. Given a map $R : \mathcal{U} \rightarrow \mathcal{U}$, Alpay et al. show that there is a continuous function $K(z, w)$, positive definite in $\mathcal{U}$, satisfying

$$K(z, w) = k(z, w)K(R(z), R(w)), \quad \text{for all } z, w \in \mathcal{U},$$

if $K(z, w)$ is not identically 0 and if there is an $l \in \mathcal{U}$ with $\lim_{n \rightarrow \infty} R^{on}(z) = l$ and $K(l, l) > 0$ [1, Proposition 3.1]. In the case that $\mathbb{C}$ is contractively contained in $\mathcal{H}_k$ and the subset

$$\Omega = \left\{ z \in \mathcal{U} : \sum_{n=0}^{\infty} |k(R^{on}(z), R^{on}(w)) - 1| < \infty \right\}$$

is nonempty, operators of the form $(S_j f)(z) = e_j(z)f(R(z))$ are shown to satisfy the Cuntz–Krieger relations if the cardinality (counting multiplicity of roots) $M(z)$ of $\{z \in \Omega : R(\zeta) = z\}$ is finite, and either

$$\sum_{R(\zeta)=z} e_j(\zeta) \overline{e_k(\zeta)} = \delta_{jk} M(z), \quad \text{for all } j, k \in J, \quad (1.1)$$

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or

$$\sum_{R(\zeta)=z} e_j(\zeta)e_k(\zeta) = \delta_{jk}M(z), \quad \text{for all } j, k \in J, \quad (1.2)$$

for all $z \in \Omega$ [1, Theorem 5.4]. From the Cuntz–Krieger relations one is able to construct an orthonormal basis for the reproducing kernel Hilbert space associated to $K(z, w)$ by letting the basis elements consist of those functions obtained via repeated composition of the $S_j f$ with each other [1, Theorem 6.1].

We are interested in the case that $\mathcal{U} = \mathbb{C}$, R is a polynomial, and $k(z, w) = 1 + z\bar{w}$, which will make Ω the basin of attraction of R at 0 if $\Omega \neq \emptyset$ and does not contain ∞ [13, Lemma 2.3]. Applying the theory from [1] outlined above, we obtain a reproducing kernel Hilbert space $\mathcal{H}_K$ associated to the basin of attraction of R at 0. Furthermore, the operators used to construct an orthonormal basis on $\mathcal{H}_k$ depend on the polynomial R . Our goal is to classify those polynomials R for which the orthonormal basis construction from [1] applies. This amounts to analyzing conditions (1.1) and (1.2) in the present setting. Our main result classifies those polynomials satisfying (1.2):

Theorem 1.1. *Suppose R is a polynomial with an attracting fixed point at 0. The polynomial R satisfies $\ddagger$ if and only if $R(z) = \sum_{k=1}^n a_k z^k$ where $n \geq 3$, $a_{n-1} = 0$, and $a_{n-2} = -na_n/2$.*

Remark 1.2. Since R has an attracting fixed point at 0, we have also that $|a_1| < 1$.

Condition (1.2) in our present setting requires that the roots of R sum to 0 and square sum to $\deg R$. We use Viète’s formulas in combination with Newton–Girard identities to prove Theorem 1.1. Condition (1.1) can be broken down in a similar fashion, however, the main difference is that we sum the square modulus of each root, as opposed to just the root itself. This difference prevents us from using Viète’s formulas and Newton–Girard identities, making classification much more elusive.

In Section 2, we collect some basic definitions and notation from complex dynamics and reproducing kernel Hilbert spaces, and conclude with a closer look at the machinery from [1]. Section 3 analyzes conditions (1.1) and (1.2), resulting in a proof for Theorem 1.1. In Section 4, we examine the dynamics of the orthonormal basis construction from [1], extending a result from [13]. Section 5 looks at two examples. We conclude with a discussion of paths for future research. We have also included a short appendix on Viète’s formulas and the Newton–Girard identities.

2. PRELIMINARY DEFINITIONS AND RESULTS

We provide the basic definitions and theory needed for the remaining sections. No claim of novelty is made to the contents of this section.

Complex Dynamics. Denote the extended complex plane by $\mathbb{C}_\infty$. Recall that a rational map $R : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ has an attracting fixed point at $z_0 \in \mathbb{C}_\infty$ if $|R'(c)| < 1$. The point z_0 is called an attracting fixed point because all points

within a certain neighborhood of z_0 approach z_0 under repeated iteration of R . The n -th iterate of R is denoted

$$R^{on}(z) = \underbrace{R \circ R \circ \cdots \circ R(z)}_{n \text{ times}},$$

and we define $R^o(z) = z$. The basin of attraction of R at the attracting fixed point z_0 is the subset of $\mathbb{C}$ defined by

$$B_{R,z_0} = \{z \in \mathbb{C} : \lim_{n \rightarrow \infty} R^{on}(z) = z_0\}.$$

We will also be interested in the subset of complex numbers

$$\Omega = \left\{ z \in \mathbb{C} : \sum_{n=0}^{\infty} |R^{on}(z)|^2 < \infty \right\} \quad (2.1)$$

when 0 is an attracting fixed point of R . In this case Ω coincides with $B_{R,0}$ if $0 \notin B_{R,0}$ [13]. For many polynomials with an attracting fixed point, the basin of attraction is a fractal.

An exceptional point of R is any point z with finite orbit. The set of exceptional points of R is denoted $E(R)$. This set has at most 2 elements if $\deg R \geq 2$. To every rational map R there exists a unique maximal-entropy measure μ whose support is the Julia set $J(R)$ [8]. This was first demonstrated by Brolin [5], in the polynomial case $R(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, who constructed such a measure as a weakly convergent limit of mass distributions μ_m defined by placing the mass k^{-m} at the k^m solutions of $R^{on}(\zeta) = z_0$, where z_0 is fixed in $\mathbb{C} - E(R)$.

We will have some interest in the Hilbert space $L^2(\mu)$ of functions where μ is the unique maximal-entropy measure associated to the rational map R .

The conjugacy class of R is the collection of rational maps S of the form $S = gRg^{-1}$, where $g(z) = (az+b)/(cz+d)$, $ad-bc \neq 0$, is a Möbius map. In this setting the maps R and S are said to be conjugates. Conjugation respects many properties, including degree, iteration, and fixed points. One often focuses on conjugacy classes when studying complex dynamics. The results presented here are easily extended to conjugacy classes, see [14] for details. The texts [6, 4, 9] serve as comprehensive references for the general theory of complex dynamics.

Reproducing Kernel Hilbert Spaces. The general theory of reproducing kernel Hilbert spaces were first laid out by Aronszajn in [2, 3], the latter of which also contains a nice historical introduction. For excellent modern references, see [10, 11]. A reproducing kernel Hilbert space on a set X is a Hilbert space of complex-valued functions on X in which every linear evaluation functional is bounded. This property, together with the Riesz representation theorem, implies the existence of a function $K : X \times X \rightarrow \mathbb{C}$ with the *reproducing property*

$$\langle f(z), K(z, w) \rangle_{\mathcal{H}} = f(w) \quad \text{for all } z, w \in X.$$

Functions with this property are called *reproducing kernels* (or *kernel functions*) and are unique when they exist.

Every reproducing kernel K is a positive definite function, that is, given $n \in \mathbb{Z}_+$, $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $x_1, \dots, x_n \in X$,

$$\sum_{i,j=1}^n \bar{\alpha}_i \alpha_j K(x_i, x_j) \geq 0.$$

Conversely, each positive definite function K on a set X corresponds to a reproducing kernel Hilbert space of complex-valued functions on X for which K is a reproducing kernel. The reproducing kernel Hilbert space associated to K will be denoted $\mathcal{H}_K$.

It is well known that every Hilbert space $\mathcal{H}$ has an orthonormal basis. In the case that $\mathcal{H}_K$ is a reproducing kernel Hilbert space with orthonormal basis $\{e_i : i \in \mathcal{I}\}$, $\mathcal{I}$ an index set, the associated reproducing kernel K is given by

$$K(z, w) = \sum_{i \in \mathcal{I}} \overline{e_i(w)} e_i(z).$$

Among the standard examples of reproducing kernel Hilbert spaces is the Hardy space $H^2(\mathbb{D})$ on the unit disk. $H^2(\mathbb{D})$ consists of functions $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with inner product $\langle f, g \rangle = \sum_{n=0}^{\infty} f_n \bar{g}_n$, so that $\|f\| = \sum_{n=0}^{\infty} |a_n|^2 < \infty$. The associated kernel function is the Szegő kernel given by

$$\begin{aligned} K(z, w) &= \sum_{n=0}^{\infty} z^n \bar{w}^n \\ &= \frac{1}{1 - \bar{w}z}. \end{aligned}$$

A simpler example that will be needed later is the reproducing kernel Hilbert space associated to $\kappa(z, w) = 1 + z\bar{w}$, which is positive definite on $\mathbb{C}$.

Kernel Functions on Basins of Attraction. We review the kernel function construction developed in [1] which this paper aims to study. Proposition 2.1 furnishes sufficient conditions for a continuous map to be positive definite on a topological space and Lemma 2.2 provides sufficient conditions for the convergence of an infinite product. In both results, R is a map from a set $\mathcal{U}$ to itself.

Proposition 2.1 ([1]). *Suppose $\mathcal{U}$ is a topological space with a positive definite function $k(z, w)$ and a continuous function $K(z, w)$, not identically 0, satisfying*

$$K(z, w) = k(z, w)K(R(z), R(w)) \quad \text{for all } z, w \in \mathcal{U}.$$

Assume further there is $l \in \mathcal{U}$ such that

$$\lim_{n \rightarrow \infty} R^{\circ n}(z) = l \quad \text{for all } z \in \mathcal{U}.$$

If $K(l, l) > 0$, then $K(z, w)$ is positive definite in $\mathcal{U}$ and

$$K(z, w) = \left(\prod_{n=0}^{\infty} k(R^{\circ n}(z), R^{\circ n}(w)) \right) K(l, l).$$

Lemma 2.2 ([1]). *If the set Ω in (2.1) is nonempty and H_k contractively includes $\mathbb{C}$ then the infinite product*

$$K(z, w) = \prod_{n=0}^{\infty} k(R^{\circ n}(z), R^{\circ n}(w))$$

converges for all $z, w \in \Omega$.

Combining Proposition 2.1 and Lemma 2.2, one may construct a kernel function, represented as an infinite product, on a subset of $B_{R,0}$. Suppose R is a rational map with an attracting fixed at 0 and $\infty \notin B_{R,0}$. Thus, $B_{R,0} = \Omega \neq \emptyset$. Since $\mathcal{H}_\kappa$ contractively includes $\mathbb{C}$, Lemma 2.2 shows that

$$K(z, w) = \prod_{n=0}^{\infty} \kappa(R^{\circ n}(z), R^{\circ n}(w))$$

converges for all $z, w \in B_{R,0}$. By definition, $\lim_{n \rightarrow \infty} R^{\circ n}(z) = 0$ for all $z \in B_{R,0}$, and $K(0, 0) = 1 > 0$. Since

$$\begin{aligned} K(z, w) &= \kappa(z, w) \prod_{n=1}^{\infty} \kappa(R^{\circ n}(z), R^{\circ n}(w)) \\ &= \kappa(z, w) K(R(z), R(w)) \end{aligned}$$

for all $z, w \in B_{R,0}$, we see that $K(z, w)$ satisfies (2.2) and we may apply Proposition 2.1 to deduce that $K(z, w)$ is a positive definite function on $B_{R,0}$. We have thus obtained the following result:

Theorem 2.3. *If R is a rational map with an attracting fixed point at 0 and $\infty \notin B_{R,0}$ then*

$$K(z, w) = \prod_{n=0}^{\infty} (1 + R^{\circ n}(z) \overline{R^{\circ n}(w)}) \quad (2.2)$$

is a kernel function on $B_{R,0}$.

An Orthonormal Basis Construction. First we recall multi-index notation. Suppose J is an index set, then

$$J^\infty = \{v : v \in J^N \text{ for some } N = 1, 2, \dots\}.$$

In what follows, we take $J = \{0, 1\}$. Denote the reproducing kernel Hilbert space associated to the kernel function in (2.2) by $\mathcal{H}$. The constant function $\mathbf{1}(z) = 1$ plays a crucial role in the construction, and in fact belongs to $\mathcal{H}$. Consider operators $S_0, S_1 : \mathcal{H} \rightarrow \mathcal{H}$, given by

$$(S_0 f)(z) = f(R(z)) \quad \text{and} \quad (S_1 f)(z) = z f(R(z)).$$

For each $v = (v_1, v_2, \dots, v_N) \in J^N$, let $S_v = S_{v_1} S_{v_2} \cdots S_{v_N}$ and define $b_v : \mathbb{C} \rightarrow \mathbb{C}$ by

$$b_v(z) = (S_v \mathbf{1})(z).$$

The next theorem is a special case of [1, Theorem 6.1], and gives sufficient conditions for the functions b_v to form an orthonormal basis.

Theorem 2.4 ([1]). *If the family of operators $\mathcal{F} = \{S_1, S_2\}$ satisfies the Cuntz–Krieger relations*

$$S_i^* S_j = \delta_{ij} I \quad \text{and} \quad S_1 S_1^* + S_2 S_2^* = I,$$

then $B = \{b_v : v \in J^\infty\}$ is an orthonormal basis for $\mathcal{H}$.

This family of operators can be shown to satisfy the Cuntz–Krieger relations when certain conditions are met, which we discuss next.

The Dagger Conditions. The family $\mathcal{F}$ of interest depends on the rational map R and the positive definite function $k(z, w)$. We are interested in the case $k(z, w) = 1 + z\bar{w}$. The functions $e_0(z) = 1$ and $e_1(z) = z$ form an orthonormal basis for the reproducing kernel Hilbert space $\mathcal{H}_k$. It can be shown that if R satisfies for all $z \in B_{R,0}$,

$$M(z) < \infty,$$

where $M(z)$ is the number of solutions to $R(\zeta) = z$ residing in $B_{R,0}$, counting multiplicity, and either

$$\frac{1}{M(z)} \sum_{R(\zeta)=z} e_i(\zeta) \overline{e_j(\zeta)} = \delta_{ij}, \quad \forall i, j \in \{0, 1\} \quad (\dagger)$$

or

$$\frac{1}{M(z)} \sum_{R(\zeta)=z} e_i(\zeta) e_j(\zeta) = \delta_{ij}, \quad \forall i, j \in \{0, 1\}, \quad (\ddagger)$$

then $\mathcal{F}$ satisfies the Cuntz–Krieger relations [1, Theorem 5.4]. To see this in the case that $\dagger$ holds, one can check that the adjoints of S_0 and S_1 are given by

$$S_0^*(f)(z) = \frac{1}{M(z)} \sum_{R(\zeta)=z} f(\zeta) \quad \text{and} \quad S_1^*(f)(z) = \frac{1}{M(z)} \sum_{R(\zeta)=z} \bar{\zeta} f(\zeta)$$

and satisfy

$$S_0^* K_w = K_{R(w)} \quad \text{and} \quad S_1^* K_w = \bar{w} K_{R(w)}.$$

Thus $S_i^* S_j(f)(z) = \delta_{ij} f(z)$ and $(S_0 S_0^* K_w)(z) + (S_1 S_1^* K_w)(z) = K(z, w)$.

For ease of exposition we will refer to the above conditions as the dagger conditions. It is natural to ask when the map R satisfies either of the above conditions.

In our current setting, $\dagger$ becomes

$$\sum_{R(\zeta)=z} 1 = M(z) \quad (\dagger_1)$$

$$\sum_{R(\zeta)=z} \zeta = 0 = \sum_{R(\zeta)=z} \bar{\zeta} \quad (\dagger_2)$$

$$\sum_{R(\zeta)=z} |\zeta|^2 = M(z), \quad (\dagger_3)$$

and $\ddagger$ becomes

$$\sum_{R(\zeta)=z} 1 = M(z) \quad (\ddagger_1)$$

$$\sum_{R(\zeta)=z} \zeta = 0 \quad (\ddagger_2)$$

$$\sum_{R(\zeta)=z} \zeta^2 = M(z). \quad (\ddagger_3)$$

In summary, we start with a rational map R that has an attracting fixed point at 0. Theorem 2.3 tells us that there is a reproducing kernel Hilbert space on the basin of attraction of R at 0, and Theorem 2.4 shows us how to construct an orthonormal basis for this space from a family of operators satisfying the Cuntz–Krieger relations. Theorem 5.4 from [1] shows that for a family of operators to satisfy the Cuntz–Krieger relations, it suffices to check that the family satisfies one of the dagger conditions.

Next we examine which polynomials R satisfy the dagger conditions and offer a classification for those satisfying the $\ddagger$ condition.

3. PARTIAL CLASSIFICATION OF THE DAGGER CONDITIONS

The purpose of the dagger conditions is to construct an orthonormal basis for the reproducing kernel Hilbert space corresponding to the kernel function on $B_{R,0}$. Thus our interest lies only with those polynomials with an attracting fixed point at 0, even though the dagger conditions do not require R to have such a property. The first two cases of either dagger condition are quite easily characterized.

Proposition 3.1. *If $R(z)$ is a degree n polynomial with an attracting fixed point at 0 then the following hold:*

- (a) $R(z)$ satisfies $\dagger_1$ and $\ddagger_1$.
- (b) $R(z)$ satisfies $\dagger_2$ and $\ddagger_2$ if and only if $a_{n-1} = 0$.

Proof. (a) Since $B_{R,0}$ is completely invariant with respect to R , we know that if $R(\zeta) = z$ for some $z \in B_{R,0}$, then we must have that $\zeta \in \Omega$. By the Fundamental Theorem of Algebra there are n solutions to $R(\zeta) = z$, counting multiplicity. Thus

$$\sum_{R(\zeta)=z} 1 = n = M(z).$$

- (b) We have that $R(z)$ satisfies $\dagger_2$ and $\ddagger_2$ if and only if

$$\sum_{R(\zeta)=z} \bar{\zeta} = 0 = \sum_{R(\zeta)=z} \zeta$$

which by Vieta's formulas is equivalent to $a_{n-1} = 0$. □

Note: The conditions $\dagger_2$ and $\ddagger_2$ are equivalent to each other since

$$\sum_{P(\zeta)=z} \bar{\zeta} = \overline{\sum_{P(\zeta)=z} \zeta}.$$

The next proposition will complete our characterization of $\ddagger$.

Proposition 3.2. *A polynomial R of degree $n \geq 3$, with an attracting fixed point at 0, satisfies $\ddagger_3$ if and only if $a_{n-2} = \frac{-na_n}{2}$.*

Proof. The polynomial R satisfies $\ddagger_3$ if and only if

$$\sum_{R(\zeta)=z} \zeta^2 = n.$$

which is equivalent to $a_{n-2} = \frac{-na_n}{2}$, since Vieta's formulas and the Newton–Girard formulas give

$$n = \sum_{R(\zeta)=z} \zeta^2 = -2e_2 = \frac{-2a_{n-2}}{a_n}. \quad \square$$

Combining the last two propositions, we characterize those polynomials with an attracting fixed point at 0 that satisfy the $\ddagger$ condition, thereby proving Theorem 1.1.

Characterizing the $\ddagger_3$ condition seems to be more challenging. It is easy to find polynomials which satisfy $\ddagger_3$ at a particular point. But the dagger conditions are required to hold for all $z \in B_{P,0}$. As such we introduce “partial” conditions, $\ddagger^c$ and $\ddagger^c$, each meaning that the corresponding set of equations hold precisely at the point $c \in \mathbb{C}$. The next proposition will help us determine which polynomials might satisfy $\ddagger$.

Proposition 3.3. *Suppose $c \in B_{P,0}$.*

- (a) *If $R(z)$ satisfies $\ddagger^c$, then $R(z)$ satisfies $\ddagger$.*
- (b) *If $R(z)$ satisfies both $\ddagger^c$ and $\ddagger^c$ then the equation $R(z) = c$ has only real solutions.*

Proof. (a) If $R(z)$ satisfies $\ddagger^c$, then we must have that

$$\sum_{P(\zeta)=c} 1 = n, \quad \sum_{P(\zeta)=c} \zeta = 0, \quad \sum_{P(\zeta)=c} \zeta^2 = n.$$

By Vieta's formulas, we have that $a_{n-1} = 0$. Thus we have that $\ddagger_1$ and $\ddagger_2$ hold by an application of Proposition 3.1. By an application of the Newton–Girard formulas we have that

$$n = \sum_{P(\zeta)=c} \zeta^2 = -2e_2 = \frac{-2a_{n-2}}{a_n}.$$

Applying the same formula again we find that

$$\sum_{P(\zeta)=z} \zeta^2 = \frac{-2a_{n-2}}{a_n} = n.$$

So R actually satisfies $\ddagger$.

- (b) Suppose that $R(z)$ satisfies both $\ddagger^c$ and $\ddagger^c$, so in particular, we have that

$$\sum_{R(\zeta)=c} \zeta^2 = \sum_{R(\zeta)=c} |\zeta^2|$$

which requires that $\sum_{R(\zeta)=c} \operatorname{Im} \zeta^2 = 0$. Thus we have that

$$\sum_{R(\zeta)=c} \operatorname{Re} \zeta^2 = \sum_{R(\zeta)=c} |\zeta^2|.$$

If ζ^2 is not real, then $|\zeta^2| > \operatorname{Re} \zeta^2$, so the above equality holds only if all solutions to $P(\zeta) = c$ are real. $\square$

So, if $R(z)$ has a non-real zero, then $R(z)$ cannot satisfy both $\dagger$ and $\ddagger$. The only interesting polynomials which might satisfy $\dagger$ are those $R(z)$ which have the property that $\ddagger^c$ is not satisfied for any $c \in \Omega$. Such a polynomial requires the property that $R(z) + c$ has at least one non-real zero for all $c \in \Omega$. An example of a polynomial with this property is any cubic of the form $R(z) = az^3 + bz + c$ with b a scalar multiple of $\bar{a}$. We show now that no polynomial can satisfy both $\dagger$ and $\ddagger$. This fact can be used in turn to show that the aforementioned property is a necessary condition for a polynomial to satisfy $\dagger$.

Proposition 3.4. *If R is a polynomial with an attracting fixed point at 0, then R cannot satisfy both $\dagger$ and $\ddagger$.*

Proof. Suppose to the contrary that R satisfies both $\dagger$ and $\ddagger$. By Proposition 3.3, $R(z) = c$ has only real solutions, for any $c \in B_{P,0}$. Thus the inverse image of $B_{P,0}$ under R must be a subset of $\mathbb{R}$. However, $B_{P,0}$ is an open set, and R is a continuous map, so that the inverse image of $B_{P,0}$ under R must be an open set. But no subset of $\mathbb{R}$ is open as a subset of $\mathbb{C}$. Thus it cannot be that R satisfies both $\dagger$ and $\ddagger$. $\square$

We can now state a necessary condition for a polynomial to satisfy $\dagger$.

Proposition 3.5. *If R is a polynomial with an attracting fixed point at 0 that satisfies $\dagger$, then for all $c \in B_{R,0}$, the equation $R(z) = c$ has at least one non-real solution.*

Proof. Suppose that R satisfies $\dagger$, and to the contrary, that $R(z) = c$ has only real solutions. If ζ is such a solution, then $|\zeta|^2 = \zeta^2$. Since R satisfies $\dagger_3$, we have also that, by the previous observation, that R satisfies $\ddagger_3$. By Proposition 3.1, R also satisfies $\ddagger_1$ and $\ddagger_2$, so that R must satisfy $\ddagger$. But R cannot satisfy both $\dagger$ and $\ddagger$, so that $R(z) = c$ must have at least one non-real solution. $\square$

Another approach to showing that a polynomial satisfies one of the dagger conditions is to write it as a product of two polynomials, each of which satisfy the same dagger condition. This is equivalent to determining whether the product of two polynomials, both satisfying the same dagger condition, will satisfy a dagger condition. This works quite well for the $\ddagger$ condition.

Proposition 3.6. *If $R(z)$ and $Q(z)$ satisfy $\ddagger$, then $R(z)Q(z)$ satisfies $\ddagger$.*

Proof. Suppose that R and Q both satisfy $\ddagger$ and let $S(z) = R(z)Q(z)$. Let a_i denote the coefficients of R , b_i denote the coefficients of Q , and suppose $\deg R = r$, $\deg Q = q$ so that $\deg S = r + q$. Theorem 1.1 tells us that $a_{r-2} = \frac{-ra_r}{2}$,

$b_{q-2} = \frac{-rb_q}{2}$, and $a_{r-1} = 0 = b_{q-1}$. If c_i denotes the coefficients of S , then we have that $c_{r+q} = a_r b_q$, $c_{r+q-1} = 0$, and $c_{r+q-2} = \frac{-(r+q)c_{r+q}}{2}$. By Theorem 1.1, the polynomial S satisfies the $\ddagger$ condition. $\square$

4. DYNAMICS OF THE BASIS CONSTRUCTION

First we introduce some notation to facilitate the discussion. Given $a \in \mathbb{C}^n / \{\vec{0}\}$ and $\rho \in \mathbb{N}^n$, let $a^\rho = a_1^{\rho_1} \cdots a_n^{\rho_n}$ and $|\rho| = \rho_1 + \cdots + \rho_n$. The set of polynomials in a_i with coefficients in $\mathbb{Q}$ and constant term 0 is given by

$$P_a = \left\{ \sum_{|\rho|=1}^m c_\rho a^\rho : c_\rho \in \mathbb{Q}, m \geq 1 \right\}.$$

We say that a polynomial has *good form (relative to a)* if its nonzero coefficients belong to P_a .

Lemma 4.1. *Suppose $a \in \mathbb{C}^n / \{\vec{0}\}$, let R_a be a polynomial satisfying Theorem 1.1 whose nonzero coefficients are the components of a . If f is a polynomial with good form relative to $a \in \mathbb{C}^n / \{\vec{0}\}$, then $S_0 f$ and $S_1 f$ have good form relative to a .*

Proof. Fix $a \in \mathbb{C}^n / \{\vec{0}\}$. Since P_a is closed under addition and multiplication, the sum and product of two polynomials with good form are also polynomials with good form. Thus, the composition of two polynomials with good form is again a polynomial with good form. Now suppose that f is a polynomial with good form, then $S_0 f = f(R_a(z))$ is a polynomial with good form since R_a is a polynomial with good form. Since $S_1 f = z S_0 f$, we have that $S_1 f$ is a polynomial with good form too. $\square$

Consider the family of polynomials

$$\mathcal{F}_n = \left\{ R_a(z) = \sum_{i=1}^n a_i z^i : a_i \in \mathbb{C}, a_{n-1} = 0, a_{n-2} = -\frac{1}{2} n a_n, |a_1| < 1 \right\},$$

with $n \geq 3$ fixed. The polynomials of $\mathcal{F}_n$ all satisfy the $\ddagger$ condition and so we may apply Theorem 2.4 to each R_a to get an orthonormal basis B_a whose elements we now write as $b_{v,a}$ to remind ourselves that the operators S_i are calculated using the polynomial R_a . An immediate result of the previous lemma is that the basis elements of B_s are the basis elements of B_t with the s_i substituted in for the t_i :

Corollary 4.2. *Let $s, t \in \mathbb{C}^n / \{\vec{0}\}$. If $v \in J^\infty$ is such that $b_{v,s}(z)$ is neither 1 nor z , then $b_{v,s}(z)$ is a polynomial with good form. Furthermore, if $\beta_i(x_1, \dots, x_n) = \sum_{|\rho|=1}^{n_i} c_\rho x^\rho$ where $n_i \in \mathbb{N}$, then*

$$b_{v,s}(z) = \sum_{i=1}^{\nu} \beta_i(s_1, \dots, s_n) z^i \quad \text{if and only if} \quad b_{v,t}(z) = \sum_{i=1}^{\nu} \beta_i(t_1, \dots, t_n) z^i.$$

Proof. With the exception of $S_0 \mathbf{1} = 1$ and $S_1 \mathbf{1} = z$, the remaining $b_{v,\vec{a}}$ are obtained by composing a nonzero and finite number of S_0 and S_1 with $S_1 \mathbf{1} = z$. Since $S_0 S_1 \mathbf{1} = R_{\vec{a}}$ and $S_1 S_1 \mathbf{1} = z R_{\vec{a}}$ are both polynomials with good form, an induction argument shows that the remaining $b_{v,\vec{a}}$ are polynomials with good form.

Since R_s can be obtained from R_t by simply substituting each s_i in for the corresponding t_i , and both $b_{v,s}$ and $b_{v,t}$ involve the same sequence of the S_i , it follows that each coefficient of $b_{v,s}$ can be obtained by making the above substitution for the corresponding coefficient of $b_{v,t}$. $\square$

We now show that the orthonormal basis elements $b_{v,a}$ vary continuously over a under the usual topology of the rational maps.

Theorem 4.3. *For each $v \in J^\infty$, the map Γ_v on*

$$\mathcal{C} = \{a \in \mathbb{C}^n : a_n \neq 0, a_{n-1} = 0, a_{n-2} = \frac{1}{2}na_n, |a_1| < 1\}$$

defined by $a \mapsto b_{v,a}$ is continuous.

Proof. Suppose $a \in \mathcal{C}$ and let $s_m \rightarrow a$ as $m \rightarrow \infty$. We have that

$$\begin{aligned} \lim_{m \rightarrow \infty} \Gamma_v(s_m) &= \lim_{m \rightarrow \infty} b_{v,s_m} \\ &= \lim_{m \rightarrow \infty} \sum_{i=1}^{\nu} \beta_i(s_{m1}, \dots, s_{mn}) z^i \\ &= \sum_{i=1}^{\nu} \beta_i(a_1, \dots, a_n) z^i \\ &= b_{v,a}, \end{aligned}$$

since the β_i are continuous as they are multivariate polynomials. $\square$

5. EXAMPLES

Here we present an example of a polynomial satisfying $\ddagger$, and an example of a polynomial satisfying $\dagger^0$.

Example 5.1. Consider the polynomial $R(z) = iz^4 - 2iz^2 - \frac{1+i}{2}z$. We see that 0 is an attracting fixed point of R since $R(0) = 0$ and $|R'(0)| = \frac{\sqrt{2}}{2} < 1$. So we have that the map

$$K(z, w) = \prod_{n=0}^{\infty} \left(1 + R^{\circ n}(z) \overline{R^{\circ n}(w)}\right)$$

is a kernel function on $B_{R,0}$, see Figure 1. The polynomial R has coefficients $a_4 = i$, $a_2 = -2i$, $a_1 = -\frac{1+i}{2}$, and $a_3 = a_0 = 0$. Since $a_3 = 0$ and $a_2 = \frac{-4a_4}{2}$, Theorem 1.1 tells us that R satisfies $\ddagger$. This in turn shows that the operators S_1 and S_2 , defined by

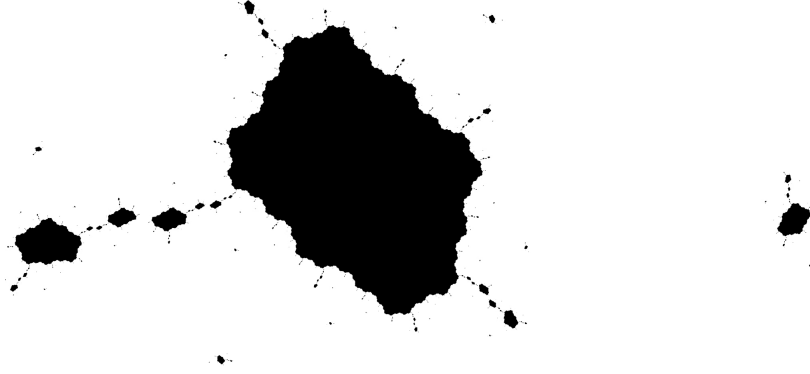
$$S_1 f(z) = f(R(z)) \quad \text{and} \quad S_2 f(z) = z f(R(z))$$

satisfy the Cuntz–Krieger relations. So we may apply Theorem 2.4 to conclude that the functions $b_v(z)$ form an orthonormal basis for the Hilbert space associated to K . Recall that

$$b_v(z) = (S_{v_1} S_{v_2} \cdots S_{v_N} \mathbf{1})(z)$$

where $v \in J^\infty$. The first few basis elements are

$$1, z, R(z), zR(z), R^{\circ 2}(z), zR^{\circ 2}(z), R(z)R^{\circ 2}(z), zR(z)R^{\circ 2}(z), \dots$$

FIGURE 1. The domain of $K(z,w)$: $B_{R,0}$ FIGURE 2. The domain of $K(z,w)$: $B_{Q,0}$

So the basis elements may be calculated recursively, but obtaining a general formula appears to require a general formula for R^{on} .

Example 5.2. Consider the polynomial $Q(z) = \frac{1}{2}z^3 + \frac{3}{4}z$, which also has an attracting fixed point at 0. So the map

$$K(z, w) = \prod_{n=0}^{\infty} \left(1 + Q^{on}(z) \overline{Q^{on}(w)} \right)$$

is a kernel function on $B_{Q,0}$, see Figure 2. There is a reproducing kernel Hilbert space associated to K , however, we can't use the dagger conditions to construct an orthonormal basis. By Theorem 1.1, Q doesn't satisfy $\ddagger$, in particular, the condition $\ddagger_3$. It turns out that Q does satisfy $\dagger^0$; this follows from Proposition 3.1 and the observation

$$\sum_{Q(\zeta)=0} |\zeta|^2 = |0|^2 + \left| i\sqrt{3/2} \right|^2 + \left| -i\sqrt{3/2} \right|^2 = 3.$$

It can be shown in a similar fashion that Q satisfies $\dagger^c$ for $c = i/(2\sqrt{2})$. However Q doesn't satisfy $\dagger^c$ for all $c \in B_{Q,0}$: for $c = i \in B_{Q,0}$ and with the aid of WolframAlpha, we have that

$$\sum_{Q(\zeta)=i} |\zeta|^2 > 3.$$

6. FUTURE WORK AND CONCLUSION

We have classified those polynomials which satisfy the $\ddagger$ condition and therefore those polynomials for which the orthonormal basis construction of [1] applies. We have also shown that the resulting orthonormal basis depends continuously on the coefficients of the given polynomial, thereby extending the main result of [13]. There are several paths future work could follow:

- (1) **Classify those polynomials satisfying the $\dagger$ condition.** We have seen that the set of polynomials satisfying $\dagger$ is disjoint from the set of polynomials satisfying $\ddagger$ and that any polynomial R satisfying $\dagger$ must necessarily have a non-real solution to $R(z) = c$ for all $c \in B_{R,0}$. The main issue lies with $\dagger_3$. Since the sum involves the modulus of the roots, Vieta's formula may not be applied. The family of cubics $az^3 + k\bar{a}z$, $k \in \mathbb{R}$ may be a good starting point in the search to find a polynomial that satisfies $\dagger_3$. If it can be shown that no such cubic satisfies $\dagger_3$, then perhaps the method can be modified to show that no such polynomial exists.
- (2) **Investigate how the present theory might apply to boundary value problems.** Recall that given a linear differential operator L and Dirac's delta function δ , a Green's function G is a solution to $LG = \delta$. If the domain of L is a Hilbert space, then G is the reproducing kernel of the Hilbert space. Thus, solutions to $Ly = f$ may be represented using the reproducing property: $y(x) = \langle y, G_x \rangle$. A simple illustrative example is Green's function for Sobolev space on $[0, 1]$. A more instructive example might be the Hardy space on the unit disk since its kernel, the Szëgo kernel, is given by Theorem 2.3 with $R(z) = z^2$. The hope is to use the ideas above to study boundary value problems with a Julia set boundary.
- (3) **Develop the multiplicity free theory.** Much of the theory presented in [1] should still work if we don't count the multiplicity of the solutions to $R(z) = c$. The biggest issue with this change would be in the application of Vieta's formula, since it does use multiplicity.
- (4) **Consider other underlying kernels.** There are other underlying kernel functions one could use besides $1 + z\bar{w}$. However, changing the underlying kernel function changes the dagger conditions. There are some kernel functions for which the approach presented here might still work, in particular, kernel functions of the form $1 + (z\bar{w})^n$, where n is a positive integer. For example, with the underlying kernel function $1 + (z\bar{w})^2$, the dagger condition $\ddagger$ becomes $\sum 1 = M(z)$, $\sum \zeta^2 = 0$, and $\sum \zeta^4 = M(z)$, where all sums are taken over solutions to $R(\zeta) = z$. The first sum will hold by virtue of the Fundamental Theorem of Algebra. The other two can be analyzed using Vieta's formula and the Newton–Girard identities: $\sum \zeta^2$ by the identity for P_2 and $\sum \zeta^4$ by the identity for P_4 . Using Vieta's formula one can then get conditions on the coefficients of R . In the case of $\sum \zeta^2 = 0$, one finds that $a_{n-1}^2/a_n^2 - 2a_{n-2}/a_n = 0$. In the case of

$\sum \zeta^4 = M(z)$, one finds that

$$\frac{-a_{n-1}^2 a_{n-2}}{a_n^3} + \frac{4a_{n-1} a_{n-3}}{a_n^2} - \frac{4a_{n-4}}{a_n} = M(z).$$

- (5) **Consider generalized Julia sets.** The constructions presented here work for an arbitrary topological space in place of $\mathbb{C}$ [1]. It may be of interest to examine these constructions in the settings of generalized Julia sets such as the normed Julia sets conceived by Sanooj and Vinodkumar [12].

7. APPENDIX: NEWTON–GIRARD IDENTITIES AND THE VIETA FORMULA

Here we take a brief look at the Newton–Girard identities and Vieta’s formula; both quintessential tools in this paper. See [7] for a more in depth historical introduction.

Theorem 7.1 (Vieta’s formula). *Suppose $R(z) = \sum_{j=0}^n a_j z^j$ satisfies $a_n \neq 0$. If $z_1, \dots, z_n$ are the roots (counting multiplicity) of R , then*

$$\sum_{1 \leq j_1 < \dots < j_k \leq n} z_{i_1} \cdots z_{i_k} = \frac{(-1)^k a_{n-k}}{a_n}.$$

The Newton–Girard identities involve symmetric polynomials, so we start with a few definitions and notational conventions. The k th power sum in n variables is the polynomial $p_{k,n} = \sum_{i=1}^n z_i^k$. The elementary symmetric polynomials in n variables are defined by

$$e_0 = 1, \quad e_1 = \sum_{i=1}^n z_i, \quad e_2 = \sum_{1 \leq i < j \leq n} z_i z_j, \quad \dots, \quad e_n = z_1 z_2 \cdots z_n, \quad e_k = 0 \quad \forall k > n.$$

Theorem 7.2 (Newton–Girard identities). *Let e_m denote the m th elementary symmetric polynomial in n variables and P_m denote the m th power sum in n variables. We have that*

$$\sum_{l=0}^{k-1} \left((-1)^l e_l P_{k-l} \right) + (-1)^k k e_k = 0 \quad \forall k, n \in \mathbb{N}.$$

The first two Newton–Girard identities are

- (1) For $k = 1$: $P_1 - e_1 = 0$.
- (2) For $k = 2$: $P_2 - e_1 P_1 + 2e_2 = 0$.

Solving the second formula for P_2 , we obtain $P_2 = e_1 P_1 - 2e_2$. Now let n be the degree of some polynomial R that satisfies $\dagger_2$ or $\ddagger_2$, and plug the n roots (counting multiplicity) of R into P_2 . The identity becomes $P_2 = -2e_2$ since $\dagger_2$ (or $\ddagger_2$) implies that $e_1 = P_1 = 0$. By Vieta’s formula, we have that $e_2 = \frac{a_{n-2}}{a_n}$. So we may conclude by stating that

$$P_2 = \sum_{R(\zeta)=0} \zeta^2 = \frac{-2a_{n-2}}{a_n}.$$

Although we only made use of the first two identities here, the other identities will likely be of use when the underlying kernel function is changed.

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