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MEAN SQUARE CONVERGENCE OF FUNCTIONAL CONDITIONAL DISTRIBUTION ESTIMATORS UNDER TWICE CENSORING

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Abstract. In this work, we propose a novel kernel estimator for the conditional distribution in the twice censored model, with functional regressors. We next analyze its mean square convergence, with an explicit rate given in Theorem 3.2. To validate our theoretical result, we conduct a simulation study that demonstrates the estimator's accuracy and performance, further supporting the robustness of our method, even with twice censored data.

1. Introduction

The conditional distribution function is of utmost importance in nonparametric statistics because it provides a flexible and versatile approach to understanding the relationship between the response variable given a value of the regressor variable. It enables the prediction of the response variable's value when encountering a new observation of the covariate, without requiring a predefined model for the data's underlying distribution. Foundational results in this area were established early on and have been continuously developed through subsequent research, as seen in [17], [9] and [15], when the covariate is scalar and multivariate.

Over the past few decades, functional data modeling has seen significant growth, resulting in a substantial increase in research aimed at enhancing our understanding and analysis of this type of data. Comprehensive overviews of the field can be found in sources such as [5], [4] and [6], which examine the asymptotic properties of stimates for conditional quantiles in the framework of functional complete data.

Regrettably, it is widely recognized that in various research fields, the variable of interest cannot be directly observed. This is especially the case when the variable of interest is subject to right-censoring, a type of censoring commonly encountered in practical settings such as healthcare, insurance, life sciences, social sciences, environmental science, and finance. See for example [8] and [7] for the functional and right-censored data.

However, the researchers may face more complex random censoring situations,

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such as the twice-censoring model, which involves both right and left censoring simultaneously. Such situations can be found in [3] and [16]. The latter introduced a product-limit distribution estimator and [10] subsequently analyzed its rate of almost-complete convergence. Additionally, [11] and [1] focused on kernel estimation of regression and conditional density when the regressors are scalar, while [13] and [14] studied kernel regression estimator when the covariates are of a functional type.

It is important to note that the presence of censoring increases the complexity of calculating the asymptotic variance and the rate of mean square convergence, and to our knowledge, no study has provided the asymptotic mean squared error for the kernel conditional distribution estimator in the functional and twice censorship model. This gap in the literature motivated us to address the issue. Specifically, in Section 2, we introduce a novel kernel conditional distribution estimate. Section 3 is devoted to establishing its rate of mean square convergence. In Section 4, we analyze the performance of our estimator.

In the following, let us consider any real random variable W. Let $F_W(t) = P(W \leq t)$ represent the distribution function, and $S_W(t) = 1 - F_W(t)$ denote the survival function. The upper and lower endpoints of the support of W are given by $T_W = \sup\{t \in \mathbb{R},\ F_W(t) < 1\}$ and $I_W = \inf\{t \in \mathbb{R},\ F_W(t) > 0\}$, respectively. For any right-continuous function $\varphi(t) : \mathbb{R} \to \mathbb{R}$, we define $\varphi(t^-) = \lim_{\epsilon \overset{>}{\to} 0} \varphi(t - \epsilon)$, whenever this limit exists. Additionally, for any set B, let 1_B denote the indicator function of the set B.

2. Constructing the new kernel estimator

Let us consider n independent pairs of random variables $(X_i, Y_i)_{i=1,\dots,n}$ which are assumed drawn from the pair (X, Y). This later is valued in $\mathcal{F} \times \mathbb{R}$, where $\mathcal{F}$ is a semi metric space equipped with a semi-metric d.

Throughout this paper, we assume that there exists a regular version of the conditional distribution of Y given X, defined by $F_Y^x(y) = P(Y \leq y | X = x)$ for a fixed point $x \in \mathcal{F}$. As noted by [5], the kernel estimator of $F_Y^x(y)$ in the complete data case is given by

$$\overline{F}_Y^x(y) = \frac{\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right) J\left(\frac{y - Y_i}{h_J}\right)}{\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right)}, \quad \left(\frac{0}{0} := 0\right),$$

where K and J are kernels, with $J(u) = \int_{-\infty}^u J_0(v)dv$ where J_0 is a kernel of type 0 (see [5]) and $h_K = h_{K,n}$ (resp. $h_J = h_{J,n}$) is a sequence of strictly positive real numbers that serves as a smoothing parameter.

Regrettably, Y is twice censored in our setting, so, we are restricted to observing a sample of $(X_i, Z_i, A_i)_{1 \leq i \leq n}$ independent and identically distributed (i.i.d.) as $(X, Z = \max(\min(Y, R), L), A)$. Here, $A = 1_{(L<R<Y)} + 2 \times 1_{(\min(Y,R) \leq L)}$ acts as the censored indicator, and R and L are nonnegative independent variables corresponding the right and left censoring variables, respectively. This model has been explored in [16].

To guarantee the identifiability of the presented model, we posit throughout the paper that (R, L) and (X, Y) are independent. It is straightforward to prove that

$$E\left(1_{\{A=0\}}|(X,Y)\right) = \frac{F_L(Y^-) - F_Z(Y^-)}{S_Y(Y^-)}, \tag{2.1}$$

Therefore, according to the idea of [11], we have the following formula for each function $g : \mathbb{R} \times \mathcal{F} \to \mathbb{R}$

$$E[g(X,Y)] = \frac{1}{n}\sum_{i=1}^{n} g(X_i, Z_i)\frac{S_Y(Z_i^-)\ 1_{\{A_i=0\}}}{F_L(Z_i^-) - F_Z(Z_i^-)}.$$

In this framework, we can construct a kernel pseudo estimate of $F_Y^x(y)$ by

$$\widetilde{F}_n^x(y) = \frac{\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right) J\left(\frac{y-Z_i}{h_J}\right) \frac{S_Y(Z_i^-)\ 1_{\{A_i=0\}}}{F_L(Z_i^-)-F_Z(Z_i^-)}}{\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right)}, \qquad \left(\frac{0}{0} := 0\right). \tag{2.2}$$

In practice, the estimator $\widetilde{F}_n^x(y)$ can not be used directly unless the functions F_Z, F_L, and S_Y are replaced by their estimators given respectively by

$$\widehat{F}_Z(t) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{Z_i \le t\}}, \tag{2.3}$$

$$\widehat{F}_L(t) = \prod_{j/Z'_j > t}\left(1 - \frac{D_j}{n\widehat{F}_Z\left(Z'_j\right)}\right), \tag{2.4}$$

and

$$\widehat{S}_Y(t) = 1 - \widehat{F}_Y(t) = \prod_{j/Z_j \le t}\left(1 - \frac{1_{\{A_j=0\}}}{n\left(\widehat{F}_L\left(Z_j^-\right) - \widehat{F}_Z\left(Z_j^-\right)\right)}\right), \tag{2.5}$$

where $\left(Z'_j\right)_{1\le j\le m}$ $(m \le n)$ denote the distinct values of $(Z_i)_{1\le i\le n}$ and $D_j = \sum_{i=1}^n 1_{\left\{Z_i = Z'_j, A_i = 2\right\}}$.
Consequently, the feasible estimator of $F_Y^x(y)$ is provided by

$$\widehat{F}_n^x(y) = \frac{\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right) J\left(\frac{y-Z_i}{h_J}\right) \frac{\widehat{S}_Y(Z_i^-)\ 1_{\{A_i=0\}}}{\widehat{F}_L(Z_i^-)-\widehat{F}_Z(Z_i^-)}}{\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right)}, \qquad \left(\frac{0}{0} := 0\right). \tag{2.6}$$

The estimators (2.2) and (2.6) can be written as follows

$$\widetilde{F}_n^x(y) = \frac{\frac{1}{nE\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]}\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right) J\left(\frac{y-Z_i}{h_J}\right) \frac{S_Y(Z_i^-)\ 1_{\{A_i=0\}}}{F_L(Z_i^-)-F_Z(Z_i^-)}}{\frac{1}{nE\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]}\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right)} := \frac{\widetilde{F}_N^x(y)}{\widehat{F}_D(x)}, \tag{2.7}$$

and

$$\widehat{F}_n^x(y) = \frac{\frac{1}{nE\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]}\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right) J\left(\frac{y-Z_i}{h_J}\right)\frac{\widehat{S}_Y(Z_i^-)\,1_{\{A_i=0\}}}{\widehat{F}_L(Z_i^-)-\widehat{F}_Z(Z_i^-)}}{\frac{1}{nE\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]}\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right)} := \frac{\widehat{F}_N^x(y)}{\widehat{F}_D(x)}. \tag{2.8}$$

3. Mean square convergence

Let $\mathcal{C} \subset]I_Z, \min(T_Y, T_R)[$ be compact on $\mathbb{R}$. The goal of this section is to prove the mean square convergence of $\widehat{F}_n^x(y)$ on $\mathcal{C}$, with an explicit rate given in theorem 3.2. To achieve this, consider the commonly required conditions in the nonparametric estimation context.

($\mathbf{H_1}$) $\max(I_L, I_R) < I_Y$.

($\mathbf{H_2}$) $\forall h > 0, P(X \in B(x,h)) = \phi_x(h) > 0$, where $B(x,h) := \{y \in \mathcal{F},\ d(x,y) \le h\}$ represents a closed ball in $\mathcal{F}$ centered at x with radius h.

($\mathbf{H_3}$) The kernel $K : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a density function and there exist two real constants $0 < C' < C < \infty$, such that

$$C' 1_{[0,1]}(t) \le K(t) \le C 1_{[0,1]}(t).$$

($\mathbf{H_4}$) $\exists b_1 > 0, \exists b_2 > 0 \quad \exists C > 0, \forall x' \in B(x, h_K), \forall (y, y') \in \mathcal{C} \times]y - h_J, y + h_J[$,

$$\left|F^x(y) - F^{x'}(y')\right| \le C\left(d^{b_1}(x,x') + |y-y'|^{b_2}\right).$$

($\mathbf{H_5}$) The bandwidths h_K and h_J satisfy

$$\lim_{n\to\infty} h_J = 0,\ \lim_{n\to\infty} h_K = 0,\ \lim_{n\to\infty} n\phi_x(h_K) = +\infty,\ \phi_x(h_K) = O\left(\frac{1}{\log n}\right) \text{ and } \lim_{n\to\infty}\frac{\log n}{n\phi_x(h_K)} = 0.$$

The condition ($\mathbf{H_1}$) plays a key role in ensuring the identifiability of the twice censored model, as discussed in works such as [10] and [2]. The condition ($\mathbf{H_2}$) is crucial to our analysis, as it supports key aspects of the study. The assumption of regularity ($\mathbf{H_4}$) enables us to evaluate the bias term, which is essential for deriving our results. Finally, The specific assumptions ($\mathbf{H_3}$) and ($\mathbf{H_5}$) are fundamental for supporting the proofs of our results.

Remark 3.1. Since J has a compact support and h_J tends to 0, we are able to take $Z_i \in [\theta', \theta]$ in both sums, for all large enough n, where $[\theta', \theta] \subset]I_Z, \min(T_Y, T_R)[$, and it is assumed that in (2.2) and (2.6), there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\forall i \in \{1, 2, ..., n\},\ \theta' \le Z_i \le \theta.$$

So, in view of ($\mathbf{H_1}$)

$$F_L(Z_i^-) - F_Z(Z_i^-) = F_L(Z_i^-) S_Y(Z_i^-) S_R(Z_i^-) \neq 0. \tag{3.1}$$

It is obvious that $S_R(\theta) > 0$ and $F_L(\theta') > 0$.

We can now present our result.

Theorem 3.2. *Given that conditions* $(\mathbf{H_1})$*–*$(\mathbf{H_5})$ *hold, then*

$$\sup_{y\in\mathcal{C}} \operatorname{MSE}\left(\widehat{F}_n^x(y)\right) = \sup_{y\in\mathcal{C}} \mathbb{E}\left(\widehat{F}_n^x(y) - F^x(y)\right)^2 = O\left(h_K^{2b_1} + h_J^{2b_2} + h_K^{b_1}h_J^{b_2}\right) + O\left(\frac{1}{n\phi_x(h_K)}\right).$$

Remark 3.3. It is possible to obtain convergence under a weaker condition than $(\mathbf{H}_4)$, which is

$(\mathbf{H'_4})$ $\forall y \in \mathcal{S}$, $\lim_{d(x',x)\to 0} F^{x'}(y) = F^x(y)$ and $\lim_{|y'-y|\to 0} F^x(y') = F^x(y)$.

However, this condition ensures mean square convergence, but it does not provide it rate. Thus, $\sup_{y\in\mathcal{C}} \operatorname{MSE}\left(\widehat{F}_n^x(y)\right) \longrightarrow 0$.

Proof. The demonstration of the theorem 3.2 follows directly from the relation

$$\mathbb{E}\left(\widehat{F}_n^x(y) - F^x(y)\right)^2 \leq 2\mathbb{E}\left[\left(\widehat{F}_n^x(y) - \widetilde{F}_n^x(y)\right)^2\right] + 4\operatorname{Var}\left(\widetilde{F}_n^x(y)\right) + 4\left(E\left(\widetilde{F}_n^x(y)\right) - F^x(y)\right)^2.$$

Thus, to complete the proof, it is necessary to examine the following lemmas in detail. □

Lemma 3.4. *Given conditions* $(\mathbf{H}_1) - (\mathbf{H}_3)$ *and* $(\mathbf{H}_5)$*, we get*

$$\sup_{y\in\mathcal{C}} \mathbb{E}\left[\left(\widehat{F}_n^x(y) - \widetilde{F}_n^x(y)\right)^2\right] = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

Proof. Referring to lemma 1-(ii) in [2], we find that there is a constant $b > 0$ satisfies

$$\begin{aligned}\sup_{y\in\mathcal{C}} \mathbb{E}\left[\left(\widehat{F}_n^x(y) - \widetilde{F}_n^x(y)\right)^2\right] =& \sup_{y\in\mathcal{C}} \int_{\mathcal{B}} \left[\widehat{F}_n^x(y) - \widetilde{F}_n^x(y)\right]^2 dP + \sup_{y\in\mathcal{C}} \int_{\mathcal{B}^c} \left[\widehat{F}_n^x(y) - \widetilde{F}_n^x(y)\right]^2 dP \\ :=& M_{n,1}(x,y) + M_{n,2}(x,y),\end{aligned}$$

where

$$\mathcal{B} = \left\{ \sup_{\theta'\leq t\leq\theta} \left| \frac{\widehat{S}_Y(t^-)}{\widehat{F}_L(t^-) - \widehat{F}_Z(t^-)} - \frac{S_Y(t^-)}{F_L(t^-) - F_Z(t^-)} \right| \leq b\sqrt{\frac{\log n}{n}} \right\},$$

and

$$\mathcal{B}^c = \left\{ \sup_{\theta'\leq t\leq\theta} \left| \frac{\widehat{S}_Y(t^-)}{\widehat{F}_L(t^-) - \widehat{F}_Z(t^-)} - \frac{S_Y(t^-)}{F_L(t^-) - F_Z(t^-)} \right| > b\sqrt{\frac{\text{logn}}{n}} \right\}.$$

The application of lemma 5 in [13], which states that

$$\sum_{n=1}^{+\infty} P\left(\widehat{F}_D(x) < \frac{1}{2}\right) < \infty, \tag{3.2}$$

gives

$$\begin{aligned}
M_{n,1}(x,y) :=& \sup_{y\in\mathcal{C}} \int_{\mathcal{B}} \left[\widehat{F}_n^x(y) - \widetilde{F}_n^x(y)\right]^2 dP \\
\leq& \frac{b^2 \log n}{n^3 \widehat{F}_D^2(x) E^2\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \sup_{y\in\mathcal{C}} E\left[\left(\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right) J\left(\frac{y-Y_i}{h_J}\right)\right)^2\right] \\
\leq& \frac{4b^2 \log n}{n^3 E^2\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \sup_{y\in\mathcal{C}} \sum_{i=1}^n E\left[K^2\left(\frac{d(x,X_i)}{h_K}\right) J^2\left(\frac{y-Y_i}{h_J}\right)\right] \\
&+ \frac{4b^2 \log n}{n^3 E^2\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \sup_{y\in\mathcal{C}} \sum_{\substack{1\leq i,j\leq n\\ i\neq j}} E\left[K\left(\frac{d(x,X_i)}{h_K}\right) J\left(\frac{y-Y_i}{h_J}\right)\right] E\left[K\left(\frac{d(x,X_j)}{h_K}\right)\right. \\
&\left. J\left(\frac{y-Y_j}{h_J}\right)\right].
\end{aligned}$$

The boundedness of J, combined with the fact that

$$\forall m \geqslant 1,\ \exists 0 < C' < C < +\infty;\ \ C'\phi_x(h_K) \leq \mathbb{E}\left[K^m\left(\frac{d(x,X_1)}{h_K}\right)\right] \leq C\phi_x(h_K),$$

imply that

$$\begin{aligned}
M_{n,1}(x,y) \leq& \frac{4Cb^2 n \log n \phi_x(h_K)}{n^3\phi_x^2(h_K)} + \frac{4Cb^2(n^2-n)\log n \phi_x^2(h_K)}{n^3\phi_x^2(h_K)} \\
\leq& \frac{C\log n}{n^2\phi_x(h_K)} + \frac{C(n-1)\log n}{n^2\phi_x(h_K)} \\
=& O\left(\frac{1}{n\phi_x(h_K)}\right),
\end{aligned}$$

Now, we proceed to examine the term $M_{n,2}(x,y)$

$$\begin{aligned}
M_{n,2}(x,y) :=& \sup_{y\in\mathcal{C}} \int_{\mathcal{B}^c} \left[\widehat{F}_n^x(y) - \widetilde{F}_n^x(y)\right]^2 dP \\
=& \sup_{y\in\mathcal{C}} \int_{\mathcal{B}^c} \frac{1}{n^2 \widehat{F}_D^2(x)\mathbb{E}^2\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \left[\sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right) J\left(\frac{y-Z_i}{h_J}\right) 1_{\{A_i=0\}}\right. \\
&\left.\times\left(\frac{\widehat{S}_Y(Z_i^-)}{\widehat{F}_L(Z_i^-) - \widehat{F}_Z(Z_i^-)} - \frac{S_Y(Z_i^-)}{F_L(Z_i^-) - F_Z(Z_i^-)}\right)\right]^2 dP.
\end{aligned}$$

Given that, for $i \in \{1,\ldots,n\}$ such that $A_i = 0$, we have

$$\frac{\widehat{S}_Y(Z_i^-)}{n\left(\widehat{F}_L(Z_i^-) - \widehat{F}_Z(Z_i^-)\right)} = \Delta\widehat{F}_Y\left(Z_i\right) \leq 1,$$

So, the fact that $Z_i \in [\theta', \theta]$ for n large enough, gives

$$\begin{aligned}
M_{n,2}(x,y) \leq & \left(n + \frac{1}{F_L(\theta'^-)S_R(\theta^-)}\right)^2 \sup_{y\in\mathcal{C}} \int_{\mathcal{B}^c} \frac{1}{\widehat{F}_D^2(x)} \left[\frac{1}{n\mathbb{E}\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \sum_{i=1}^n K\left(\frac{d(x,X_i)}{h_K}\right)\right. \\
& \left. J\left(\frac{y-Z_i)}{h_J}\right)\right]^2 dP \\
& \leq C\left(n + \frac{1}{F_L(\theta'^-)S_R(\theta^-)}\right)^2 \times \\
& P\left(\sup_{\theta'\leq t\leq\theta}\left|\frac{\widehat{S}_Y(t^-)}{\widehat{F}_L(t^-)-\widehat{F}_Z(t^-)} - \frac{S_Y(t^-)}{F_L(t^-)-F_Z(t^-)}\right| > b\sqrt{\frac{\log n}{n}}\right),
\end{aligned}$$

thanks to the relation (3.2).
In view of the lemma 1-(ii) in [2] and considering that $\gamma > 3$, we get

$$P\left(\sup_{\theta'\leq t\leq\theta}\left|\frac{\widehat{S}_Y(t^-)}{\widehat{F}_L(t^-)-\widehat{F}_Z(t^-)} - \frac{S_Y(t^-)}{F_L(t^-)-F_Z(t^-)}\right| > b\sqrt{\frac{\log n}{n}}\right) = O\left(n^{-\gamma}\right). \tag{3.3}$$

Thus,

$$M_{n,2}(x,y) \leq C\left(n + \frac{1}{F_L(\theta'^-)S_R(\theta^-)}\right)^2 n^{-\gamma}.$$

The hypothesis $(\mathbf{H_5})$ leads to

$$M_{n,2}(x,y) = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

□

Lemma 3.5. *Under assumptions* $(\mathbf{H}_1) - (\mathbf{H}_3)$ *and* $(\mathbf{H}_5)$, *we derive*

$$\sup_{y\in\mathcal{C}} \mathrm{Var}\left(\widetilde{F}_n^x(y)\right) = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

Proof. Considering assumption $(\mathbf{H_1})$ and the relation (3.1), we obtain

$$\widetilde{F}_n^x(y) = \frac{\sum_{i=1}^n K(h_K^{-1}d(X_i,x))J(h_J^{-1}(y-Z_i))\frac{1_{\{A_i=0\}}}{F_L(Z_i^-)S_R(Z_i^-)}}{\sum_{i=1}^n K(h_K^{-1}d(X_i,x))}.$$

Based on the expression (8.12) in [18] and since $\mathbb{E}\left(\widehat{F}_D(x)\right) = 1$, we get

$$\begin{aligned}
\mathrm{Var}\left(\widetilde{F}_n^x(y)\right) &= \frac{\mathrm{Var}\left(\widetilde{F}_N^x(y)\right)}{\left(\mathbb{E}\left(\widehat{F}_D(x)\right)\right)^2} - 2\mathbb{E}\left(\widetilde{F}_N^x(y)\right)\frac{\mathrm{Cov}\left(\widetilde{F}_N^x(y),\widehat{F}_D(x)\right)}{\left(\mathbb{E}\left(\widehat{F}_D(x)\right)\right)^3} + \mathrm{Var}\left(\widehat{F}_D(x)\right)\frac{\mathbb{E}^2\left(\widetilde{F}_N^x(y)\right)}{\left(\mathbb{E}\left(\widehat{F}_D(x)\right)\right)^4} \\
&= \mathrm{Var}\left(\widetilde{F}_N^x(y)\right) - 2\mathbb{E}\left(\widetilde{F}_N^x(y)\right)\mathrm{Cov}\left(\widetilde{F}_N^x(y),\widehat{F}_D(x)\right) + \mathrm{Var}\left(\widehat{F}_D(x)\right)\mathbb{E}^2\left(\widetilde{F}_N^x(y)\right).
\end{aligned}$$

Thus, it is straightforward to write that

$$\sup_{y\in\mathcal{C}} \operatorname{Var}\left(\widetilde{F}_n^x(y)\right) \leq C\left[\sup_{y\in\mathcal{C}}\left(\operatorname{Var}\left(\widetilde{F}_N^x(y)\right)\right) + \sup_{y\in\mathcal{C}}\left(\mathbb{E}\left(\widetilde{F}_N^x(y)\right)\operatorname{Cov}\left(\widetilde{F}_N^x(y), \widehat{F}_D(x)\right)\right) + \operatorname{Var}\left(\widehat{F}_D(x)\right) \right.$$
$$\left. \times \sup_{y\in\mathcal{C}}\left(\mathbb{E}^2\left(\widetilde{F}_N^x(y)\right)\right)\right].$$

- **Analysis of the term** $\sup_{y\in\mathcal{C}} \operatorname{Var}\left(\widetilde{F}_N^x(y)\right)$.

  According to the definition of $\widetilde{F}_N^x(y)$ in (2.7) and given the conditions $(\mathbf{H}_1)-(\mathbf{H}_3)$ and $(\mathbf{H}_5)$, we can deduce

$$\begin{aligned}
\sup_{y\in\mathcal{C}} \operatorname{Var}\left(\widetilde{F}_N^x(y)\right) &= \frac{1}{nE^2\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \sup_{y\in\mathcal{C}} \operatorname{Var}\left(K\left(\frac{d(x,X_1)}{h_K}\right) J\left(\frac{y-Z_1}{h_J}\right) 1_{\{A_1=0\}} \frac{1}{F_L(Z_1^-)S_R(Z_1^-)}\right) \\
&\leq \frac{1}{n\mathbb{E}^2\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \frac{1}{F_L^2(\theta'^-)S_R^2(\theta^-)} \sup_{y\in\mathcal{C}} \mathbb{E}\left[K^2\left(\frac{d(x,X_1)}{h_K}\right) J^2\left(\frac{y-Y_1}{h_J}\right)\right] \\
&\leq \frac{C}{nF_L^2(\theta'^-)S_R^2(\theta^-)\,\mathbb{E}^2\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \mathbb{E}\left[K^2\left(\frac{d(x,X_1)}{h_K}\right)\right] \\
&\leq \frac{C\phi_x(h_K)}{nF_L^2(\theta'^-)S_R^2(\theta^-)\,\phi_x^2(h_K)} \\
&= O\left(\frac{1}{n\phi_x(h_K)}\right).
\end{aligned} \tag{3.4}$$

- **Analysis of the term** $\operatorname{Var}\left(\widehat{F}_D(x)\right)$.

  The hypotheses $(\mathbf{H}_2)-(\mathbf{H}_3)$ and $(\mathbf{H}_5)$ and the definition of $\widehat{F}_D(x)$ in (2.8) provide

$$\begin{aligned}
\operatorname{Var}\left(\widehat{F}_D(x)\right) &= \frac{1}{n\mathbb{E}^2\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \operatorname{Var}\left(K\left(\frac{d(x,X_1)}{h_K}\right)\right) \\
&\leq \frac{1}{n\mathbb{E}^2\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]} \mathbb{E}\left(K^2\left(\frac{d(x,X_1)}{h_K}\right)\right) \\
&\leq \frac{C\phi_x(h_K)}{n\phi_x^2(h_K)} \\
&= O\left(\frac{1}{n\phi_x(h_K)}\right).
\end{aligned} \tag{3.5}$$

- **Analysis of the term** $\sup_{y\in\mathcal{C}} \mathrm{Cov}\left(\widetilde{F}_N^x(y), \widehat{F}_D(x)\right)$.
  The Cauchy–Schwarz inequality and $\lim n\phi_x(h_K) = +\infty$, give

$$\begin{aligned}\sup_{y\in\mathcal{C}} \mathrm{Cov}\left(\widetilde{F}_N^x(y), \widehat{F}_D(x)\right) &\le \sup_{y\in\mathcal{C}} \sqrt{\mathrm{Var}\left(\widetilde{F}_N^x(y)\right)\mathrm{Var}\left(\widehat{F}_D(x)\right)} \\ &= O\left(\frac{1}{n\phi_x(h)}\right).\end{aligned} \tag{3.6}$$

The proof is completed by noting that

$$\begin{aligned}\mathbb{E}\left(\widetilde{F}_N^x(y)\right) &= \frac{1}{\mathbb{E}\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]}\mathbb{E}\left(K\left(\frac{d(x,X_1)}{h_K}\right)J\left(\frac{y-Z_1}{h_J}\right)1_{\{A_1=0\}}\frac{1}{F_L(Z_1^-)S_R(Z_1^-)}\right) \\ &\le \frac{C}{F_L\,(\theta'^-)S_R(\theta^-)}.\end{aligned}$$

□

Lemma 3.6. *Under hypotheses of theorem 3.2, we obtain*

$$\sup_{y\in\mathcal{C}}\left[E\left(\widetilde{F}_n^x(y)\right) - F^x(y)\right]^2 = O\left(h_K^{2b_1} + h_J^{2b_2} + h_K^{b_1}h_J^{b_2}\right) + O\left(\frac{1}{n\phi_x(h_K)}\right).$$

Proof. By employing the usual identity $\frac{1}{u} = \frac{1}{v} - \frac{u-v}{v^2} + \frac{(u-v)^2}{uv^2}$ ($u \neq 0$ and $v \neq 0$) with $u = \widehat{F}_D(x)$ and $v = \mathbb{E}(\widehat{F}_D(x)) = 1$, we get

$$\begin{aligned}\sup_{y\in\mathcal{C}}\left(E(\widetilde{F}_n^x(y)) - F^x(y)\right)^2 &= \sup_{y\in\mathcal{C}}\left\{\left(\mathbb{E}(\widetilde{F}_N^x(y)) - F^x(y)\right) - \mathbb{E}\left(\widetilde{F}_N^x(y)\left(\widehat{F}_D(x)-1\right)\right) + \right.\\ &\left.\mathbb{E}\left(\widetilde{F}_n^x(y)\left(\widehat{F}_D(x)-1\right)^2\right)\right\}^2 \\ &= \sup_{y\in\mathcal{C}}\left\{\left(\mathbb{E}(\widetilde{F}_N^x(y)) - F^x(y)\right) - \mathrm{Cov}\left(\widetilde{F}_N^x(y), \widehat{F}_D(x)\right) + \right.\\ &\left.\mathbb{E}\left(\widetilde{F}_n^x(y)\left(\widehat{F}_D(x)-1\right)^2\right)\right\}^2 \\ &\le 3\sup_{y\in\mathcal{C}}\left\{\left(\mathbb{E}(\widetilde{F}_N^x(y)) - F^x(y)\right)\right\}^2 + 3\sup_{y\in\mathcal{C}}\left\{\mathrm{Cov}\left(\widetilde{F}_N^x(y), \widehat{F}_D(x)\right)\right\}^2 \\ &+ 3\sup_{y\in\mathcal{C}}\left\{\mathbb{E}\left(\widetilde{F}_n^x(y)\left(\widehat{F}_D(x)-1\right)^2\right)\right\}^2 \\ &:= J_{n,1}(x,y) + J_{n,2}(x,y) + J_{n,3}(x,y).\end{aligned}$$

- **Analysis of the term** $J_{n,1}(x,y) = 3\sup_{y\in\mathcal{C}}\left\{\left(\mathbb{E}(\tilde{F}_N^x(y)) - F^x(y)\right)\right\}^2$.
  On the one hand, the property of conditional expectation together with

the formula (2.1) give

$$\mathbb{E}(\widetilde{F}_N^x(y) = \mathbb{E}\left(\frac{1}{nE\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]}\sum_{i=1}^{n} K\left(\frac{d(x,X_i)}{h_K}\right) J\left(\frac{y-Z_i}{h_J}\right)\frac{1_{\{A_i=0\}}}{F_L(Z_i^-)S_R(Z_i^-)}\right)$$
$$= \frac{1}{\mathbb{E}\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]}\mathbb{E}\left[K\left(\frac{d(x,X_1)}{h_K}\right)\mathbb{E}\left(J\left(\frac{y-Y_i}{h_J}\right)\mathbb{E}\left(\frac{1_{\{A_i=0\}}}{F_L(Y_1^-)S_R(Y_1^-)} \mid (X_1,Y_1)\right) \mid X_1\right)\right]$$
$$= \frac{1}{\mathbb{E}\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]}\mathbb{E}\left[K\left(\frac{d(x,X_1)}{h_K}\right)\mathbb{E}\left(J\left(\frac{y-Y_i}{h_J}\right) \mid X_1\right)\right],$$

thus,

$$\left\{\mathbb{E}(\widetilde{F}_N^x(y) - F^x(y)\right\}^2 = \left\{\frac{1}{\mathbb{E}\left[K\left(\frac{d(x,X_1)}{h_K}\right)\right]}\mathbb{E}\left(K\left(\frac{d(x,X_1)}{h_K}\right)\left[\mathbb{E}\left(J\left(\frac{y-Y_i}{h_J}\right) \mid X_1\right) - F^x(y)\right]\right)\right\}^2.$$

On the other hand, as outlined in [12], we have

$$\sup_{y\in\mathcal{C}}\left|\mathbb{E}\left(J\left(\frac{y-Y_i}{h_J}\right) \mid X_1\right) - F^x(y)\right| = O\left(h_K^{b_1} + h_J^{b_2}\right).$$

Therefore, we can conclude

$$J_{n,1}(x,y) = 3\sup_{y\in\mathcal{C}}\left\{\left(\mathbb{E}(\widetilde{F}_N^x(y)) - F^x(y)\right)\right\}^2 = O\left(h_K^{2b_1} + h_J^{2b_2} + h_K^{b_1}h_J^{b_2}\right).$$

- **Analysis of the term** $J_{n,2}(x,y) = 3\sup_{y\in\mathcal{C}}\left\{\mathrm{Cov}\left(\widetilde{F}_N^x(y), \widehat{F}_D(x)\right)\right\}^2$.
  By applying equation (3.6) and condition ($\mathbf{H}_5$), we obtain

$$J_{n,2}(x,y) = O\left(\frac{1}{n^2\phi_x^2(h_K)}\right) = O\left(\frac{1}{n\phi_x(h_K)}\right).$$

- **Analysis of the term** $J_{n,3}(x,y) = 3\sup_{y\in\mathcal{C}}\left\{\mathbb{E}\left[\widetilde{F}_n^x(y)\left(\widehat{F}_D(x) - 1\right)^2\right]\right\}^2$.
  Given that J is bounded, we get

$$\widetilde{F}_n^x(y) \le \frac{C}{F_L\left(\theta'^-\right)S_R\left(\theta^-\right)}.$$

Therefore, noting that $\lim_{n\to\infty} n\phi_x(h_K) = +\infty$, and by invoking equation (3.5), we deduce that

$$J_{n,3}(x,y) \le \frac{C}{F_L^2\left(\theta'^-\right)S_R^2\left(\theta^-\right)}\left\{\mathbb{E}\left(\widehat{F}_D(x) - 1\right)^2\right\}^2$$
$$\le C\left\{\mathrm{Var}\left(\widehat{F}_D(x)\right)\right\}^2$$
$$= O\left(\frac{1}{n\phi_x(h_K)}\right).$$

□

4. Simulation study

This section seeks to evaluate the performance of our estimate (KFE). For this aim, we present the proposed nonparametric functional regression model

$$Y = m(X) + \epsilon,$$

where X and ϵ are independent and the function m is defined by

$$m(X) = \frac{1}{4}\exp\left\{2 - \frac{1}{\left(\int_0^1 X'(t)dt\right)^2}\right\},$$

where the functional covariate $X(t)$ is defined, for $t \in [0, 2\pi/3]$ by

$$X(t) = 2 - \cos\left(W\left(t - \frac{2\pi}{3}\right)\right),$$

with $W \rightsquigarrow \mathcal{N}(0,1)$. The curves are sampled on the same grid which is composed of 200-equidistant values in $[0, 2\pi/3]$ which is represented in the Figure 1.
Then, we follow the steps below.

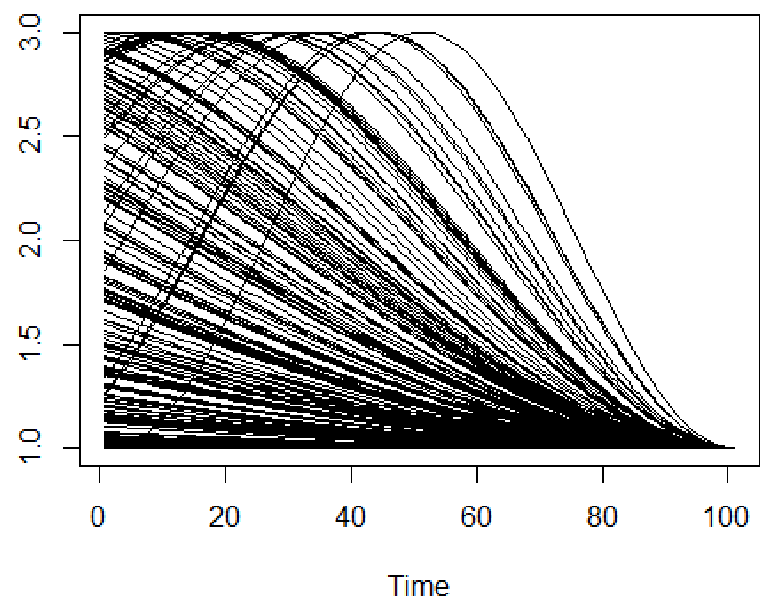

Figure 1. A sample of 200 curves representing a realization of the functional random variable X.

• **Step 1.** For different values of the sample size $n = 100, 300, 500, 800$, we simulate n i.i.d. ϵ_i, the regressors X_i and we calcule Y_i. We also simulate the censored variables Ri R_i and L_i from exponential distribution with parameters λ_1 and λ_2 respectively which are adjusted to calibrate the rates of censoring (CR). Next, we compute the observed data $Z_i = \max(\min(Y_i, R_i), L_i)$ and the censored indicator $A_i = 1_{(L_i<R_i<Y_i)} + 2 \times 1_{(\min(Y_i,R_i)\leq L_i)}$ for all i.

• **Step 2.** We divide the data into two parts.
- $(X_i, Z_i, A_i)_{1\le i\le n_1}$: The learning sample utilized for constructing the estimator, where $n_1 = n/2$.
- $(X_i, Z_i, A_i)_{n_1+1\le i\le n}$: The testing sample designated for the comparison.

• **Step 3.** After calculating the estimates $\widehat{F}_Z(Z_i^-)$, $\widehat{F}_L(Z_i^-)$ and $\widehat{S}_Y(Z_i^-)$ according to (2.3), (2.4) and (2.5), we compute the estimate $\widehat{F}_n^x(X_i)$ in the formula (2.6) based on the learning sample.

• **Step 4.** To assess good performance, we compute the prediction errors (EMSE) for various values of n, as given by

$$EMSE := \frac{1}{n-n_1}\sum_{i=n_1+1}^{n}\Big(m(X_i) - \widehat{F}_n^x(X_i)\Big)^2,$$

then, we plot the EMSE against sample sizes in Figure 2.
In order to compute our estimate, we utilize the quadratic kernel $K(x) = \frac{3}{2}(1-x^2)1_{[0,1]}(x)$ and the distribution function $J(x) = \int_{-\infty}^{x}\frac{3}{4}(1-t^2)1_{[-1,1]}(t)dt$. The bandwidths h_K and h_J are selected by the 2-fold cross-validation method and the semi-metric d is constructed from the derivative provided in [5] (see routines "semimetric.deriv" in the websitehttp://www.lsp.upstlse.fr/staph/npfda).
The obtained results are represented in table 1.
The Figure 2 and Table 1 demonstrate that, similar to both complete and right

Table 1. $EMSE$ of the KFE estimate for varying sample sizes n.

n	100	300	500	800	1000
$EMSE$	0.0845	0.0584	0.0517	0.0476	0.0309

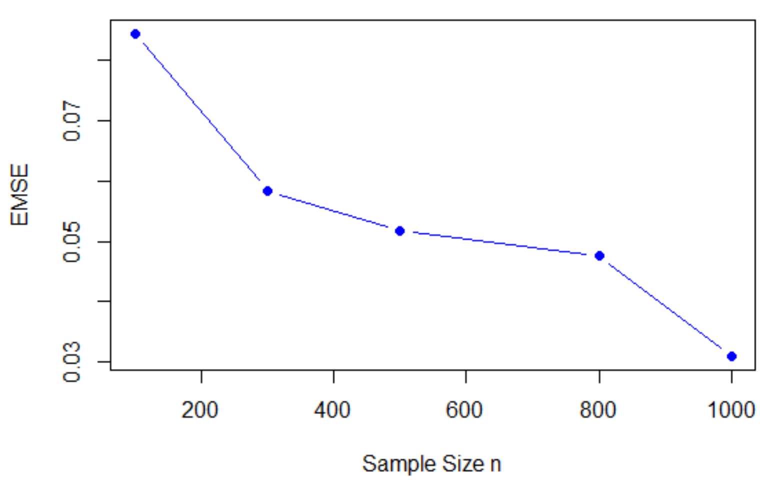

Figure 2. $EMSE$ for varying sample sizes n.

censored data, our kernel estimator performs better with greater sample sizes n, even in the case of twice censored data. These results highlight the accuracy of our estimator.

5. Conclusion

To summarize, the theoretical and practical analysis confirms that the functional conditional distribution kernel estimator performs reliably and is based on robust theoretical foundations, making it a valuable tool for addressing complex censored statistical contexts.

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