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S-STRONGLY 0-DIMENTIONAL RINGS

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Abstract. Let R be a commutative ring with identity, and let S be a not necessarily saturated multiplicative subset of R. We introduce the notions of strongly S-prime ideals and S-strongly 0-dimensional rings as S-analogues of strongly prime ideals and strongly 0-dimensional rings. We establish fundamental properties of these structures and provide characterizations supported by illustrative examples. We further obtain new criteria for S-Artinian rings, showing that such rings are always S-strongly 0-dimensional and finitely S-cogenerated. Finally, we examine several equivalent conditions under which a ring is compactly S-packed.

Keywords. S-strongly 0-dimensional ring, (strongly) S-prime ideal, S-Artinian ring, compactly S-packed ring, trivial ring extension.
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1. Introduction

Throughout this article, we focus exclusively on commutative rings with identity, which we denote by R. The notion of prime ideals and their generalizations play a central role in Commutative Algebra and Algebraic Geometry. They are used to characterize a large variety of rings and have applications in other areas such as General Topology and Graph Theory. Of course, a proper ideal $\mathfrak{p}$ of R is said to be a prime ideal if, whenever $ab \in \mathfrak{p}$ for some $a, b \in R$, then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Additionally, the sets of all prime ideals and maximal ideals are denoted by $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$, respectively.

A subset S of R is called multiplicative if it satisfies the following conditions: $1 \in S$, $0 \notin S$, and for every $a, b \in S$, we have $ab \in S$. For instance, the set of regular elements $\mathrm{reg}(R)$ and $S_P = R \setminus P$ (where $P \in \mathrm{Spec}(R)$) are both examples of multiplicative subsets.

One well-known property of prime ideals is that for a commutative ring R and a prime ideal P of R, if P contains the intersection of a finite family of ideals, then it contains at least one of those ideals (see [22, Theorem 1.4.3]). In [5], Gilmer examined when this property holds for an infinite family of ideals.

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In [12], Jayaram et al. defined prime ideals satisfying this property for any infinite family of ideals as strongly prime ideals. That is, a prime ideal P is called strongly prime if, whenever an infinite intersection of a family of ideals is contained in P, at least one of the ideals in that family is in P. Jayaram et al. called a ring strongly 0-dimensional if every prime ideal of the ring is strongly prime. They proved that strongly 0-dimensional rings are zero-dimensional and examined some properties of these rings, including their relation with von Neumann regular, Artinian, and Noetherian rings. In [7], Gottlieb conducted a further study on strongly prime ideals and strongly 0-dimensional rings. He provided some equivalent conditions for a ring to be strongly 0-dimensional and identified a class of strongly 0-dimensional rings, namely strongly n-regular rings. Additionally, the authors in [3, 12, 17] extended some of these properties to the case of R-modules, introducing and studying strongly prime submodules and strongly 0-dimensional modules.

We note that another type of ideal in commutative ring theory, also called strongly prime ideals, was introduced by Hedstrom and Houston in [10]. According to their definition, a prime ideal of a domain R with quotient field K is strongly prime if, for any $x, y \in K$, the condition $xy \in P$ implies that either $x \in P$ or $y \in P$. This concept is unrelated to the strongly prime ideals considered in this paper.

One of the most significant methods for generalizing classical rings and modules involves the use of multiplicative subsets S of a ring R (see, for example, [1, 8, 9, 13]). At the beginning of the 21st century, Anderson and Dumitrescu introduced the concept of an S-Noetherian ring in 2002 [1]. A ring R is defined as S-Noetherian if, for every ideal I of R, there exists a finitely generated sub-ideal $K \subseteq I$ such that $sI \subseteq K$ for some $s \in S$.

More recently, Sevim et al. [19] investigated the notion of S-prime ideals, which generalizes the concept of prime ideals. This notion was applied to characterize integral domains, certain prime ideals, fields, and S-Noetherian rings. Let S be a nonempty subset of R. An ideal $\mathfrak{p}$ of R is said to be an S-prime ideal if $P \cap S = \emptyset$ and there exists a fixed $s \in S$ such that whenever $ab \in P$ for some $a, b \in R$, then either $sa \in P$ or $sb \in P$ [4, 19]. In particular, the set of all S-prime ideals of R is denoted by $\mathrm{Spec}_S(R)$, which is called the S-spectrum of R. Note that according to [14, Proposition 2.1], $\mathrm{Spec}_S(R)$ is not empty.

An ideal $\mathfrak{p}$ of R is said to be an S-maximal ideal if $P \cap S = \emptyset$ and there exists a fixed $s \in S$ such that whenever $P \subseteq Q$ for some ideal Q of R, then either $sQ \subseteq P$ or $Q \cap S \neq \emptyset$ [24]. We denote by $\mathrm{Max}_S(R)$ the set of all S-maximal ideals of R. More recently, Yildiz et al. constructed a topology on $\mathrm{Spec}_S(R)$, and this topology is a generalization of classical Zariski topology [24]. They investigated some topological properties of $\mathrm{Spec}_S(R)$ such as connectedness, compactness, and separation axioms.

The principal aim of this paper is to introduce and investigate the S-versions of strongly prime ideals and strongly 0-dimensional rings. Let R be a commutative ring, and let S be a multiplicative subset of R.

An S-prime ideal P is defined to be S-strongly prime if there exists $s \in S$ such that, for any family of ideals $\{I_\alpha \mid \alpha \in \Gamma\}$ of R, whenever $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$, it follows that $sI_\beta \subseteq P$ for some $\beta \in \Gamma$.

A ring R is said to be S-strongly 0-dimensional if every S-prime ideal of R is S-strongly prime.

Several equivalent conditions are established for an S-prime ideal to be S-strongly prime and for a ring to be S-strongly 0-dimensional (see Proposition 2.3, Corollaries 2.5 and 2.4).

Notably, it is shown that every strongly 0-dimensional ring is S-strongly 0-dimensional, although the converse does not hold in general (see Corollary 2.6 and Example 2.7).

The paper further develops an S-version of Krull's theorem (Theorem 2.9) and demonstrates that every S-strongly 0-dimensional S-domain is an S-field (Proposition 2.13).

Additionally, we establish that if P is a strongly S-prime ideal of a ring R, then its localization P_S is a strongly prime ideal of R_S (Proposition 2.17). Consequently, if R is S-strongly 0-dimensional, R_S is strongly 0-dimensional by Corollary 2.18, although the converse does not hold in general (see Example 2.19).

In Theorem 2.22, various equivalent conditions are derived for R to be S-strongly 0-dimensional.

Moreover, it is proven that a ring R is S-Artinian if and only if, for any family of ideals $\{I_i \mid i \in \Gamma\}$ of R, there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $s(\bigcap_{i\in\Gamma_0} I_i) \subseteq \bigcap_{i\in\Gamma} I_i$ (Theorem 2.24).

As a corollary, all S-Artinian rings are S-strongly 0-dimensional and finitely S-cogenerated (Corollaries 2.25 and 2.26).

Additionally, the behavior of the S-strongly 0-dimensional property is analyzed in the context of trivial extensions (Theorem 2.27).

Finally, Theorem 2.32 provides several equivalent characterizations for a ring R to be compactly S-packed.

2. Mains results

Let I be an ideal of R, and let J be a nonempty subset of R. The residual of I by J is defined as $(I : J) = \{x \in R : xJ \subseteq I\}$. For the special case where $J = \{a\} \subseteq R$, we use the notation $(I : a)$ instead of $(I : \{a\})$.

Recall from [4, Proposition 1] that for a commutative ring R with a multiplicative subset $S \subseteq R$, an ideal $\mathfrak{p}$ of R disjoint from S is an S-prime ideal of R if and only if $(\mathfrak{p} : s)$ is a prime ideal of R for some $s \in S$.

Throughout this paper, the element s corresponding to an S-prime ideal $\mathfrak{p}$ will be denoted by $s_\mathfrak{p}$.

Proposition 2.1. *Let R be a ring with a multiplicative subset S, $I_1, I_2, \ldots, I_m$ be a finite number of ideals of R, and $\mathfrak{p}$ be an S-prime ideal of R such that $\bigcap_{i=1}^m I_i \subseteq \mathfrak{p}$. Then, $s_\mathfrak{p} I_j \subseteq \mathfrak{p}$ for some $j \in \{1, \ldots, m\}$.*

Proof. Suppose $s_\mathfrak{p} I_i \nsubseteq \mathfrak{p}$ for all i. Then, there exist $x_i \in I_i$ such that $s_\mathfrak{p} x_i \notin \mathfrak{p}$ for $1 \leqslant i \leqslant m$. Therefore, $\prod x_i \in \prod I_i \subseteq \bigcap I_i$, but $\prod x_i \notin \mathfrak{p}$ (since $\mathfrak{p}$ is S-prime according to [4, Proposition 4]). Hence, $\bigcap I_i \nsubseteq \mathfrak{p}$, a contradiction. □

Now, we aim to extend this statement to the case of infinite intersections. To this end, we introduce the notions of S-strongly prime ideals and S-strongly 0-dimensional rings and characterize them. We begin with the following definition.

Definition 2.2. An S-prime ideal P is said to be an S-strongly prime ideal if there exists a fixed $s \in S$ such that whenever $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$ for some family of ideals $\{I_\alpha \mid \alpha \in \Gamma\}$ of R, we have $sI_\beta \subseteq P$ for some $\beta \in \Gamma$.

A ring R is said to be S-strongly 0-dimensional if all S-prime ideals of R are S-strongly prime ideals.

Proposition 2.3. *Let R be a ring, S be a multiplicative subset of R and P be an S-prime ideal of R. Then the following assertions are equivalent:*

(1) *P is an S-strongly prime ideal of R;*
(2) *$(P : s_p)$ is a strongly prime ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let $\{I_\alpha \mid \alpha \in \Gamma\}$ be any family of ideals of R, such that $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq (P : s_p)$, so $\bigcap_{\alpha\in\Gamma} s_pI_\alpha \subseteq P$ and so there exists $s \in S$ such that $ss_pI_\beta \subseteq P$ for some $\beta \in \Gamma$. Therefore, $sI_\beta \subseteq (P : s_p)$. On the other hand, as $(P : s_p)$ is a prime ideal disjoint with S, we get that $I_\beta \subseteq (P : s_p)$. Whence $(P : s_p)$ is a strongly prime ideal of R.

(2) $\Rightarrow$ (1) Let $\{I_\alpha \mid \alpha \in \Gamma\}$ be any family of ideals of R, such that $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$. Thus

$$\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P \subseteq (P : s_p).$$

Therefore, $I_\beta \subseteq (P : s_p)$ for some $\beta \in \Gamma$. Thus

$$s_pI_\beta \subseteq s_p(P : s_p) \subseteq P.$$

Whence P is an S-strongly prime ideal of R. $\square$

In light of the previous theorem, the element s in the definition of S-strongly prime ideals can be assumed to be $s_{\mathfrak{p}}$.

Corollary 2.4. *An S-prime ideal P is S-strongly prime if and only if whenever $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$ for some family $\{I_\alpha \mid \alpha \in \Gamma\}$ of ideals of R, $s_pI_\beta \subseteq P$ for some $\beta \in \Gamma$.*

Corollary 2.5. *A ring R is S-strongly 0-dimensional if and only if every prime ideal of R disjoint with S is strongly prime.*

Corollary 2.6. *Every strongly 0-dimensional ring is S-strongly 0-dimensional.*

Proof. Let R be a strongly 0-dimensional ring with a multiplicative subset S. Let $\mathfrak{p}$ be an S-prime ideal of R. Then, $(\mathfrak{p} : s_{\mathfrak{p}})$ is a prime ideal of R, and since R is strongly 0-dimensional, it follows that $(\mathfrak{p} : s_{\mathfrak{p}})$ is strongly prime. Hence, by Proposition 2.3, $\mathfrak{p}$ is an S-strongly prime ideal of R. Thus, R is S-strongly 0-dimensional. $\square$

Example 2.7. Let $R = R_1 \times R_2$, where R_1 is a strongly 0-dimensional ring (e.g., an Artinian ring) and R_2 is a non-strongly 0-dimensional ring (e.g., a non-0-dimensional ring). Set $S = \{1\} \times R_2$. Then, R is an S-strongly 0-dimensional ring, but it is not strongly 0-dimensional.

Proof. Since the homomorphic image $R_2 \cong R/R_1 \times (0)$ is not strongly 0-dimensional, it follows from [12, Lemma 2.5] that R is not strongly 0-dimensional. On the other hand, to prove that R is S-strongly 0-dimensional, it suffices to show that all prime ideals of R disjoint from S are strongly prime.

Let P be a prime ideal of R disjoint from S. Then, P must be of the form $P = \mathfrak{p}_1 \times R_2$ for some prime ideal $\mathfrak{p}_1$ of R_1. Let $\{I_\alpha \mid \alpha \in \Gamma\}$ be any family of ideals of R such that $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$. Since $I_\alpha = I_{1,\alpha} \times I_{1,\alpha}$, it follows that $\bigcap_{\alpha\in\Gamma} I_{1,\alpha} \subseteq \mathfrak{p}_1$. As $\mathfrak{p}_1$ is a strongly prime ideal of R_1, there exists some $\beta \in \Gamma$ such that $I_{1,\beta} \subseteq \mathfrak{p}_1$. Consequently, $I_\alpha = I_{1,\alpha} \times I_{1,\alpha} \subseteq \mathfrak{p}_1 \times R_2 = P$. □

The next result provides a local characterization of strongly 0-dimensional rings.

Proposition 2.8. *Let R be a ring and S be a multiplicative subset of R. Then the following statements are equivalent:*

(1) *R is a strongly 0-dimensional ring;*
(2) *R is an $(R \setminus \mathfrak{p})$-strongly 0-dimensional ring for every prime ideal $\mathfrak{p}$ of R;*
(3) *R is an $(R \setminus \mathfrak{m})$-strongly 0-dimensional ring for every maximal ideal $\mathfrak{m}$ of R.*

Proof. (1) ⇒ (2) This follows from Corollary 2.6.

(2) ⇒ (3) is obvious.

(3) ⇒ (1) Let P be a prime ideal of R. Then $P \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. Thus, P is a prime ideal of R disjoint with $R \setminus \mathfrak{m}$. By Corollary 2.5, since R is $(R \setminus \mathfrak{m})$-strongly 0-dimensional, P is strongly prime. Therefore, R is strongly 0-dimensional. □

Theorem 2.9. (S-version of Krull's theorem) *Let R be an S-Noetherian ring where $S \subseteq \mathrm{Reg}(R)$, and let $\mathfrak{m}$ be an ideal of R disjoint with S. Then, $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$ if and only if no element of $1 - \mathfrak{m}$ is a zero-divisor in R.*

Proof. If there exists an element $1-z \in 1-\mathfrak{m}$ that is a zero-divisor, say $(1-z)y = 0$ for some $y \neq 0$, then we have $y = zy = z^2y = \cdots = z^ny$, implying $y \in \bigcap_{n=1}^{\infty} \mathfrak{m}^n$.

Conversely, assume that no element of $1 - \mathfrak{m}$ is a zero-divisor in R. Since $S \subseteq \mathrm{Reg}(R)$, no element of $1 - \mathfrak{m}_S$ is a zero-divisor in R_S. Thus,

$$\bigcap_{n=1}^{\infty} (\mathfrak{m}_S)^n = (0)$$

because R_S is a Noetherian ring according to [25, Corollary 1, p. 216]. Now, since

$$\left(\bigcap_{n=1}^{\infty} \mathfrak{m}^n\right)_S = \bigcap_{n=1}^{\infty} (\mathfrak{m}_S)^n = 0,$$

and given that $S \subseteq \mathrm{Reg}(R)$, we conclude that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. □

Corollary 2.10. *Let D be an S-Noetherian domain. If $\mathfrak{m}$ is an ideal of D disjoint with S, then $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$.*

Remark 2.11. The above corollary implies that if $(0) \neq \mathfrak{m}$ is an ideal of D disjoint with S and D is an S-Noetherian domain, then $\mathfrak{m}^p \neq \mathfrak{m}^q$ for $p \neq q$. Otherwise, if $p < q$, we would have $\mathfrak{m}^p = \mathfrak{m}^{p+1}$, and multiplying by $\mathfrak{m}$, we get $\mathfrak{m}^{p+2} = \mathfrak{m}^{p+1} = \mathfrak{m}^p$. By induction, $\mathfrak{m}^{p+n} = \mathfrak{m}^p$ for all n. This leads to the contradiction $\mathfrak{m}^p = (0)$.

Now, we present an example of a ring that is an integral domain but not an S-strongly 0-dimensional ring.

Example 2.12. Let D be an S-principal ideal domain that is not an S-field. Then, (0) is not an S-maximal ideal of D, and according to [14, Theorem 2.3], $(0) = (0 : s)$ is not a maximal element of the set of all ideals of D disjoint from S for any $s \in S$. Consequently, there must exist a nonzero ideal $\mathfrak{m}$ of D disjoint from S. By Corollary 2.10, $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. However, $s\mathfrak{m}^n \neq (0)$ for all positive integers n and all $s \in S$. Thus, D is not an S-strongly 0-dimensional ring.

Proposition 2.13. *Let R be an S-domain. If R is an S-strongly 0-dimensional ring, then R is an S-field.*

Proof. Assume that there exists an S-domain R such that R is an S-strongly 0-dimensional ring but not an S-field. Let s_0 be the element in S such that (0) is S-prime with respect to s_0. Since (0) is not an S-maximal ideal of R, it follows from [14, Theorem 2.3] that $(0 : s)$ is not a maximal element of the set of all ideals of R disjoint from S for any $s \in S$. Consequently, there exists an ideal $\mathfrak{m}$ of R disjoint from S such that $s_0\mathfrak{m} \neq 0$.

Define Γ as the set of all such ideals. Let I be the intersection of all ideals in Γ. We claim that $I \neq (0)$. If this were not the case, then there would exist $s \in S$ such that $sJ = 0$ for some $J \in \Gamma$, since (0) is an S-strongly prime ideal. Consequently, $sJ = 0 \subseteq (0 : s_0)$, and since $(0 : s_0)$ is a prime ideal of R, we obtain $s_0 J = 0$, leading to a contradiction. Hence, we have $I \neq (0)$.

Choose a nonzero element $x \in I$. Since I is the smallest ideal in Γ, we obtain $(x) = I = (x^2)$ (as both (x) and (x^2) belong to Γ). Thus, $x = rx^2$ for some $r \in R$. This implies that $x(1 - rx) = 0 \in (0 : s_0)$, and since $(0 : s_0)$ is a prime ideal of R and $s_0 x \neq 0$, we deduce that $s_0(1 - rx) = 0$. Therefore, $s_0 = s_0 rx \in I \cap S$, a contradiction. □

Proposition 2.14. *Let $f : R \to T$ be a surjective homomorphism of rings, and let S be a multiplicative subset of R such that $\ker(f) \cap S = \emptyset$. If R is an S-strongly 0-dimensional ring, then T is an $f(S)$-strongly 0-dimensional ring.*

Proof. Assume that $\bigcap_{\alpha\in\Gamma} J_\alpha \subseteq \mathfrak{Q}$ for some ideals J_α of T and $\mathfrak{Q} \in \mathrm{Spec}_{f(S)}(T)$. There exist ideals I_α of R and $P \in \mathrm{Spec}_S(R)$ such that $\ker(f) \subseteq I_\alpha$, $\ker(f) \subseteq P$, and $f(I_\alpha) = J_\alpha$, $f(P) = \mathfrak{Q}$. Thus, we obtain $f(\bigcap_{\alpha\in\Gamma} I_\alpha) \subseteq f(P)$. Hence, $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$. Since R is an S-strongly 0-dimensional ring, it follows that $sI_\alpha \subseteq P$ for some $\alpha \in \Gamma$ and $s \in S$. Consequently,

$$f(s)J_\alpha = f(s)f(I_\alpha) = f(sI_\alpha) \subseteq f(P) = \mathfrak{Q},$$

and so T is an $f(S)$-strongly 0-dimensional ring. □

Corollary 2.15. *Let R be a ring with S a multiplicative subset of R, and let I be an ideal of R disjoint from S. If R is an S-strongly 0-dimensional ring, then R/I is a $\bar{S}$-strongly 0-dimensional ring with $\bar{S} = S + I$.*

Corollary 2.16. *Let $\mathfrak{p}$ be an S-strongly prime ideal. Then $\mathfrak{p}$ is an S-maximal ideal.*

Proof. Let $\mathfrak{p}$ be an S-strongly prime ideal of R. Then, as in the proof of Proposition 2.14, (0) is an $\bar{S}$-strongly prime ideal of $R/\mathfrak{p}$. Since $R/\mathfrak{p}$ is an S-domain, applying the same reasoning as in the proof of Theorem 2.13, we conclude that $R/\mathfrak{p}$ is an S-field. Thus, $\mathfrak{p}$ is an S-maximal ideal of R. □

Proposition 2.17. *Let R be a ring, S be a multiplicative subset of R, and P be an S-prime ideal of R. If P is strongly S-prime, then P_S is a strongly prime ideal of R_S.*

Proof. Let $\bigcap_{\alpha\in T}(I_\alpha)_S \subseteq P_S$ for any ideals I_α. Set $\mathfrak{p} = (P : s_p)$, then $\mathfrak{p}$ is a prime ideal of R. Since

$$\left(\bigcap_{\alpha\in T} I_\alpha\right)_S \subseteq \bigcap_{\alpha\in T}(I_\alpha)_S \subseteq \mathfrak{p}_S,$$

we have $\bigcap_{\alpha\in T} I_\alpha \subseteq \mathfrak{p}$ as $\mathfrak{p}$ is a prime ideal of R disjoint from S. Since P is strongly S-prime, it follows from Proposition 2.3 that $\mathfrak{p}$ is strongly S-prime. Thus, $I_\beta \subseteq \mathfrak{p}$ for some $\beta \in T$, which implies $(I_\beta)_S \subseteq \mathfrak{p}_S = P_S$ for some $\beta \in T$. □

Corollary 2.18. *Let R be a ring and S be a multiplicative subset of R. If R is an S-strongly 0-dimensional ring, then R_S is a strongly 0-dimensional ring.*

The following example provides an S-prime ideal $\mathfrak{p}$ that is not S-strongly prime, while $\mathfrak{p}_S$ is a strongly prime ideal of R_S. Additionally, it illustrates a ring R that is not S-strongly 0-dimensional, whereas R_S is strongly 0-dimensional.

Example 2.19. Let $R = \mathbb{Z}$ and $S = \mathbb{N}^*$. Then $R_S = \mathbb{Q}$ is a field; however, the S-prime ideal (0) is not strongly S-prime since $\bigcap_{n\in\mathbb{N}^*} I^n \subseteq (0)$ and $sI^n \neq (0)$ for every $n \in \mathbb{N}^*$, where $I = 2\mathbb{Z}$. Thus, R_S is a strongly 0-dimensional ring, whereas R is not S-strongly 0-dimensional.

Definition 2.20. (R, S) is said to satisfy the condition $(*)$ if, for any family $\{I, I_i\}_{i\in\Delta}$ of ideals of R, the condition $I + I_i = R$ for all $i \in \Delta$ implies that $\big(I + \big(\bigcap_{i\in\Delta} I_i\big)\big) \cap S \neq \emptyset$.

Lemma 2.21. ([22, Theorem 1.4.7]) *Let I be an ideal of a ring R, and let S be a multiplicative subset of R with $I \cap S = \emptyset$. Then there exists a prime ideal $\mathfrak{p}$ of R such that $I \subseteq \mathfrak{p}$ and $\mathfrak{p} \cap S = \emptyset$.*

We now establish some equivalent conditions for R to be an S-strongly 0-dimensional ring. Recall from [14] that a ring R is zero S-dimensional if and only if every S-prime ideal of R is S-maximal.

Theorem 2.22. *Let R be a ring and S be a multiplicative subset of R. Then the following statements are equivalent:*

(1) *R is an S-strongly 0-dimensional ring;*
(2) *R is a zero S-dimensional ring and (R, S) satisfies the condition $(*)$;*
(3) *Every S-prime ideal is S-maximal, and every S-maximal ideal is an S-strongly prime ideal;*

(4) *For all prime ideals P of R disjoint from S and any subset $\{a_i \in R \mid i \in \Delta\}$ of R, if $\bigcap_{i\in\Delta}(a_i) \subseteq P$, then $a_j \in P$ for some $j \in \Delta$;*
(5) *For any family $\{I, I_i\}_{i\in\Delta}$ of ideals of R, and for any family $\{P_i\}_{i\in\Gamma}$ of prime ideals of R disjoint from S, R satisfies the following conditions:*
  (a) *If $\bigcap_{i\in\Gamma} P_i \subseteq P$ and P is a prime ideal of R, then $P_j \subseteq P$ for some $j \in \Gamma$.*
  (b) $(\sqrt{I + (\bigcap_{i\in\Delta} I_i)})_S = (\bigcap_{i\in\Delta}\sqrt{I + I_i})_S$.

Proof. (1) $\Rightarrow$ (2) Let R be an S-strongly 0-dimensional ring, and let P be an S-prime ideal of R. Since R is an S-strongly 0-dimensional ring, by Corollary 2.15, R/P is an S-strongly 0-dimensional ring. As R/P is an $(S+P)$-domain, Proposition 2.13 implies that R/P is an $(S+P)$-field. Hence, P is an S-maximal ideal of R, and so R is a zero S-dimensional ring.

Now, suppose $I + I_i = R$ for all $i \in \Delta$ and
$$\left(I + \left(\bigcap_{i\in\Delta} I_i\right)\right) \cap S \neq \emptyset$$
for any family $\{I, I_i\}_{i\in\Delta}$ of ideals of R. Then,
$$I + \left(\bigcap_{i\in\Delta} I_i\right) \subseteq P$$
for some prime ideal P of R disjoint from S. Since P is an S-strongly prime ideal and $\bigcap_{i\in\Delta} I_i \subseteq P$, it follows that $sI_i \subseteq P$ for some $i \in \Delta$ and $s \in S$. Thus, $I_i \subseteq P$, and so $R = I + I_i \subseteq P$, a contradiction. Therefore, (2) holds.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (5) We show that every prime ideal is a strongly prime ideal. Let P be a prime ideal of R disjoint from S. Suppose $\bigcap_{i\in\Delta} I_i \subseteq P$ for some family $\{I_i\}_{i\in\Delta}$ of ideals of R. Assume that $I_i \nsubseteq P$ for all $i \in \Delta$. Then, there exists an element $a_i \in I_i$ such that $a_i \notin P$. Since
$$\bigcap_{i\in\Delta}(a_i) \subseteq \bigcap_{i\in\Delta} I_i \subseteq P,$$
by (4), $a_i \in P$ for some $i \in \Delta$, a contradiction. Thus, P is a strongly prime ideal and satisfies condition (a).

Now, let $\{I, I_i \mid i \in \Delta\}$ be a family of ideals of R. We claim that
$$(\bigcap_{i\in\Delta}\sqrt{I + I_i})_S \subseteq (\sqrt{I + \left(\bigcap_{i\in\Delta} I_i\right)})_S.$$

If this is not the case, then there exists an element $a \in \bigcap_{i\in\Delta}\sqrt{I + I_i}$ such that $sa \notin \sqrt{I + (\bigcap_{i\in\Delta} I_i)}$ for every $s \in S$. Since $\left(I + (\bigcap_{i\in\Delta} I_i)\right) \cap S = \emptyset$, it follows that there exists a prime ideal P disjoint from S such that $I + \left(\bigcap_{i\in\Delta} I_i\right) \subseteq P$. Since P is a strongly prime ideal, we obtain $I + I_j \subseteq P$ for some $j \in \Delta$, implying $a \in P$.

Thus, in R_S, we have $\frac{a}{1} \in P_S$ for every prime ideal P_S of R_S containing $I + \left(\bigcap_{i\in\Delta} I_i\right)_S$. Therefore, there exists $s' \in S$ such that $s'a \in \sqrt{I + \left(\bigcap_{i\in\Delta} I_i\right)}$, a contradiction. Hence, R satisfies condition (b).

(5) $\Rightarrow$ (1) Suppose R satisfies conditions (a) and (b). To show that R is an S-strongly 0-dimensional ring, it suffices to prove that all prime ideals of R disjoint from S are strongly prime, according to Proposition 2.3.

Let P be a prime ideal of R, and assume that $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$, where I_α $(\alpha \in \Gamma)$ are ideals of R. Since

$$(\sqrt{\bigcap_{\alpha\in\Gamma} I_\alpha})_S = (\bigcap_{\alpha\in\Gamma} \sqrt{I_\alpha})_S,$$

for every $x \in \bigcap_{\alpha\in\Gamma} \sqrt{I_\alpha}$, there exists $s \in S$ such that

$$sx \in \sqrt{\bigcap_{\alpha\in\Gamma} I_\alpha} \subseteq P,$$

and so $x \in P$ as P is prime and $s \notin P$ (since $S \cap P = \emptyset$). Thus, we obtain $\bigcap_{\alpha\in\Gamma} \sqrt{I_\alpha} \subseteq P$. But $\sqrt{I_\alpha} = \bigcap_{i\in\Omega} P_{\alpha,i}$, where $P_{\alpha,i}$ are prime ideals of R containing I_α for an index set Ω. Hence, we get $\bigcap_{\alpha\in\Gamma, i\in\Omega} P_{\alpha,i} \subseteq P$. Then, by condition (a), there exist $\alpha \in \Gamma$ and $i \in \Omega$ such that $P_{\alpha,i} \subseteq P$. Therefore, $I_\alpha \subseteq P$ for some $\alpha \in \Gamma$, and hence R is an S-strongly 0-dimensional ring. $\square$

Proposition 2.23. *Let R be a ring and S be a multiplicative subset of R, satisfying the property that for any family of ideals $\{I_\alpha\}_{\alpha\in\Gamma}$ of R and any index set Γ, there exists a finite subset Γ' of Γ such that*

$$\sqrt[S]{\bigcap_{\alpha\in\Gamma} I_\alpha} = \bigcap_{\alpha\in\Gamma'} \sqrt[S]{I_\alpha},$$

then R is an S-strongly 0-dimensional ring. The converse is true when R_S is Noetherian.

Proof. Let $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$ for a family of ideals $\{I_\alpha\}_{\alpha\in\Gamma}$ and a prime ideal P of R disjoint from S. By our assumption, we obtain

$$\sqrt[S]{\bigcap_{\alpha\in\Gamma} I_\alpha} = \bigcap_{\alpha\in\Gamma'} \sqrt[S]{I_\alpha} \subseteq P$$

for some finite subset Γ' of Γ, implying $I_\beta \subseteq \sqrt[S]{I_\beta} \subseteq P$ for some $\beta \in \Gamma'$.

For the converse, let $\{I_\alpha\}_{\alpha\in\Gamma}$ be a family of ideals of R. Then,

$$\sqrt[S]{\bigcap_{\alpha\in\Gamma} I_\alpha} \subseteq \bigcap_{\alpha\in\Gamma'} \sqrt[S]{I_\alpha}$$

for all finite subsets Γ' of Γ. On the other hand, we have

$$\sqrt[S]{\bigcap_{\alpha\in\Gamma} I_\alpha} = \bigcap_{P_\gamma\in V_S(\bigcap_{\alpha\in\Gamma} I_\alpha)} (P_\gamma : s_{P_\gamma}) = \bigcap_{P_\gamma\in \mathrm{Min}^S(\bigcap_{\alpha\in\Gamma} I_\alpha)} P_\gamma$$

where $\mathrm{Min}^S(J)$ denotes the set of all minimal prime ideals disjoint from S.

Moreover, the map $\xi : \mathfrak{p} \mapsto \mathfrak{p}_S$ is a one-to-one correspondence from $\mathrm{Min}^S(J)$ to $\mathrm{Min}(J_S)$. Since R_S is a Noetherian ring, the set $\mathrm{Min}^S\left(\bigcap_{\alpha\in\Gamma} I_\alpha\right)$ is finite, say

$$\mathrm{Min}^S\left(\bigcap_{\alpha\in\Gamma} I_\alpha\right) = \{P_{\gamma_1}, P_{\gamma_2}, \dots, P_{\gamma_n}\}.$$

Therefore, we obtain $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P_{\gamma_j}$ for all $\gamma_j \in \Gamma' = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$.

Since R is an S-strongly 0-dimensional ring, there exists an $\alpha_j \in \Gamma$ such that $I_{\alpha_j} \subseteq P_{\gamma_j}$ according to Theorem 2.22, and hence $\sqrt[S]{I_{\alpha_j}} \subseteq P_{\gamma_j}$. Thus,

$$\sqrt[S]{I_{\alpha_1}} \cap \cdots \cap \sqrt[S]{I_{\alpha_n}} \subseteq P_{\gamma_1} \cap \cdots \cap P_{\gamma_n},$$

which leads to

$$\bigcap_{\alpha_j\in\Gamma'} \sqrt[S]{I_{\alpha_j}} \subseteq \bigcap_{\gamma_j\in\Gamma'} P_{\gamma_j} = \sqrt[S]{\bigcap_{\alpha\in\Gamma} I_\alpha}. \qquad \square$$

Recall that a ring R is said to be an Artinian ring if every descending chain of ideals

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

of R stabilizes, i.e., there exists $k \in \mathbb{N}$ such that $I_k = I_n$ for all $n \geq k$.

Similarly, a commutative ring R with a multiplicative subset S is called an S-Artinian ring if, for every descending chain of ideals $\{I_n\}_{n\in\mathbb{N}}$ of R, there exist $s \in S$ and $k \in \mathbb{N}$ such that $sI_k \subseteq I_n$ for all $n \geq k$ [20].

Let Ω be a nonempty family of ideals of R, and let $S \subseteq R$ be a multiplicative subset. According to [20], an ideal $I \in \Omega$ is called an S-minimal element of Ω if there exists $s \in S$ such that whenever $J \subseteq I$ for some $J \in \Omega$, then $sI \subseteq J$. In particular, R satisfies the (S-MIN) condition if every nonempty family of ideals Ω of R has an S-minimal element.

Theorem 2.24. *The following statements are equivalent for a ring R with a multiplicative subset S:*

1. *R is an S-Artinian ring;*
2. *R satisfies the (S-MIN) condition;*
3. *If $\{I_i \mid i \in \Gamma\}$ is a family of ideals of R, then there exists a finite subset Γ_0 of Γ such that $s(\bigcap_{i\in\Gamma_0} I_i) \subseteq \bigcap_{i\in\Gamma} I_i$ for some $s \in S$.*

Proof. (1) $\Leftrightarrow$ (2) follows from [20, Theorem 2.1].

(2) $\Rightarrow$ (3) We may assume that Γ is nonempty. Define $B = \bigcap_{i\in\Gamma} I_i$ and let Ω be the set of intersections of finitely many I_i. Since each $I_i \in \Omega$, we have that Ω is nonempty. By hypothesis, Ω has an S-minimal element, say $A = \bigcap_{i\in\Gamma_0} I_i$, where Γ_0 is a finite subset of Γ. That is, there exists $s \in S$ such that whenever $J \subseteq A$ for some $J \in \Omega$, then $sA \subseteq J$. In particular, for each $i \in \Gamma$, we obtain $sA \subseteq A \cap I_i \subseteq I_i$ by the S-minimality of A. Hence, $sA \subseteq \bigcap_{i\in\Gamma} I_i = B$.

(3) $\Rightarrow$ (1) Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of ideals of R. By the hypothesis, there exists a positive integer m and $s \in S$ such that

$$sI_m = s\bigcap_{i=1}^{m} I_i \subseteq \bigcap_{i=1}^{\infty} I_i.$$

Therefore, $sI_m \subseteq I_m$ for all $n \geqslant m$. Thus, R is S-Artinian. ☐

Let R be a ring and $S \subseteq R$ be a multiplicative subset. Then, R is called a finitely S-cogenerated ring if, for any family of ideals $\{I_i\}_{i\in\Delta}$ of R, where Δ is an index set, $\bigcap_{i\in\Delta} I_i = 0$ implies that $s\left(\bigcap_{i\in\Delta'} I_i\right) = 0$ for some $s \in S$ and a finite subset $\Delta' \subseteq \Delta$ [20].

Corollary 2.25. *Every S-Artinian ring is finitely S-cogenerated.*

Corollary 2.26. *Every S-Artinian ring is an S-strongly 0-dimensional ring.*

Proof. Let R be an S-Artinian ring. Suppose $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$ for some ideals I_α of R and $P \in \mathrm{Spec}_S(R)$. Since R is an S-Artinian ring, there exist $s \in S$ and a finite subset $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Gamma$ such that

$$s\bigcap_{i=1}^{n} I_{\alpha_i} \subseteq \bigcap_{\alpha\in\Gamma} I_\alpha.$$

As $s\bigcap_{i=1}^{n} I_{\alpha_i} \subseteq P$, we obtain $s_p\bigcap_{i=1}^{n} I_{\alpha_i} \subseteq P$. Consequently, $\bigcap_{i=1}^{n} I_{\alpha_i} \subseteq (P : s_p)$, and thus $s_P I_{\alpha_k} \subseteq P$ for some k. Therefore, R is an S-strongly 0-dimensional ring. ☐

Let A be a ring and M be an A-module. The following ring construction, called the *trivial extension* of A by M (also referred to as the *idealization* of M), denoted by $A \ltimes M$, is the ring whose additive structure is that of the external direct sum $A \oplus M$ and whose multiplication is defined by

$$(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + r_2m_1)$$

for all $r_1, r_2 \in A$ and all $m_1, m_2 \in M$ (this construction is also known by other terminologies and notations, such as the idealization $A(+)M$).

The purpose of idealization is to embed M inside a commutative ring R such that the structure of M as an A-module remains essentially the same as that of M as an R-module, making M an ideal of R. The fundamental properties of trivial ring extensions are summarized in [6, 11]. Trivial ring extensions have been instrumental in solving various open problems and conjectures in both commutative and non-commutative ring theory (see, for example, [2]).

It is clear that if S is a multiplicative subset of $A \ltimes M$, then $S_0 = \pi_A(S) = \{s \in A \mid (s, m) \in S \text{ for some } m \in M\}$ is a multiplicative subset of A.

Theorem 2.27. *Let M be an A-module and S be a multiplicative subset of the trivial ring extension $R = A \ltimes M$. Then the following statements are equivalent:*

(1) *A is an S_0-strongly 0-dimensional ring;*
(2) *R is an S-strongly 0-dimensional ring.*

Proof. (1) $\Rightarrow$ (2) Assume that A is an S_0-strongly 0-dimensional ring. To prove that R is an S-strongly 0-dimensional ring, it suffices to show that every prime ideal P of R disjoint from S is S-strongly prime, by Corollary 2.5.

Suppose that $\{I_\alpha\}_{\alpha\in\Gamma}$ is a family of ideals of $A \ltimes M$, and let P be a prime ideal of R disjoint from S such that $\bigcap_{\alpha\in\Gamma} I_\alpha \subseteq P$. Then there exists a prime ideal $\mathfrak{p}$ of A such that $P = \mathfrak{p} \ltimes M$. Since $P = \mathfrak{p} \ltimes M$ is disjoint from S for all $i \in \Gamma$, it follows that $\mathfrak{p} \cap S_0 = \emptyset$.

Define $J_\alpha = \pi_A(I_\alpha)$ for every $\alpha \in \Gamma$. Note that $I_\alpha \subseteq J_\alpha \ltimes M$ and $J_\alpha \ltimes M = I_\alpha + 0 \ltimes M$. Since $0 \ltimes M \subseteq P$, we get that

$$\Big(\bigcap_{\alpha\in\Gamma} J_\alpha\Big) \ltimes M \cap \bigcap_{\alpha\in\Gamma}(J_\alpha \ltimes M) = \bigcap_{\alpha\in\Gamma} I_\alpha + 0 \ltimes M \subseteq P = \mathfrak{p} \ltimes M,$$

which implies that

$$\bigcap_{\alpha\in\Gamma} J_\alpha \subseteq \mathfrak{p}.$$

Since $\mathfrak{p}$ is a prime ideal of A disjoint from S_0, it follows from Corollary 2.5 that $\mathfrak{p}$ is a strongly prime ideal of A. Thus, there exists $\alpha \in \Gamma$ such that $J_\alpha \subseteq \mathfrak{p}$. Therefore,

$$I_\alpha \subseteq J_\alpha \ltimes M \subseteq \mathfrak{p} \ltimes M = P.$$

Thus, P is a strongly prime ideal of R, and hence R is an S-strongly 0-dimensional ring.

(2) $\Rightarrow$ (1) follows directly from Proposition 2.15. $\square$

When S is a subset of units of R, we obtain the following result:

Corollary 2.28. ([15, Proposition 3.17] or [3, Theorem 9]) *Let M be an A-module. Then the following statements are equivalent:*

(1) *A is a strongly 0-dimensional ring;*
(2) *$A \ltimes M$ is a strongly 0-dimensional ring.*

Theorem 2.29. (S-version of Prime Avoidance Theorem) *Let $I, \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ be ideals of a ring R, and suppose that at most two of the $\mathfrak{p}_i$ are not S-prime. If*

$$I \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_n,$$

then there exists $s \in S$ and i with $1 \leq i \leq n$ such that $sI \subseteq \mathfrak{p}_i$.

Proof. If $n = 2$, set $x = x_1 + x_2$. If $x \in \mathfrak{p}_1$, then $x_2 = x - x_1 \in \mathfrak{p}_1$, which is impossible. By the same argument, it is impossible that $x \in \mathfrak{p}_2$. Therefore, we assume $n > 2$. We can also assume that $I \nsubseteq \mathfrak{p}_i$ for $i = 1, 2$, because if this is not the case, the proof is complete by taking $s = 1$.

Now, we suppose that $\mathfrak{p}_3, \ldots, \mathfrak{p}_n$ are all S-prime ideals. Define $\mathfrak{q}_i = (\mathfrak{p}_i : s_{\mathfrak{p}_i})$ for $i = 3, \ldots, n$. By [4, Proposition 1], all $\mathfrak{q}_i$ are S-prime ideals. Moreover, since

$$I \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_n \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \mathfrak{q}_3 \cup \cdots \cup \mathfrak{q}_n,$$

we deduce that there exists some i, with $3 \leq i \leq n$, such that $I \subseteq \mathfrak{q}_i$. Since $\mathfrak{q}_i = (\mathfrak{p}_i : s_{\mathfrak{p}_i})$, it follows that $s_{\mathfrak{p}_i} I \subseteq \mathfrak{p}_i$. $\square$

Corollary 2.30. *([4, Theorem 2]) Let R be a commutative ring and $S \subseteq R$ a multiplicative set. Let I be an ideal of R and let $P_1, \ldots, P_n$ be S-prime ideals of R. If $I \subseteq \bigcup_{i=1}^{n} P_i$, then there exist $s \in S$ and $j \in \{1, \ldots, n\}$ such that $sI \subseteq P_j$.*

The natural question that arises is whether Theorem 2.29 can be extended to the case of an infinite union. In this context, Reis and Viswanathan [18] considered the following property:

(*) If an ideal $I \subseteq \bigcup_{\alpha\in\Gamma} P_\alpha$ where the $P_\alpha(\alpha \in \Gamma)$ are prime ideals of R, then $I \subseteq P_\alpha$ for some $\alpha \in \Gamma$.

The principal result of [18] is that a Noetherian ring satisfies (*) if and only if every prime ideal is the radical of a principal ideal. Subsequently, Smith in [21] demonstrated that the assumption of the ring being Noetherian is not required.

More recently, Yildiz et al. constructed a topology on $\mathrm{Spec}_S(R)$, generalizing the classical Zariski topology [24]. They investigated several topological properties of $\mathrm{Spec}_S(R)$ such as connectedness, compactness, and separation axioms. Among other things, they defined the S-variety $V_S(E)$ of a subset E of R as follows:

$$V_S(E) = \{P \in \mathrm{Spec}_S(R) : sE \subseteq P \text{ for some } s \in S\},$$

where sE denotes the subset $\{sa : a \in E\}$. They also introduced the concept of the S-radical $\sqrt[S]{I}$ of an ideal I of R, defined as follows:

$$\sqrt[S]{I} = \{a \in R : sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N}\}.$$

Notably, they demonstrated that

$$\sqrt[S]{I} = \bigcap_{P\in V_S(I)} (P : s_P),$$

(see [24, Proposition 5]). Observe that $\{(P : s_P) \mid P \in V_S(I)\}$ is precisely the set of all prime ideals of R containing I and disjoint from S. Thus, [24, Proposition 5] demonstrates that the S-radical of an ideal I of R is the intersection of all prime ideals of R containing I and disjoint from S.

Definition 2.31. ([23, Definition 2]) Let R be a ring and S be a multiplicative subset of R. Then, R is called a compactly S-packed ring if whenever $Q \subseteq \bigcup_{i\in\Gamma} P_i$, where Q is an ideal of R and $\{P_i\}_{i\in\Gamma}$ is a family of S-prime ideals of R, there exists $k \in \Gamma$ and $s \in S$ such that $sQ \subseteq P_k$.

Note that if $sQ \subseteq P_k$ where P_k is an S-prime ideal of R, then $s_{P_k}Q \subseteq P_k$.

Theorem 2.32. *Let R be a ring and S a multiplicative subset of R. Then the following assertions are equivalent:*

1. *R is a compactly S-packed ring;*
2. *If an ideal $I \subseteq \bigcup_{\alpha\in\Gamma} P_\alpha$, where the $P_\alpha(\alpha \in \Gamma)$ are prime ideals of R disjoint from S, then $I \subseteq P_\alpha$ for some $\alpha \in \Gamma$;*
3. *For every S-prime ideal P of R, there exists $r \in R$ such that*

$$s_P \sqrt[S]{(r)} \subseteq P \subseteq \sqrt[S]{(r)};$$

(4) *Every prime ideal of R disjoint from S is the S-radical of a principal ideal.*

Proof. (1) $\Rightarrow$ (2) Let I be an ideal of R such that

$$I \subseteq \bigcup_{\alpha\in\Gamma} P_\alpha,$$

where the $P_\alpha(\alpha \in \Gamma)$ are prime ideals of R disjoint from S. Since all P_α are S-prime, it follows from (1) that $I \subseteq P_\alpha$ for some $\alpha \in \Gamma$. Moreover, since P_α is a prime ideal of R, we have $s_{P_\alpha} = 1$, which implies $I \subseteq P_\alpha$.

(2) $\Rightarrow$ (3) Let P be an S-prime ideal of R. Then $\mathfrak{p} := (P : s_P)$ is a prime ideal of R. We claim that $\mathfrak{p}$ is the S-radical of a principal ideal. Suppose, for contradiction, that for each $r \in \mathfrak{p}$, we have $\mathfrak{p} \neq \sqrt[S]{(r)}$. Since $\sqrt[S]{(r)}$ is the intersection of all prime ideals of R disjoint from S that contain r, there exists a prime ideal P_r disjoint from S such that $r \in P_r$ but $\mathfrak{p} \nsubseteq P_r$. Clearly, $\mathfrak{p} \subseteq \bigcup_{r\in P} P_r$, which leads to a contradiction. Therefore, there exists $r \in R$ such that $\mathfrak{p} = \sqrt[S]{(r)}$. Since $\mathfrak{p} = (P : s_P)$, we obtain

$$s_P \sqrt[S]{(r)} \subseteq P \subseteq \sqrt[S]{(r)}.$$

(3) $\Rightarrow$ (4) This implication is straightforward, as P being prime implies $s_P = 1$.

(4) $\Rightarrow$ (1) Let I be an ideal of R such that $I \subseteq \bigcup_{\alpha\in\Gamma} P_\alpha$, where the $P_\alpha(\alpha \in \Gamma)$ are S-prime ideals of R. Set $\mathfrak{p}_\alpha := (P_\alpha : s_{P_\alpha})$ for every $\alpha \in \Gamma$, so that all $\mathfrak{p}_\alpha$ are prime ideals of R disjoint from S. We have

$$I \subseteq \bigcup_{\alpha\in\Gamma} P_\alpha \subseteq \bigcup_{\alpha\in\Gamma} \mathfrak{p}_\alpha.$$

Now, consider the multiplicative set $T = R \setminus \bigcup_{\alpha\in\Gamma} \mathfrak{p}_\alpha$. Since $I \cap T = \emptyset$, there exists a prime ideal $\mathfrak{p}$ of R such that $I \subseteq \mathfrak{p}$ and $T \cap \mathfrak{p} = \emptyset$ by Lemma 2.21. Thus,

$$I \subseteq \mathfrak{p} \subseteq \bigcup_{\alpha\in\Gamma} \mathfrak{p}_\alpha.$$

Since $\mathfrak{p} \subseteq \bigcup_{\alpha\in\Gamma} \mathfrak{p}_\alpha$, it follows that $\mathfrak{p}$ is a prime ideal of R disjoint from S. By (4), there exists $r \in \mathfrak{p}$ such that $\mathfrak{p} = \sqrt[S]{(r)}$, implying that $r \in \bigcup_{\alpha\in\Gamma} \mathfrak{p}_\alpha$. Hence, $r \in \mathfrak{p}_\alpha$ for some $\alpha \in \Gamma$.

Now, if $x \in \mathfrak{p} = \sqrt[S]{(r)}$, then there exists $s \in S$ and $n \in \mathbb{N}$ such that $sx^n \in (r) \subseteq \mathfrak{p}_\alpha$. Since $s \notin \mathfrak{p}_\alpha$ and $\mathfrak{p}_\alpha$ is a prime ideal of R, it follows that $x \in \mathfrak{p}_\alpha$. Therefore,

$$I \subseteq \mathfrak{p} \subseteq \mathfrak{p}_\alpha = (P_\alpha : s_{P_\alpha}),$$

and thus $s_{P_\alpha} I \subseteq P_\alpha$. $\square$

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