

## REPDIGITS AS ABSOLUTE VALUE OF DIFFERENCE BETWEEN LUCAS AND FIBONACCI NUMBERS

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ABSTRACT. In this paper, we investigate the Diophantine equation

$$|L_{n_1} - F_{n_2}| = d \left( \frac{10^m - 1}{9} \right),$$

where  $(L_n)_{n \geq 0}$  and  $(F_n)_{n \geq 0}$  are respectively Lucas and Fibonacci sequences,  $d \in \{1, \dots, 9\}$  and  $m \geq 1$ . All solutions of the aforementioned equation are provided.

### 1. INTRODUCTION

Let  $g \geq 2$  be an integer. Recall that a natural number  $N$  is a  $g$  *rep-digit* if all of its base  $g$ -digits are equal; that is, if

$$N = a \left( \frac{g^m - 1}{g - 1} \right), \quad \text{for some } m \geq 1 \text{ and } a \in \{1, 2, \dots, g - 1\}.$$

Such integer  $N$  is simply called rep-digit when  $g = 10$ .

Let  $F_0 = 0$  and  $F_1 = 1$ . The Fibonacci sequence  $\{F_m\}_{m \in \mathbb{N}}$  is given by

$$F_{m+2} = F_{m+1} + F_m, \tag{1.1}$$

for each  $m \geq 0$ . The first terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1507, \dots$$

We let  $\{L_m\}_{m \geq 0}$  be the Lucas sequence given by

$$L_{m+2} = L_{m+1} + L_m, \tag{1.2}$$

for  $m \geq 0$ , where  $L_0 = 2$  and  $L_1 = 1$ . The following numbers are the first terms of this sequence:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, \dots$$

Many papers on Diophantine equations already studied the three above types of numbers: rep-digits, Fibonacci and Lucas [2], [3] [4], [6], [8], [9], [10], [11], [12], [13], [14], [15], [17], [18], [19]. Most of these works considered a number of a first

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kind as a sum or a difference of two numbers of the same type. In this paper we follow such ideas but we consider difference of two numbers which are not in the same kind. More precisely, we determine all solutions of the Diophantine equations

$$d \left( \frac{10^m - 1}{9} \right) = F_{n_1} - L_{n_2} \quad (1.3)$$

and

$$d \left( \frac{10^m - 1}{9} \right) = L_{n_1} - F_{n_2} \quad (1.4)$$

in integers  $(n_1, n_2, m, d)$  with  $m \geq 1$  and  $1 \leq d \leq 9$ . In fact, our paper is positionned in the general context of the diophantine equation

$$S_m^{(1)} + \varepsilon S_n^{(2)} = S_r^{(3)}. \quad (1.5)$$

where  $\varepsilon \in \{1, -1\}$  and each one of  $S^{(1)}$ ,  $S^{(2)}$  and  $S^{(3)}$  can be Fibonacci numbers  $F$  or Lucas numbers  $L$  or repdigits  $D$ . Paper [8] investigated  $S^{(1)} = S^{(2)} = D$ ,  $S^{(3)} = F$  with  $\varepsilon = 1$  while authors in [2] considered the same equation with  $S^{(3)} = L$ . The assumptions of these two works were continued in [10] with  $\varepsilon = -1$ . One can deduce solutions for  $S^{(1)} = S^{(2)} = F$ ,  $S^{(3)} = D$  and  $\varepsilon = 1$  from [14] while the equation was solved with  $S^{(1)} = S^{(2)} = L$ ,  $S^{(3)} = D$  and  $\varepsilon = 1$  in [19]. The problems of the two last articles were considered with  $\varepsilon = -1$  in [3]. So these aforementioned works always assumed that <sup>1</sup>  $S^{(1)} = S^{(2)}$ . In the present paper, we suppose  $S^{(1)} \notin \{S^{(2)}, S^{(3)}\}$ . Our main results on equation (1.3) and (1.4) are the following.

**Theorem 1.1.** *The largest repdigit which is a difference of Fibonacci and Lucas numbers is*

$$222 = 233 - 11 = F_{13} - L_5. \quad (1.6)$$

**Theorem 1.2.** *The largest repdigit which is a difference of Lucas and Fibonacci numbers is*

$$55 = 199 - 144 = L_{11} - F_{12}. \quad (1.7)$$

The paper is organized as follows. In Section 2, we recall some elementary properties of Fibonacci and Lucas numbers, Matveev's result on lower bound for linear forms in logarithms of algebraic numbers, as well as a variant of reduction method due to Baker and Davenport. The proofs of Theorems 1.1 and 1.2 are achieved in Sections 3 and 4 respectively.

## 2. PRELIMINARIES

**2.1. Fibonacci and Lucas sequences.** The Binet formula of  $F_n$  and  $L_n$  is given respectively by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{for all } n \geq 0 \quad (2.1)$$

and

$$L_n = \alpha^n + \beta^n \quad \text{for all } n \geq 0 \quad (2.2)$$

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<sup>1</sup>i.e., for example, if  $S^{(1)}$  is a Fibonacci number so is  $S^{(2)}$ . Similarly when  $S^{(1)}$  is a Lucas number or a repdigit.

where  $\alpha = (1 + \sqrt{5})/2$  and  $\beta = (1 - \sqrt{5})/2$  are the roots of the characteristic equation  $x^2 - x - 1$  of  $F_n$  and  $L_n$ . Furthermore, we have

$$\alpha^{n-2} < F_n < \alpha^{n-1} \quad (2.3)$$

and

$$\alpha^{n-1} < L_n < \alpha^{n+1} \quad (2.4)$$

**2.2. Linear forms in logarithms.** Let  $\eta$  be an algebraic number such that  $f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(k)}) \in \mathbb{Z}[X]$  be its minimal polynomial over  $\mathbb{Z}$  where  $\eta^{(1)}, \dots, \eta^{(k)}$  are its conjugate roots and  $a_0$  is a positive integer. The Weil height of  $\eta$  is given by

$$h(\eta) = \frac{1}{k} \left( \log a_0 + \sum_{j=1}^k \max\{0, \log |\eta^{(j)}|\} \right).$$

Let  $\mathbb{L}$  be an algebraic number field of degree  $d_{\mathbb{L}}$ . Let  $\eta_1, \eta_2, \dots, \eta_l \in \mathbb{L}$  not 0 or 1 and  $d_1, \dots, d_l$  be nonzero integers. We set

$$D = \max\{|d_1|, \dots, |d_l|\} \text{ and } \Gamma = \prod_{i=1}^l \eta_i^{d_i} - 1.$$

Let  $A_1, \dots, A_l$  be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for } j = 1, \dots, l$$

With these notations, we get the following consequences of Matveev's theorem.

**Theorem 2.1** ([5], Theorem 9.4). *If  $\Gamma \neq 0$  and  $\mathbb{L} \subseteq \mathbb{R}$ , then*

$$\log |\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}}) (1 + \log D) A_1 A_2 \cdots A_l.$$

**2.3. The Baker-Davenport lemma.** Here, we present de Weger's variant [20] of the reduction method of Baker and Davenport.

Let  $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$  be given, and let  $x_1, x_2 \in \mathbb{Z}$  be unknowns. Let

$$\Lambda = \beta + x_1 \vartheta_1 + x_2 \vartheta_2. \quad (2.5)$$

Let  $c, \delta$  be positive constants. Set  $X = \max\{|x_1|, |x_2|\}$ . Let  $X_0, Y$  be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \quad (2.6)$$

$$X \leq X_0. \quad (2.7)$$

When  $\beta = 0$  in (2.5), we get

$$\Lambda = x_1 \vartheta_1 + x_2 \vartheta_2.$$

Put  $\vartheta = -\vartheta_1/\vartheta_2$ . We assume that  $x_1$  and  $x_2$  are coprime. Let the continued fraction expansion of  $\vartheta$  be given by

$$[a_0, a_1, a_2, \dots],$$

and let the  $k$ th convergent of  $\vartheta$  be  $p_k/q_k$  for  $k = 0, 1, 2, \dots$ . We may assume without loss of generality that  $|\vartheta_1| < |\vartheta_2|$  and that  $x_1 > 0$ . We have the following results.

**Lemma 2.2.** (See Lemma 3.2 in [20]) Let

$$A = \max_{0 \leq k \leq Y_0} a_{k+1}.$$

where  $k$  is an integer such that  $k \leq -1 + \frac{\log(1 + X_0\sqrt{5})}{\log(\frac{1+\sqrt{5}}{2})} := Y_0$ .

If (2.6) and (2.7) hold for  $x_1, x_2$  and  $\beta = 0$ , then

$$Y < \frac{1}{\delta} \log \left( \frac{c(A+2)X_0}{|\vartheta_2|} \right). \quad (2.8)$$

When  $\beta \neq 0$  in (2.5), put  $\vartheta = -\vartheta_1/\vartheta_2$  and  $\psi = \beta/\vartheta_2$ . Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let  $p/q$  be a convergent of  $\vartheta$  with  $q > X_0$ . For a real number  $x$  we let  $\|x\| = \min\{|x - n|, n \in \mathbb{Z}\}$  be the distance from  $x$  to the nearest integer. We have the following result.

**Lemma 2.3.** (See Lemma 3.3 in [20]) Suppose that

$$\|q\psi\| > \frac{2X_0}{q}.$$

Then, the solutions of (2.6) and (2.7) satisfy

$$Y < \frac{1}{\delta} \log \left( \frac{q^2 c}{|\vartheta_2| X_0} \right).$$

### 3. THE PROOF OF THEOREM 1.1

Let us assume (1.3). We verified with Maple that there is no solutions in the interval  $n_1 \in [15, 300]$ . So, from now on, we may assume  $n_1 > 300$ . On the other hand, we have  $n_2 \geq 2$  since by [3],  $F_{n_1} - L_0 = F_{n_1} - F_3 = d\left(\frac{10^{n_1}-1}{9}\right)$  and  $F_{n_1} - L_1 = F_{n_1} - F_1 = d\left(\frac{10^{n_1}-1}{9}\right)$  have no solution when  $n_1 > 300$ .

**Lemma 3.1.** For  $n \geq 2$  we have  $F_{n+1} < L_n$

**Corollary 3.2.** If  $F_{n_1} - L_{n_2} \geq 0$  then  $n_1 \geq n_2 + 2$ .

**Lemma 3.3.** If  $n_1 = n_2 + 2$  the equation (1.3) has no a solution.

*Proof.* It's enough to solve  $F_{n+2} - L_n = d \left( \frac{10^m - 1}{9} \right)$ . We have

$$\begin{aligned} F_{n+2} - L_n &= \alpha^{n+2} + \beta^{n+2} - \frac{\alpha^n - \beta^n}{\sqrt{5}} \\ &= \frac{\alpha^n \left( \frac{3-\sqrt{5}}{2} \right) - \beta^n \left( \frac{3+\sqrt{5}}{2} \right)}{\sqrt{5}} \\ &= \frac{\alpha^n \beta^2 - \beta^n \alpha^2}{\sqrt{5}} \\ &= \frac{\alpha^{n-2} - \beta^{n-2}}{\sqrt{5}} \\ &= F_{n-2}. \end{aligned}$$

Thus  $F_{n-2} - F_0 = d \left( \frac{10^m - 1}{9} \right)$ . By [3], Theorem 2, this last equation has no solution.  $\square$

**Corollary 3.4.** *If the equation (1.3) has a solution, then  $n_1 \geq n_2 + 3$ .*

**Lemma 3.5.** *All solutions of the equation (1.3) satisfy*

$$0.20n_1 - 1 < m < 0.21n_1 + 1. \quad (3.1)$$

*Proof.* We have

$$d \left( \frac{10^m - 1}{9} \right) = \frac{\alpha^{n_1} - \beta^{n_1}}{\sqrt{5}} - (\alpha^{n_2} + \beta^{n_2})$$

By Corollary 3.4 we have  $n_1 \geq n_2 + 3$ . Thus

$$\begin{aligned} \sqrt{5}d \left( \frac{10^m - 1}{9} \right) &= \alpha^{n_1} - \beta^{n_1} - \sqrt{5}\alpha^{n_2} - \sqrt{5}\beta^{n_2} \\ &\geq \alpha^{n_1} - \alpha^{n_2+2} - |\beta| (1 + \sqrt{5}) \\ &\geq \alpha^{n_1} - \alpha^{n_2+2} - 2 \\ &\geq (1 - \alpha^{n_2+2-n_1})\alpha^{n_1} - 2 \geq (1 - \alpha^{-1})\alpha^{n_1} - 2 \\ &\geq (\alpha - 1)\alpha^{n_1-1} - 2 \end{aligned}$$

Hence

$$\begin{aligned} 10^m &\geq \frac{9((\alpha - 1)\alpha^{n_1-1} - 2)}{\sqrt{5}d} + 1 \geq \frac{(\alpha - 1)\alpha^{n_1-1}}{\sqrt{5}} \\ &\geq (\alpha - 1)\alpha^{n_1-3} \end{aligned}$$

so that

$$\begin{aligned} m \log(10) &\geq (n_1 - 3) \log(\alpha) + \log(\alpha - 1) \\ &\geq (n_1 - 4) \log(\alpha) \end{aligned}$$

which leads to

$$m \geq \frac{(n_1 - 4) \log(\alpha)}{\log(10)} \geq 0.20n_1 - 1 \quad (3.2)$$

On the other hand, we have  $\sqrt{5}d \left( \frac{10^m - 1}{9} \right) = \alpha^{n_1} - \beta^{n_1} - \sqrt{5}\alpha^{n_2} - \sqrt{5}\beta^{n_2} < \alpha^{n_1}$  since  $n_2 \geq 2$ ,  $n_1 > 300$ . Hence

$$10^m < \frac{9\alpha^{n_1}}{\sqrt{5}d} + 1 < 4.025\alpha^{n_1} + 1 < 4.65\alpha^{n_1}$$

It follows that

$$m < \frac{\ln(4.65)}{\ln(10)} + n_1 \frac{\ln(\alpha)}{\ln(10)} < 0.21n_1 + 1$$

□

**Lemma 3.6.** *All solutions of the equation (1.3) satisfy*

$$n_2 < n_1 < 6.4 \cdot 10^{29} \quad \text{and} \quad m < 1.3 \cdot 10^{29}.$$

*Proof.* We first use the Binet formulas (2.1) and (2.2) on equation (1.3) to get

$$d \left( \frac{10^m - 1}{9} \right) = \frac{\alpha^{n_1} - \beta^{n_1}}{\sqrt{5}} - \alpha^{n_2} - \beta^{n_2}$$

i.e.,

$$\sqrt{5}d(10^m - 1) = 9(\alpha^{n_1} - \beta^{n_1} - \sqrt{5}\alpha^{n_2} - \sqrt{5}\beta^{n_2}). \quad (3.3)$$

We examine (3.3) in two different steps as follows.

**Step 1.** Equation (3.3) gives

$$\sqrt{5}d10^m - 9\alpha^{n_1} = \sqrt{5}d - 9\beta^{n_1} - 9\sqrt{5}\alpha^{n_2} - 9\sqrt{5}\beta^{n_2}$$

which we rewrite as

$$\left| \sqrt{5}d \cdot 10^m - 9\alpha^{n_1} \right| = \left| \sqrt{5}d - 9\beta^{n_1} - 9\sqrt{5}\alpha^{n_2} - 9\sqrt{5}\beta^{n_2} \right|$$

Thus, dividing both sides by  $9\alpha^{n_1}$ , we get

$$\left| \frac{\sqrt{5}d}{9} 10^m \alpha^{-n_1} - 1 \right| \leq \frac{\sqrt{5}\beta + \beta^2 + \sqrt{5} + \sqrt{5}\beta^2}{\alpha^{n_1-n_2}} < \frac{4.86}{\alpha^{n_1-n_2}} \quad (3.4)$$

Put

$$\Gamma_1 := \frac{\sqrt{5}d}{9} 10^m \alpha^{-n_1} - 1. \quad (3.5)$$

We compare this upper bound with the lower bound on the quantity  $\Gamma_1$  given by Theorem 2.1. Observe first that  $\Gamma_1$  is not zero, for if it were, then  $\sqrt{5} = q\alpha^n$ , with some  $q \in \mathbb{Q}$ , and hence  $\alpha^{2n} = 5q^{-2} \in \mathbb{Q}$ , which is false for any  $n > 0$ . We take

$$\eta_1 = \frac{\sqrt{5}d}{9}, \quad \eta_2 = \alpha, \quad \eta_3 = 10, \quad d_1 = 1, \quad d_2 = -n_1, \quad d_3 = m,$$

where  $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{5})$  and  $d_1, d_2, d_3 \in \mathbb{Z}$ . The degree of  $\mathbb{L} := \mathbb{Q}(\sqrt{5})$  is  $d_{\mathbb{L}} = 2$ .

By Lemma 3.5, we deduce that  $m < n_1$ . Therefore, we can take  $D = n_1$ . We have

$$h(\eta_1) \leq h(9) + h(d\sqrt{5}) \leq 2h(9) + h(\sqrt{5}),$$

which implies that

$$h(\eta_1) < 5.2.$$

On the other hand

$$h(\eta_3) = \log(10) \text{ and } h(\eta_2) = \frac{\log(\alpha)}{2}.$$

Thus, we can take

$$A_1 := 10.4, \quad A_2 := 0.5 \text{ and } A_3 := 4.7.$$

By Theorem 2.1, we get

$$\log |\Gamma_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2) (1 + \log n_1) 10.4 \cdot 0.5 \cdot 4.7 > -2.38 \cdot 10^{13} (1 + \log n_1).$$

Comparing this last inequality with (3.4) leads to

$$(n_1 - n_2) \log \alpha < 2.4 \cdot 10^{13} (1 + \log n_1). \quad (3.6)$$

**Step 2.** Equation (3.3) becomes

$$10^m - \frac{9}{\sqrt{5}d} (\alpha^{n_1} - \sqrt{5}\alpha^{n_2}) = -\frac{9}{\sqrt{5}d} (\beta^{n_1} + \sqrt{5}\beta^{n_2}) + 1,$$

which we rewrite as

$$\left| 10^m - 9\alpha^{n_1} \left( \frac{1 - \sqrt{5}\alpha^{n_2-n_1}}{\sqrt{5}d} \right) \right| = \left| -\frac{9}{\sqrt{5}d} (\beta^{n_1} + \sqrt{5}\beta^{n_2}) + 1 \right| < 14.1.$$

Thus, dividing both sides by  $10^m$ , we get

$$\left| 1 - 9\alpha^{n_1} 10^{-m} \left( \frac{1 - \sqrt{5}\alpha^{n_2-n_1}}{\sqrt{5}d} \right) \right| = \frac{1}{10^m} \left| -\frac{9}{\sqrt{5}d} (\beta^{n_2} + \sqrt{5}\beta^{n_1}) + 1 \right| < \frac{14.1}{10^m}. \quad (3.7)$$

Put

$$\Gamma'_1 := 1 - 9\alpha^{n_1} 10^{-m} \left( \frac{1 - \sqrt{5}\alpha^{n_2-n_1}}{\sqrt{5}d} \right). \quad (3.8)$$

Suppose that  $\Gamma'_1 = 0$ . We then have

$$\sqrt{5}d = 9 \cdot 10^{-m} (\alpha^{n_1} - \sqrt{5}\alpha^{n_2}).$$

Conjugating in  $\mathbb{Q}(\sqrt{5})$ , we get

$$-\sqrt{5}d = 9 \cdot 10^{-m} (\beta^{n_1} + \sqrt{5}\beta^{n_2}).$$

Consequently, we obtain

$$\frac{2d10^m}{9\sqrt{5}} = F_{n_1} - L_{n_2}.$$

ie.,  $\sqrt{5} = \frac{9(F_{n_1} - L_{n_2})}{2d10^m}$  which leads to a contradiction since  $\sqrt{5} \notin \mathbb{Q}$ . Thus,  $\Gamma'_1 \neq 0$ . With the notations of Theorem 2.1, we take

$$\eta_1 = 9 \left( \frac{1 - \sqrt{5}\alpha^{n_2-n_1}}{\sqrt{5}d} \right), \quad \eta_2 = \alpha, \quad \eta_3 = 10, \quad d_1 = 1, \quad d_2 = n_1, \quad d_3 = -m,$$

where  $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{5})$  and  $d_1, d_2, d_3 \in \mathbb{Z}$ . The degree of  $\mathbb{L} := \mathbb{Q}(\sqrt{5})$  is  $d_{\mathbb{L}} = 2$

Since  $1 \leq n_2 \leq n_1$  and  $m < n_1$ , we can take  $D = n_1$ . We now discuss the  $A_j$ 's for  $j = 1, 2, 3$ . We have

$$\begin{aligned} h(\eta_1) &\leq h \left( 9 \left( \frac{1 - \sqrt{5}\alpha^{n_2-n_1}}{\sqrt{5}d} \right) \right) \\ &\leq h(9) + h(\sqrt{5}) + h(d) + h(1 - \sqrt{5}\alpha^{n_2-n_1}) \\ &\leq h(9) + h(\sqrt{5}) + h(9) + h(\sqrt{5}) + (n_1 - n_2)h(\alpha) + \log 2 \\ &\leq 6.7 + \frac{(n_1 - n_2)}{2} \log(\alpha); \end{aligned}$$

i.e.,

$$2h(\eta_1) < 13.4 + (n_1 - n_2) \log(\alpha).$$

Thus, using inequality (3.6) we obtain

$$A_1 := 2.5 \cdot 10^{13}(1 + \log n_1).$$

As before, we can choose

$$A_2 := 0.5 \quad \text{and} \quad A_3 := 4.7.$$

Theorem 2.1 implies that

$$\log |\Gamma'_1| > -5.7 \cdot 10^{25}(1 + \log n_1)^2.$$

Comparing this last inequality with (3.7) leads to

$$m < 2.5 \cdot 10^{25}(1 + \log n_1)^2$$

By Lemma 3.5, we have hence

$$0.20n_1 < 2.5 \cdot 10^{25}(1 + \log n_1)^2 + 1$$

which implies

$$n_1 < 1.3 \cdot 10^{26}(1 + \log n_1)^2$$

so that

$$n_1 < 6.4 \cdot 10^{29}$$

□

**Lemma 3.7.** *All solutions of (1.3) satisfy  $n_1 - n_2 \leq 170$ .*

*Proof.* We rewrite equation (3.3) into the form

$$\begin{aligned} 9\alpha^{n_1} - \sqrt{5}d10^m &= 9\beta^{n_1} + 9\sqrt{5}\alpha^{n_2} + (9\sqrt{5}\beta^{n_2} - \sqrt{5}d) \\ &> 9\sqrt{5}\alpha^2 - 0.01 - 9\sqrt{5}\beta^2 - 9\sqrt{5} \end{aligned}$$

since  $n_1 > 300$ ,  $n_2 \geq 2$ . Hence

$$\frac{9\alpha^{n_1}10^{-m}}{\sqrt{5}d} - 1 = -10^{-m} + \frac{9\beta^{n_1}10^{-m}}{\sqrt{5}d} + \frac{9\alpha^{n_2}10^{-m}}{d} + \frac{9\beta^{n_2}10^{-m}}{d}.$$

By (3.4)

$$\left| \frac{\sqrt{5}d\alpha^{-n_1}10^m}{9} \left| 1 - \frac{9\alpha^{n_1}10^{-m}}{\sqrt{5}d} \right| \right| < \frac{4.86}{\alpha^{n_1-n_2}}$$

so that

$$0 < \frac{9\alpha^{n_1}10^{-m}}{\sqrt{5}d} - 1 < \frac{9 \times 4.86}{\sqrt{5}d10^m\alpha^{-n_2}} \quad (3.9)$$

On the other hand, by (3.3) we have

$$\begin{aligned} \alpha^{n_1} &= \sqrt{5} \left( d \left( \frac{10^m - 1}{9} \right) + \alpha^{n_2} + \beta^{n_2} \right) + \beta^{n_1} \\ &\leq \sqrt{5} (10^m - 1 + \alpha^{n_2} + 1) + \beta^{n_1} \\ &\leq \sqrt{5} (10^m + \alpha^{n_2}) + \beta^{n_1}. \end{aligned}$$

Thus

$$\frac{\alpha^{n_1} - \beta^{n_1}}{\sqrt{5}} \leq 10^m + \alpha^{n_2}$$

so that

$$\alpha^{n_1-n_2} \left( \frac{1}{\sqrt{5}} - \frac{(-1)^{n_1}}{\sqrt{5}} \beta^{2n_1} - \frac{1}{\alpha^{n_1-n_2}} \right) \leq 10^m \alpha^{-n_2}.$$

Since  $n_1 \geq n_2 + 3$  and  $n_1 > 10$  we have

$$10^m \alpha^{n_2} \geq \left( \frac{1}{\sqrt{5}} - 0.01 - \frac{1}{\alpha^3} \right) \alpha^{n_1-n_2} > 0.2 \alpha^{n_1-n_2}$$

i.e.,

$$\frac{1}{10^m \alpha^{-n_2}} < \frac{1}{0.2 \alpha^{n_1-n_2}} = \frac{5}{\alpha^{n_1-n_2}}.$$

Hence by (3.9)

$$0 < \frac{9\alpha^{n_1}10^{-m}}{\sqrt{5}d} - 1 < \frac{9 \times 4.86\sqrt{5}}{\alpha^{n_1-n_2}} < \frac{97.9}{\alpha^{n_1-n_2}}.$$

So we have

$$0 < \eta_1^{-1} \eta_2^{n_1} \eta_3^{-m} - 1 < \frac{97.9}{\alpha^{n_1-n_2}}.$$

Put

$$\Lambda = -m \log 10 + n_1 \log \alpha + \log \left( \frac{9}{\sqrt{5}d} \right).$$

We obtain that

$$\frac{\alpha^{n_1}}{\sqrt{5}} - \frac{d10^m}{9} = \frac{d10^m}{9} (e^\Lambda - 1) > 0,$$

so

$$0 < \Lambda < e^\Lambda - 1 = \Gamma < \frac{97.9}{\alpha^{n_1-n_2}},$$

which implies that

$$0 < -m \log 10 + n_1 \log \alpha + \log \left( \frac{9}{\sqrt{5}d} \right) < \frac{97.9}{\alpha^{n_1-n_2}} < 10^2 \exp(-0.49 \cdot (n_1 - n_2)).$$

Thus, we see that

$$|\Lambda| < 10^2 \exp(-0.49 \cdot Y)$$

holds with  $Y := n_1 - n_2 < n_1 < 6.4 \cdot 10^{29}$  by Lemma 3.6.

We also have

$$\frac{\Lambda}{\log 10} = \frac{\log(9/(d\sqrt{5}))}{\log 10} + n_1 \frac{\log \alpha}{\log 10} - m.$$

Thus, we take

$$c = 10^2, \delta = 0.49, X_0 = 6.4 \cdot 10^{29}, \psi = \frac{\log(9/(d\sqrt{5}))}{\log 10},$$

$$\vartheta = -\frac{\log \alpha}{\log 10}, \vartheta_1 = \log \alpha, \vartheta_2 = \log 10, \beta = \log(9/d\sqrt{5}).$$

The smallest value of  $q > X_0$  is  $q = q_{63}$ . We find that  $q_{64}$  satisfies the hypothesis of Lemma 2.3 for  $d = 1, \dots, 9$ . Applying it, we get  $n_1 - n_2 = Y \leq 170$ .  $\square$

We end this section with the proof of Theorem 1.1

**Proof of Theorem 1.1.** We have

$$10^m - \frac{9}{\sqrt{5}d}(\alpha^{n_1} - \sqrt{5}\alpha^{n_2}) = \frac{-9(\beta^{n_1} + \sqrt{5}\beta^{n_2}) + \sqrt{5}d}{\sqrt{5}d} > \frac{-9(0.01 + \sqrt{5} \times 0.01) + \sqrt{5}}{\sqrt{5}} > 0,$$

since  $n_1 > n_2 > 10$ . By (3.7)

$$\begin{aligned} \left| \frac{\sqrt{5}d}{9\alpha^{n_1}10^{-m}(1 - \sqrt{5}\alpha^{n_2-n_1})} - 1 \right| &< \frac{14.1}{10^m} \times \frac{\sqrt{5}d}{9\alpha^{n_1}10^{-m}(1 - \sqrt{5}\alpha^{n_2-n_1})} = \frac{14.1\sqrt{5}d}{9\alpha^{n_1}} \times \frac{1}{1 - \sqrt{5}\alpha^{n_2-n_1}} \\ &< \frac{14.1\sqrt{5}}{\alpha^{n_1}(1 - \sqrt{5}\alpha^{-3})} \end{aligned}$$

since  $n_2 - n_1 \leq -3$ . Thus

$$\left| \frac{\sqrt{5}d}{9\alpha^{n_1}10^{-m}(1 - \sqrt{5}\alpha^{n_2-n_1})} - 1 \right| < \frac{66.8}{\alpha^{n_1}} \quad (3.10)$$

Put

$$\Lambda' = -n_1 \log \eta_2 + m \log \eta_3 - \log \eta_1$$

and

$$\Gamma' := \frac{\sqrt{5}d\alpha^{-n_1}10^m}{9(1 - \sqrt{5}\alpha^{n_2-n_1})} - 1 = e^{\Lambda'} - 1.$$

From equation (3.10), we have that

$$0 < \Lambda' < e^{\Lambda'} - 1 = |\Gamma'| < \frac{66.8}{\alpha^{n_1}}.$$

Hence, we have

$$0 < -n_1 \log \eta_2 + m \log \eta_3 - \log \eta_1 < 66.8 \exp(-0.49n_1).$$

We consider  $X_0 = 6.4 \cdot 10^{29}$ . Let

$$\psi' = \frac{\log \left( \frac{\sqrt{5}d}{9(1 - \sqrt{5}\alpha^{n_2-n_1})} \right)}{\log(10)} \quad c = 66.8, \quad \delta = 0.49,$$

$$\vartheta = \frac{\log \alpha}{\log 10}, \quad \vartheta_1 = -\log \alpha, \quad \vartheta_2 = \log 10, \quad \beta = \log \left( \frac{\sqrt{5}d}{9(1 - \sqrt{5}\alpha^{n_2-n_1})} \right).$$

The integer  $q = q_{62} > X_0$  obeys all conditions of Lemma 2.3. As  $n_1 > n_2$ , we obtain  $n_1 \leq 144$ . This contradicts the assumption that  $n_1 > 300$ . Therefore, the theorem is proved.  $\square$

#### 4. PROOF OF THEOREM 1.2

We use same the method as proof of Theorem 1.1. So let us assume (1.4). Similarly, we use Maple to verify that there is no solutions in the interval  $n_1 \in [15, 300]$  which allows us to assume  $n_1 > 300$  for the following. On the other hand, by F. Lucas [12], Theorem 2, and K. Bhoi, P. K. Ray [3], equation (1.4) has no solution when  $n_2 \in \{0, 1, 2\}$ . So we also suppose for the following that  $n_2 \geq 3$ .

**Lemma 4.1.** *For  $n \geq 3$  we have  $F_{n+2} > L_n$*

**Corollary 4.2.** *If the equation (1.4) has a solution then  $n_2 \leq n_1 + 2$ .*

**Lemma 4.3.** *If  $n_2 \in \{n_1 + 1, n_1 + 2\}$ , the equation (1.4) has no solution.*

*Proof.* First, it is enough to solve the equation  $L_n - F_{n+1} = d \left( \frac{10^m - 1}{9} \right)$ . We have

$$\begin{aligned} d \left( \frac{10^m - 1}{9} \right) &= \frac{\alpha^n(\sqrt{5} - \alpha) - \beta^n(\sqrt{5} + \beta)}{\sqrt{5}} \\ &= \frac{-\beta\alpha^n + \alpha\beta^n}{\sqrt{5}} \\ &= \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} \\ &= F_{n-1} \end{aligned}$$

which lead to contradiction since F. Luca [12] proved that  $F_0, F_1, F_2, F_3, F_4, F_5, F_6$  and  $F_{10}$  are the only repdigit in the Fibonacci numbers. On other hand,  $L_n - F_{n+2} = d\left(\frac{10^m-1}{9}\right)$  leads to

$$\begin{aligned} d\left(\frac{10^m-1}{9}\right) &= \frac{\alpha^n(\sqrt{5}-\alpha^2) - \beta^n(\sqrt{5}+\beta^2)}{\sqrt{5}} \\ &= \frac{-\beta^2\alpha^n + \alpha^2\beta^n}{\sqrt{5}} \\ &= \frac{-\alpha^{n-2} + \beta^{n-2}}{\sqrt{5}} \\ &= -F_{n-2}. \end{aligned}$$

We get a contradiction since  $d\left(\frac{10^m-1}{9}\right) > 0$  and  $-F_{n-2} \leq 0$ .  $\square$

**Corollary 4.4.** *If the equation (1.4) has a solution,  $n_2 \leq n_1$ .*

**Lemma 4.5.** *All solutions of equation (1.4) satisfy*

$$0.20n_1 - 1 < m < 0.21n_1 + 2. \quad (4.1)$$

*Proof.*

$$d\left(\frac{10^m-1}{9}\right) = \alpha^{n_1} + \beta^{n_1} - \frac{(\alpha^{n_2} - \beta^{n_2})}{\sqrt{5}}$$

By Corollary 4.4, we have  $n_1 \geq n_2$  when  $n_1 \geq 2$ . So

$$\begin{aligned} \sqrt{5}d\left(\frac{10^m-1}{9}\right) &= \sqrt{5}\alpha^{n_1} + \sqrt{5}\beta^{n_1} - \alpha^{n_2} + \beta^{n_2} \\ &\geq \alpha^{n_1+1} - \alpha^{n_2} \\ &\geq (1 - \alpha^{n_2-(n_1+1)})\alpha^{n_1+1} \geq (1 - \alpha^{-1})\alpha^{n_1+1} \\ &\geq (\alpha - 1)\alpha^{n_1} \end{aligned}$$

Hence

$$\begin{aligned} 10^m &\geq \frac{9(\alpha-1)\alpha^{n_1}}{\sqrt{5}d} + 1 \geq \frac{(\alpha-1)\alpha^{n_1}}{\sqrt{5}} + 1 \\ &\geq (\alpha-1)\alpha^{n_1-2} \end{aligned}$$

so that

$$\begin{aligned} m \ln(10) &\geq (n_1 - 2) \ln(\alpha) + \ln(\alpha - 1) \\ &\geq (n_1 - 3) \ln(\alpha) \end{aligned}$$

i.e.,

$$m \geq \frac{(n_1 - 3) \ln(\alpha)}{\ln(10)} \geq 0.20n_1 - 1 \quad (4.2)$$

On the other hand, we have

$$d\left(\frac{10^m-1}{9}\right) = \alpha^{n_1} + \beta^{n_1} - \frac{\alpha^{n_2} - \beta^{n_2}}{\sqrt{5}} < \alpha^{n_1} + |\beta| + \frac{|\beta|}{\sqrt{5}} < \alpha^{n_1} + 1.$$

Hence

$$10^m < \frac{9\alpha^{n_1} + 9}{d} + 1 < 9\alpha^{n_1} + 10 < 9\alpha^{n_1+1}$$

since  $n_1 > 300$ . It follows that

$$m < \frac{\ln(9)}{\ln(10)} + (n_1 + 1) \frac{\ln(\alpha)}{\ln(10)} < 0.21n_1 + 2$$

□

**Lemma 4.6.** *All solutions of equation (1.4) satisfy*

$$n_2 \leq n_1 < 5.4 \cdot 10^{29} \quad \text{and} \quad m < 1.14 \cdot 10^{29}.$$

*Proof.* We use the Binet formulas (2.1) and (2.2) on equation (1.4) to get

$$d \left( \frac{10^m - 1}{9} \right) = \alpha^{n_1} + \beta^{n_1} - \frac{\alpha^{n_2} - \beta^{n_2}}{\sqrt{5}}$$

i.e.,

$$\sqrt{5}d(10^m - 1) = 9(\sqrt{5}\alpha^{n_1} + \sqrt{5}\beta^{n_1} - \alpha^{n_2} + \beta^{n_2}). \quad (4.3)$$

We examine (4.3) in two different steps as follows.

**Step 1.** Equation (4.3) gives

$$\sqrt{5}d10^m - 9\sqrt{5}\alpha^{n_1} = \sqrt{5}d + 9\sqrt{5}\beta^{n_1} - 9\alpha^{n_2} + 9\beta^{n_2}$$

which we rewrite as

$$\left| \sqrt{5}d10^m - 9\sqrt{5}\alpha^{n_1} \right| = \left| \sqrt{5}d + 9\sqrt{5}\beta^{n_1} - 9\alpha^{n_2} + 9\beta^{n_2} \right|$$

Thus, dividing both sides by  $9\sqrt{5}\alpha^{n_1}$ , we get

$$\left| \frac{d}{9}10^m\alpha^{-n_1} - 1 \right| < \frac{1.8}{\alpha^{n_1-n_2}} \quad (4.4)$$

Put

$$\Gamma_2 := \frac{d}{9}10^m\alpha^{-n_1} - 1. \quad (4.5)$$

We apply Theorem 2.1 with  $\Gamma_2$ . Observe first that  $\Gamma_2$  is not zero, for if it were, then  $1 = q\alpha^{-n_1}$ , with some  $q \in \mathbb{Q}$ , and hence  $\alpha^n = q \in \mathbb{Q}$ , which is false since  $n_1 > 0$ . We take

$$\eta_1 = \frac{d}{9}, \eta_2 = \alpha, \eta_3 = 10, d_1 = 1, d_2 = -n_1, d_3 = m,$$

where  $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{5})$  and  $d_1, d_2, d_3 \in \mathbb{Z}$ . The degree of  $\mathbb{L} := \mathbb{Q}(\sqrt{5})$  is  $d_{\mathbb{L}} = 2$ .

By Lemma 4.5, we deduce that  $m < n_1$ . Therefore, we can take  $D = n_1$ . We have  $h(\eta_1) \leq h(9) + h(9) \leq 2h(9)$  which implies that  $2h(\eta_1) < 8.8$ . Thus, as in the previous section we can take

$$A_1 := 8.8, A_2 := 0.5, A_3 := 4.7.$$

From Theorem 2.1, we obtain

$$\log |\Gamma_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2) (1 + \log n_1) 8.8 \cdot 0.5 \cdot 4.7 > -2.01 \cdot 10^{13} (1 + \log n_1).$$

Comparing this last inequality with (4.4) leads to

$$(n_1 - n_2) \log(\alpha) < 2.019 \cdot 10^{13} (1 + \log n_1). \quad (4.6)$$

**Step 2.** Equation (4.3) becomes

$$10^m - \frac{9}{\sqrt{5}d} (\sqrt{5}\alpha^{n_1} - \alpha^{n_2}) = \frac{9}{\sqrt{5}d} (\sqrt{5}\beta^{n_1} + \beta^{n_2}) + 1,$$

which we rewrite as

$$\left| 10^m - \frac{9}{d} \alpha^{n_1} \left( 1 - \frac{\alpha^{n_2-n_1}}{\sqrt{5}} \right) \right| = \left| \frac{9}{\sqrt{5}d} (\sqrt{5}\beta^{n_1} + \beta^{n_2}) + 1 \right| < 14.1.$$

Thus, dividing both sides by  $10^m$ , we get

$$\left| 1 - \frac{9}{d} \alpha^{n_1} 10^{-m} \left( 1 - \frac{\alpha^{n_2-n_1}}{\sqrt{5}} \right) \right| = \frac{1}{10^m} \left| \frac{9}{\sqrt{5}d} (\sqrt{5}\beta^{n_1} + \beta^{n_2}) + 1 \right| < \frac{14.1}{10^m}. \quad (4.7)$$

Put

$$\Gamma'_2 := 1 - \frac{9}{d} \alpha^{n_1} 10^{-m} \left( 1 - \frac{\alpha^{n_2-n_1}}{\sqrt{5}} \right). \quad (4.8)$$

Suppose that  $\Gamma'_2 = 0$ . We then have  $10^m - \frac{9}{d\sqrt{5}} (\sqrt{5}\alpha^{n_1} - \alpha^{n_2}) = 0$  so that

$$\alpha^{n_1} - \frac{\alpha^{n_2}}{\sqrt{5}} = \frac{d_1 10^m}{9}.$$

Conjugating in  $\mathbb{Q}(\sqrt{5})$  leads to  $\beta^{n_1} + \frac{\beta^{n_2}}{\sqrt{5}} = \frac{d_1 10^m}{9}$ . Hence  $L_{n_1} - F_{n_2} = \frac{2d_1 10^m}{9}$  which is absurd since  $L_{n_1} - F_{n_2} = \frac{d_1(10^m-1)}{9}$ . Thus,  $\Gamma_2 \neq 0$ . With the notations of Theorem 2.1, we take

$$\eta_1 = \frac{9}{d} \left( 1 - \frac{\alpha^{n_2-n_1}}{\sqrt{5}} \right), \quad \eta_2 = \alpha, \quad \eta_3 = 10, \quad d_1 = 1, \quad d_2 = n_1, \quad d_3 = -m,$$

where  $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{5})$  and  $d_1, d_2, d_3 \in \mathbb{Z}$ . The degree of  $\mathbb{L} := \mathbb{Q}(\sqrt{5})$  is  $d_{\mathbb{L}} = 2$

Since  $1 \leq n_2 \leq n_1$  and  $m < n_1$ , we can take  $D = n_1$ . We have

$$\begin{aligned} h(\eta_1) &\leq h \left( \frac{9}{d} \left( 1 - \frac{\alpha^{n_2-n_1}}{\sqrt{5}} \right) \right) \\ &\leq h(9) + h(d) + h(\sqrt{5}) + h(\alpha^{n_2-n_1}) + \log(2) \\ &\leq h(9) + h(\sqrt{5}) + h(9) + (n_1 - n_2)h(\alpha) + \log 2 \\ &\leq 5.9 + \frac{(n_1 - n_2)}{2} \log(\alpha); \end{aligned}$$

i.e.,

$$2h(\eta_1) < 11.18 + (n_1 - n_2) \log(\alpha).$$

Thus, by using inequality (4.6) we obtain

$$A_1 := 2.02 \cdot 10^{13}(1 + \log n).$$

As before, we can choose

$$A_2 := 0.5 \quad \text{and} \quad A_3 := 4.7.$$

Theorem 2.1 implies that

$$\log |\Gamma'_2| > -4.7 \cdot 10^{25}(1 + \log n_1)^2.$$

Comparing this last inequality with (4.7) leads to

$$m < 2.1 \cdot 10^{26}(1 + \log n_1)^2$$

By Lemma 4.5, we have hence

$$0.20n_1 < 2.1 \cdot 10^{26}(1 + \log n_1)^2 + 1$$

which implies

$$n_1 < 1.1 \cdot 10^{26}(1 + \log n_1)^2$$

so that

$$n_1 < 5.4 \cdot 10^{29}$$

By Lemma 4.5, we have

$$m \leq 0.21n + 2 \leq 1.14 \cdot 10^{29}.$$

□

**Lemma 4.7.** *All solutions of equation (1.4) satisfy  $0 \leq n_1 - n_2 \leq 168$ .*

*Proof.* We rewrite (4.3) into the form

$$\begin{aligned} \sqrt{5}(9\alpha^{n_1} - d10^m) &= -\sqrt{5}d - 9\sqrt{5}\beta^{n_1} + 9\alpha^{n_2} - 9\beta^{n_2} \\ &\geq -9\sqrt{5} - 1 + 9\alpha^{n_2} - 9|\beta|^3 \quad \text{since } n_2 \geq 3, n_1 > 300 \\ &\geq 9\alpha^3 - (9\sqrt{5} + 1 + 9\beta^3) > 0 \end{aligned}$$

Hence

$$\frac{9\alpha^{n_1}10^{-m}}{d} - 1 = -10^{-m} - \frac{9\beta^{n_1}10^{-m}}{d} + \frac{9\alpha^{n_2}10^{-m}}{d\sqrt{5}} - \frac{9\beta^{n_2}10^{-m}}{d\sqrt{5}}$$

By (4.4)

$$\left| \frac{d\alpha^{-n_1}10^m}{9} \left| 1 - \frac{9\alpha^{n_1}10^{-m}}{d} \right| \right| < \frac{1.8}{\alpha^{n_1-n_2}}$$

so that

$$0 < \frac{9\alpha^{n_1}10^{-m}}{d} - 1 < \frac{9 \times 1.8}{d \times 10^m \alpha^{-n_2}} \leq \frac{9 \times 1.8}{10^m \alpha^{-n_2}} \quad (4.9)$$

On the other hand, by (4.3) we have

$$\begin{aligned}\alpha^{n_1} &= d \left( \frac{10^m - 1}{9} \right) - \beta^{n_1} + \frac{\alpha^{n_2} - \beta^{n_2}}{\sqrt{5}} \\ &\leq 10^m - 1 - \frac{\beta^{n_2}}{\sqrt{5}} + \frac{\alpha^{n_2}}{\sqrt{5}} - \beta^{n_1} \\ &\leq 10^m + \frac{\alpha^{n_2}}{\sqrt{5}}.\end{aligned}$$

Thus

$$10^m \geq \alpha^{n_1} - \frac{\alpha^{n_2}}{\sqrt{5}} = \alpha^{n_1} \left( 1 - \frac{\alpha^{n_2}}{\alpha^{n_1} \sqrt{5}} \right) > \left( 1 - \frac{1}{\sqrt{5}} \right) \alpha^{n_1}$$

so that

$$\alpha^{n_1 - n_2} \left( 1 - \frac{1}{\sqrt{5}} \right) < 10^m \alpha^{-n_2}.$$

Hence by (4.9)

$$0 < \frac{9\alpha^{n_1} 10^{-m}}{d} - 1 < \frac{1.8 \times 9 \times \sqrt{5}}{\sqrt{5} - 1} \times \frac{1}{\alpha^{n_1 - n_2}} < \frac{29.31}{\alpha^{n_1 - n_2}}$$

We have

$$0 < \eta_1^{-1} \eta_2^{n_1} \eta_3^{-m} - 1 < \frac{29.31}{\alpha^{n_1 - n_2}}$$

where

$$\eta_1 = \frac{d}{9}, \eta_2 = \alpha, \eta_3 = 10.$$

Put

$$\Lambda = -m \log 10 + n_1 \log \alpha + \log \left( \frac{9}{d} \right).$$

We obtain that

$$\alpha^{n_1} - \frac{d10^m}{9} = \frac{d10^m}{9} (e^\Lambda - 1) > 0,$$

so

$$0 < \Lambda < e^\Lambda - 1 = \Gamma < \frac{29.31}{\alpha^{n_1 - n_2}},$$

which implies that

$$0 < -m \log 10 + n_1 \log \alpha + \log \left( \frac{9}{d} \right) < 29.31 \exp(-0.49 \cdot (n_1 - n_2)).$$

Thus, we see that

$$|\Lambda| < 29.31 \exp(-0.49 \cdot Y)$$

holds with  $Y := n_1 - n_2 < n_1 < 5.4 \cdot 10^{29}$  according with Lemma 4.6.

We also have

$$\frac{\Lambda}{\log 10} = \frac{\log(9/d)}{\log 10} + n_1 \frac{\log \alpha}{\log 10} - m.$$

Thus, we take

$$c = 29.31, \delta = 0.49, X_0 = 5.4 \cdot 10^{29}, \psi = \frac{\log(9/d)}{\log 10},$$

$$\vartheta = -\frac{\log \alpha}{\log 10}, \vartheta_1 = \log \alpha, \vartheta_2 = \log 10, \beta = \log(9/d) \neq 0 \quad \text{if } d = 1, \dots, 8.$$

The smallest value of  $q > X_0$  is  $q = q_{63}$ . We find that  $q_{64}$  satisfies the hypothesis of Lemma 2.3 for  $d = 1, \dots, 8$ . Applying it, we get  $n_1 - n_2 = Y \leq 168$ . We now take  $0 \leq n_1 - n_2 \leq 168$ .

When  $d = 9$ ,  $\beta = 0$ . We must apply Lemma 2.2. Then, we take  $Y_0 := -1 + \frac{\log(1+X_0\sqrt{5})}{\log \alpha}$  and find that  $A = 106$  satisfies the hypothesis of Lemma 2.2. Applying it, we get  $0 \leq n_1 - n_2 = Y \leq 155$   $\square$

**Proof of Theorem 1.2.** We have

$$10^m - \frac{9}{\sqrt{5}d}(\sqrt{5}\alpha^{n_1} - \alpha^{n_2}) = 1 \frac{9\beta^{n_1}}{d} + \frac{9\beta^{n_2}}{d\sqrt{5}} > 0,$$

since  $n_1 > 300$ ,  $n_2 \geq 3$ . By (4.7)

$$\begin{aligned} \left| \frac{\sqrt{5}d}{9\alpha^{n_1}10^{-m}(\sqrt{5} - \alpha^{n_2-n_1})} - 1 \right| &< \frac{14.1}{10^m} \times \frac{\sqrt{5}d}{9\alpha^{n_1}10^{-m}(\sqrt{5} - \alpha^{n_2-n_1})} = \frac{14.1\sqrt{5}d}{9\alpha^{n_1}} \times \frac{1}{\sqrt{5} - \alpha^{n_2-n_1}} \\ &< \frac{14.1\sqrt{5}}{\alpha^{n_1}(\sqrt{5} - 1)} \end{aligned}$$

since  $n_2 \leq n_1$ . Thus

$$\left| \frac{\sqrt{5}d}{9\alpha^{n_1}10^{-m}(\sqrt{5} - \alpha^{n_2-n_1})} - 1 \right| < \frac{25.6}{\alpha^{n_1}} \quad (4.10)$$

Put

$$\Lambda' = -n_1 \log \eta_2 + m \log \eta_3 - \log \eta_1$$

and

$$\Gamma' := \frac{\sqrt{5}d\alpha^{-n_1}10^m}{9(\sqrt{5} - \alpha^{n_2-n_1})} - 1 = e^{\Lambda'} - 1.$$

From equation (4.10), we have that

$$0 < \Lambda' < e^{\Lambda'} - 1 = |\Gamma'| < \frac{25.6}{\alpha^{n_1}}.$$

Hence, we have

$$0 < -n_1 \log \eta_2 + m \log \eta_3 - \log \eta_1 < 25.6 \exp(-0.49n_1)$$

where

$$\eta_1 = \frac{d\sqrt{5}}{9(\sqrt{5} - \alpha^{n_2-n_1})}, \eta_2 = \alpha, \eta_3 = 10.$$

We consider  $X_0 = 5.4 \cdot 10^{29}$ . Let

$$\psi' = \frac{\log \left( \frac{\sqrt{5}d}{9(\sqrt{5} - \alpha^{n_2-n_1})} \right)}{\log(10)}, \quad c = 25.6, \quad \delta = 0.49,$$

$$\vartheta = \frac{\log \alpha}{\log 10}, \vartheta_1 = -\log \alpha, \vartheta_2 = \log 10, \beta = \log \left( \frac{\sqrt{5}d}{9(\sqrt{5} - \alpha^{n_2 - n_1})} \right).$$

The integer  $q = q_{63} > X_0$  obeys all conditions of Lemma 2.3. As  $n_1 \geq n_2$ , we obtain  $n_1 \leq 167$ . This contradicts the assumption that  $n_1 > 300$ . Therefore, the theorem is proved.  $\square$

## 5. CONCLUSION

Our paper completely solve two diophantine equations which are part of the diophantine equation

$$S_m^{(1)} + \varepsilon S_n^{(2)} = S_r^{(3)}. \quad (5.1)$$

where  $\varepsilon \in \{1, -1\}$  and each one of  $S^{(1)}$ ,  $S^{(2)}$  and  $S^{(3)}$  can be Fibonacci numbers  $F$  or Lucas numbers  $L$  or repdigits  $D$ . Many authors already studied such equation. The assumption  $S^{(1)} \notin \{S^{(2)}, S^{(3)}\}$  is the fundamental difference between our work and the previous which dealt with the equality  $S^{(1)} = S^{(2)}$ . However, despite many papers on the subject, the equation (5.1) is not totally solved. It will be interesting to investigate more in future works.

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## REFERENCES

1. Abdelalim, S., El Adlouni, H., and Diany, H. (2016). The Diophantine equation  $x^3 + y^3 = 2z^2$ . Gulf Journal of Mathematics 4 (4). <https://doi.org/10.56947/gjom.v4i4.266>
2. Adegbindin C., Luca F., & Togbé A. (2019). Lucas numbers as sums of two repdigits, Lith Math J, 59, 295–304. <https://doi.org/10.1007/s10986-019-09451-y>
3. Bhoi, K., & Ray, P. K. (2021). Repdigits as difference of two Fibonacci or Lucas numbers. Mat. Stud., 56, 124–132. <https://doi.org/10.30970/ms.56.2.124-132>
4. Bravo J. J., & Luca F. (2013). On a conjecture about repdigits in k-generalized Fibonacci sequences. Publ. Math. Debrecen, 82, 623–639. DOI:10.5486/PMD.2013.5390
5. Bugeaud Y., Mignotte M., & Siksek S. (2006). Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers. Ann. of Math., 163, 969–1018. DOI:10.4007/annals.2006.163.969
6. Bugeaud Y., & Mignotte M. (1999). On integers with identical digits. Mathematika, 46, 411–417. DOI:<https://doi.org/10.1112/S0025579300007865>
7. Carmichael, R. D. (1913). On the numerical factors of the arithmetic forms  $\alpha^n \pm \beta^n$ . Ann. Math., (2) 15, 30–70. <https://doi.org/10.2307/1967797>
8. Díaz-Alvarado, S., Luca F. (2011). Fibonacci numbers which are sums of two repdigits. In F. Luca and P. Stănică (Eds.), Proceedings of the 14th International Conference on Fibonacci Numbers and Their Applications, Morelia, Mexico, July, 5-9, 2010, Aportaciones Mat., Investig., Vol. 20, Sociedad Matemática, México, 2011, pp. 97-108.
9. Dossavi-Yovo, A., Luca F., & Togbé A. (2016). On the  $x$ -coordinates of Pell equations which are rep-digits. Publ. Math. Debrecen, (9) 88, 3–4. <https://DOI:10.5486/PMD.2016.7378>
10. Erduvan F., Keskin R., & Luca F. (2022). Fibonacci and Lucas numbers as difference of two repdigits. Rend. Circ. Mat. Palermo, II. Ser, 71 , 575–589. <https://doi.org/10.1007/s12215-021-00645-3>

11. Faye, B., & Luca, F. (2015). Pell and Pell-Lucas numbers with only one distinct digit, *Ann. Math. Inform.*, 45 pp. 55–60. <https://api.semanticscholar.org/CorpusID:54846151>
12. Luca, F. (2000). Fibonacci and Lucas numbers with only one distinct digit. *Port. Math.*, 57.2, 243–254. <http://eudml.org/doc/48677>
13. Luca, F. (2000). Distinct digits in base  $b$  expansions of linear recurrence sequences, *Quaest. Math.*, 23, 389–404. <https://doi.org/10.2989/16073600009485986>
14. Luca, F. (2012). Repdigits which are sums of at most three Fibonacci number. *Math. Comm.*, 17, 1–11. <https://api.semanticscholar.org/CorpusID:11491735>
15. Marques, D., & Togbé, A. (2012). On repdigits as product of consecutive Fibonacci numbers. *Rend. Istit. Mat. Univ. Trieste*, 44, (2012), 393–397. <https://api.semanticscholar.org>
16. Matveev, E. M. (2000). An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, II, *Izv. Ross. Akad. Nauk Ser. Mat.* 64, 125–180. English translation in *Izv. Math.* 64 (2000), 1217–1269. DOI:10.1070/IM2000v064n06ABEH000314
17. Nikiema, S., Odjoumani, J. (2023), Lucas generalized numbers in Narayana’s cow sequence. *Gulf Journal of Mathematics*, 15 (1), 84-92. <https://doi.org/10.56947/gjom.v15i1.1381>
18. Normenyo, B.V., Kafle, B. & Togbé, A. (2019). Repdigit as sum of two Fibonacci numbers and two Lucas numbers, *Integers*, 19, 1–16. <https://api.semanticscholar.org>
19. Siary, Z., & Keskinz, R. (2020). Repdigits As Sums Of Two Lucas Numbers. *Applied Mathematics E-Notes*, 20, 33-38. <https://api.semanticscholar.org>
20. de Weger, B.M.M. (1989). Algorithms for diophantine equations, *Stichting Mathematisch Centrum*, <https://api.semanticscholar.org>

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