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IDEMPOTENT GRAPH OF 2×2 MATRIX RING WITH INVOLUTION

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Abstract. Let $R = M_2(\mathbb{F})$, where $\mathbb{F}$ is a finite field. In this paper, we investigate the idempotent graph of a ring R denoted by $I^*(R)$. We demonstrate that $I^*(R)$ is disconnected, having the components either complete bipartite graphs or complete graphs. A characterization is obtained for the regularity of $I^*(R)$. We determine the adjacency and Laplacian spectrum, the energy of $I^*(R)$ and prove Beck's conjecture.

1. Introduction

Exploring the relationship between different structures and analyzing the algebraic properties of rings through graph-theoretical properties is an interesting area of research. Although commutative rings have been extensively studied, expanding these ideas to non-commutative rings, notably *-rings, introduces significant challenges. The idempotent graph links ring theory and combinatorics by visually representing idempotent relationships in non-commutative rings. Compared to other graphs like the zero-divisor and unit graphs, it offers a new perspective. This study helps bridge algebra and graph theory. Studying the zero-divisor graph of $*$-rings is a difficult task, particularly for non-commutative rings. By examining the structure of idempotent graphs, we can try to explore the structure of graphs in non-commutative rings.

A mapping $*$ is called an *involution* on R, if $(a+b)^* = a^* + b^*, (ab)^* = b^*a^*$, and $(a^*)^* = a$ for any $a, b \in R$. A $*$-ring is a ring equipped with an involution $*$. An element $a \in R$ with $a^2 = a$ is called an *idempotent*. The elements 0 and 1 are termed trivial idempotents, while any idempotent other than 0 or 1 is considered non-trivial. I Beck [5] initiated the concept of zero-divisor graphs for commutative rings, primarily for the purpose of studying graph coloring. He conjectured that its chromatic number and clique numbers are same. However, Anderson *et al.* [2] demonstrated that the conjecture is not valid by providing a counterexample. Anderson and Livingston [3] defined this graph for commutative rings. Redmond [16] extended this concept to non-commutative rings. Patil and Waphare [15] assigned a simple graph for $*$-rings, denoted by $\Gamma^*(R)$, whose vertex

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set consists of all nonzero left zero-divisors in R, and two vertices x and y are adjacent if $xy^* = 0$. The set of idempotent elements in R is denoted by $Id(R)$. Akbari *et al.* [1] introduced the *graph* for a ring R, denoted by $I(R)$. In [14], authors studied the idempotent graph denoted by $GId(R)$. The study of these graphs is explored in [1, 4, 6, 8, 9, 10, 11, 13, 14]. In [9], authors studied the generalized projections in $\mathbb{Z}_n$.

We begin by introducing the necessary definitions required for the subsequent discussion. A graph G is called *connected* if there is a path between any two vertices of G. If degree of every vertex is k, then a graph is called *k-regular*, where k is referred to as the valency of G. If there is an edge between any pair of vertices, then a graph is called *complete*. If the vertices of a graph can be divided into two disjoint sets, such that no two vertices within the same set are adjacent, then it is called *bipartite graph*. Let K_n and $K_{m,n}$ denotes the complete graph and complete bipartite graph respectively. For a simple graph G with n vertices labeled as $v_1, v_2, \ldots, v_n$, the *adjacency matrix* $A(G) = [a_{ij}]$ is an $n \times n$ matrix, where: $a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$
The multiset of eigenvalues of $A(G)$ is called *adjacency spectrum*. Let D denotes the diagonal matrix with entries as degrees of vertices, then $L(G) = D - A(G)$ is *Laplacian matrix* of G. The multiset of eigenvalues of $L(G)$ is *Laplacian spectrum* of G. The energy of graph is the sum of the absolute values of its adjacency eigenvalues, and is denoted by $\mathcal{E}(G)$. The minimum number of colors require to color the vertices of G in such a way that no two adjacent vertices receive the same color is called *chromatic number* of G, and is denoted by $\chi(G)$. The size of the largest clique (a complete subgraph) within G is denoted by $\omega(G)$, and is called *clique number* of G. For further details on graph theory concepts, readers may refer to [7].

Throughout this paper, $\mathbb{F}$ denotes a finite field of order $n = |\mathbb{F}| = p^k$, for prime p and positive integer k, and $R = M_2(\mathbb{F})$. Let A^t denote the transpose of a matrix A. In this paper, we assign a simple graph $I^*(R)$ to a ring R, whose vertices are non-trivial idempotents, and two vertices A and B are adjacent if $AB^t = 0$. A characterization is obtained for the regularity of $I^*(R)$. We show that $I^*(R)$ is disconnected, and its components form either complete bipartite graphs or complete graphs. Furthermore, we explicitly determine the adjacency and Laplacian spectrum, and calculate the energy of $I^*(R)$. Additionally we provide a proof of Beck's conjecture for $I^*(R)$.

2. Idempotent elements in $M_2(\mathbb{F})$

We use the following notations:

$$E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ E^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ E_a = \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}, \ E^a = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, F^a = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix},$$

$$F_a = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}, \ E_{b,c} = \begin{pmatrix} b & c \\ b(1-b)c^{-1} & 1-b \end{pmatrix}, \ M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$N_k = \begin{pmatrix} 1 & k \\ -\frac{1}{k} & -1 \end{pmatrix}$, where $a,\ b,\ c,\ k \in \mathbb{F} \setminus \{0\}$, and $b \neq 1$.
Let $Z(R)$ denote the set of non-trivial idempotent elements in the ring R. Recent studies [12, 14] have examined idempotent elements in $M_2(\mathbb{F})$. The classification of idempotent elements in the ring $M_2(\mathbb{F})$ is given below.

Lemma 2.1. [12] *Every idempotent element in $Z(M_2(\mathbb{F}))$ has one of the following forms.*

$$(i)\ E_0, E^0, E_a, E^a, F_a, F^a \text{ for some } a \in \mathbb{F}\backslash\{0\},$$
$$(ii)\ E_{b,c} \text{ for some nonzero } \ b \in \mathbb{F}\backslash\{0,1\},\ c \in \mathbb{F}\backslash\{0\}.$$

Next, we count the idempotent elements in $Z(M_2(\mathbb{F}))$.

Lemma 2.2. *The number of non-trivial idempotent elements in $M_2(\mathbb{F})$ is $n(n+1)$.*

Proof. The non-trivial idempotent elements in $Z(M_2(\mathbb{F}))$ are listed in Lemma 2.1. We count their cardinalities: $|E_0| = |E^0| = 1$, since $a \in \mathbb{F} \setminus \{0\}$, $|E_a| = |E^a| = |F_a| = |F^a| = n-1$. Now $c \in \mathbb{F}\backslash\{0\},\ b \in \mathbb{F}\backslash\{0,1\}$, therefore $|E_{b,c}| = (n-1)(n-2)$. By adding all these, the count of non-trivial idempotent elements in $M_2(\mathbb{F})$ is $n(n+1)$. □

Note the following observations:

(1) $(E^0)^t = E^0,\ (E_0)^t = E_0,\ (E_a)^t = F^a,\ (E^a)^t = F_a, M^t = N, (N_k)^t = N_{-1/k}$.
(2) $F_{a_j}E_{a_k} = F^{a_j}E^{a_k} = 0$.
(3) $E_0E^{a_j} = F_{a_j}E_0 = E^0E_{a_j} = F^{a_j}E^0 = ME_{a_j} = F_{a_j}M = NE^{a_j} = E_{a_j}N = E_0E^0 = ME_0 = NE^0 = 0$.

3. The Idempotent Graph

The study of adjacency spectra, Laplacian spectra, and graph energy helps us understand graph structures, connections, and stability in different fields. These properties give important mathematical insights and have practical uses in science, engineering, and technology. The second largest adjacency eigenvalue affects how a graph expands, which is important in expander graphs. The second smallest Laplacian eigenvalue, called algebraic connectivity, shows how well the graph is connected. In chemistry, energy helps analyze the stability of molecules. It is also used in physics, biology, and sociology to study complex systems.

Akbari *et al.* [1] assigned a simple *graph* $I(R)$ to a ring R. The vertex set of this graph contains the non-trivial idempotents of R, and vertices a and b being adjacent if $ab = ba = 0$. Patil *et al.* [14] studied idempotent graph $GId(R)$, having vertices as non-trivial idempotents of R, and two distinct vertices a and b are adjacent if $ab = 0$ or $ba = 0$. Note that $I(R)$ is a subgraph of $GId(R)$.

In this section, we extend the concept of idempotent graphs to $*$-rings and determine the spectrum and energy of $I^*(R)$.

Definition 3.1. We assign a simple graph $I^*(R)$ to a $*$-ring R, having vertices non-trivial idempotent elements of R, and the vertices a and b are adjacent if $ab^* = 0$.

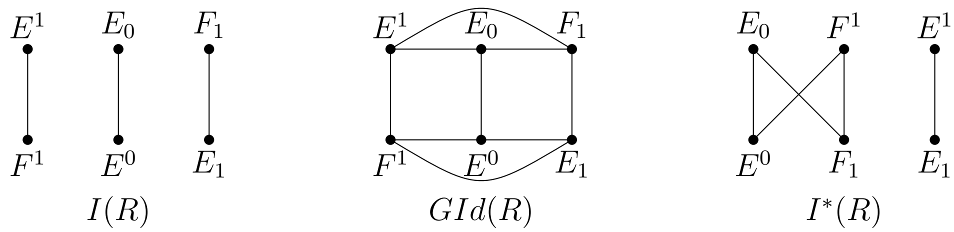


FIGURE 1. The graphs $I(R), GId(R), I^*(R)$

If R is a commutative ring, then $I(R), GId(R)$, and $I^*(R)$ are identical graphs. The following example demonstrate that $I(R), GId(R)$, and $I^*(R)$ are not identical.

Example 3.2. Let $R = M_2(\mathbb{Z}_2)$. Here, the non-trivial idempotent are given by: $\{E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, E^1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, F_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$ $F^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\}$. The graphs $I(R)$, $GId(R)$ and $I^*(R)$ are shown in Figure 1.

Recall the following result from Number Theory. A positive integer a is said to be a quadratic residue modulo n if there is an integer x such that $x^2 \equiv a \pmod{n}$. For a prime p, -1 is quadratic residue modulo p if and only if $a^2 = -1$ for some $a \in \mathbb{F}_{p^k} \setminus \{0\}$.

Remark 3.3. For a prime p, let $\mathbb{F}_{p^k}$ denotes a field with p^k elements. Then:
(1) If $p = 4r + 1$, then there are exactly 2 elements in $\mathbb{F}$ with $a^2 = -1$.
(2) If $p = 4r + 3$, then there is no element in $\mathbb{F}$ with $a^2 = -1$.
(3) If $p = 2$, then $(a + 1)^2 = a^2 + 1 = 0$ implies $a = -1 = 1$.

In [14], it was demonstrated that the graph $GId(R)$ exhibits regularity. However, the graph $I^*(R)$ is not regular, as illustrated in Example 3.2 (Figure 1). Let $\mathcal{N}(A) = \{B \in Z(M_2(R))\backslash\{A\} : AB^* = AB^t = 0\}$ be the neighborhood of A in $I^*(R)$. If $p = 4r + 3$, the following result shows that $I^*(R)$ is regular.

Lemma 3.4. *Let $R = M_2(\mathbb{F})$ and $p = 4r + 3$. Then $I^*(R)$ is n-regular graph.*

Proof. We determine the degrees of the elements in $I^*(R)$.
Since $p = 4r + 3$, there is no element in $\mathbb{F}$ satisfying $a^2 = -1$.
(1) $\mathcal{N}(E_0) = \{A \in Z(M_2(\mathbb{F})) : E_0A^t = 0,\ i.e.,\ AE_0 = 0\}$.
Therefore $\mathcal{N}(E_0) = \{E^0, F_a : a \in \mathbb{F} \setminus \{0\}\}$. Hence $|\mathcal{N}(E_0)| = n$.
(2) $\mathcal{N}(E^0) = \{A \in Z(M_2(\mathbb{F})) : E^0A^t = 0,\ i.e.,\ AE^0 = 0\}$.
Therefore $\mathcal{N}(E^0) = \{E_0, F^a : a \in \mathbb{F} \setminus \{0\}\}$. Hence $|\mathcal{N}(E^0)| = n$.
(3) $\mathcal{N}(E_a) = \{A \in Z(M_2(\mathbb{F})) : E_aA^t = 0\ i.e.,\ A(E_a)^t = AF^a = 0\}$.
Here $E^{-a}F^a = 0$. Observe that $E_{-a^{-1}}F^a = 0$. Thus, $E_{-a^{-1}} \in \mathcal{N}(E_a)$ if $E_{-a^{-1}} \neq E_a$ *i.e.*, if $a^2 \neq -1$ in $\mathbb{F}$. Next, if $E_{b,c} \in \mathcal{N}(E_a)$, then $E_{b,c}F^a = 0$. This results in the equation $ba + c = 0$ or $c = -ba$. Therefore,
$\mathcal{N}(E_a) = \{E^{-a}, E_{-a^{-1}}, E_{b,-ba} : b \in \mathbb{F} \setminus \{0, 1\}\}$, for $a \neq -a^{-1}$. Hence $|\mathcal{N}(E_a)| = n$.

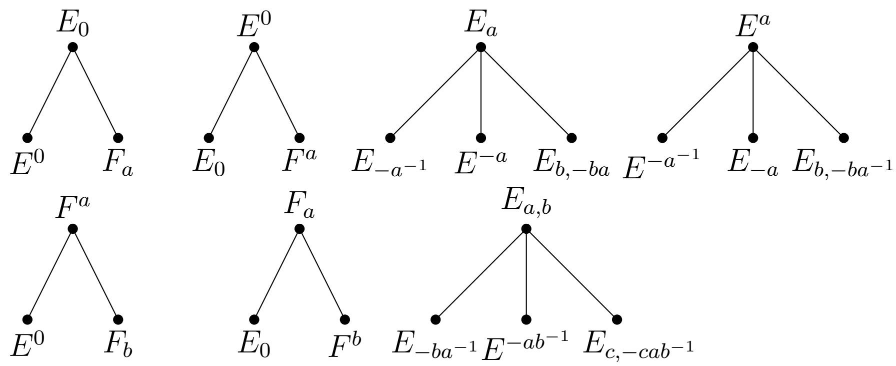


FIGURE 2. Neighborhoods of idempotent elements in $I^*(R)$

(4) $\mathcal{N}(E^a) = \{A \in Z(M_2(\mathbb{F}))\colon E^aA^t = 0 \ i.e.,\ A(E^a)^t = AF_a = 0\}$.
Note that $E_{-a}F_a = 0$ and $E^{-a^{-1}}F_a = 0$. Next, if $E_{b,c} \in \mathcal{N}(E_a)$, then $E_{b,c}F_a = 0$, which yields $c = -ba^{-1}$. Therefore
$\mathcal{N}(E^a) = \{E_{-a}, E^{-a^{-1}}, E_{b,-ba^{-1}}\colon b \in \mathbb{F} \setminus \{0,1\}\}$. Hence $|\mathcal{N}(E^a)| = n$.

(5) $\mathcal{N}(F^a) = \{A \in Z(M_2(\mathbb{F}))\colon F^aA^t = 0 \ i.e.,\ A(F^a)^t = AE_a = 0\}$.
Note that $E^0E_a = 0$ and $F_cE_a = 0$, for $c \in \mathbb{F} \setminus \{0\}$. If $E_{b,c}E_a = 0$, this yields $c = 0$, which is a contradiction. Hence $E_{b,c}E_a \neq 0$ for any b, c. Therefore $\mathcal{N}(F^a) = \{E^0, F_c\colon b \in \mathbb{F} \setminus \{0\}\}$. Hence $|\mathcal{N}(F^a)| = n$.

(6) $\mathcal{N}(F_a) = \{A \in Z(M_2(\mathbb{F}))\colon F_aA^t = 0 \ i.e.,\ A(F_a)^t = AE^a = 0\}$.
Note that $E_0E^a = 0$ and $F^bE^a = 0$ for $b \in \mathbb{F} \setminus \{0\}$, and $E_{b,c}E^a \neq 0$ for any b, c. Therefore $\mathcal{N}(F_a) = \{E_0, F^c\colon b \in \mathbb{F} \setminus \{0\}\}$. This gives $|\mathcal{N}(F_a)| = n$.

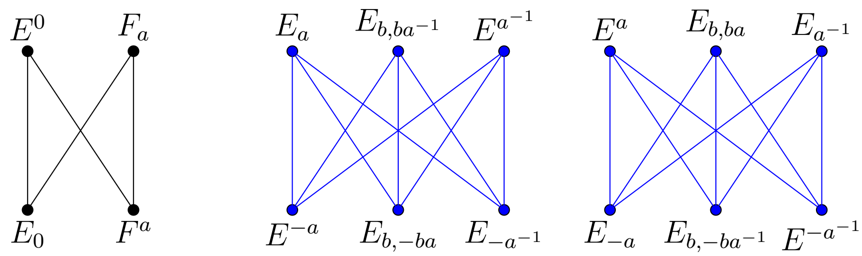
(7) $\mathcal{N}(E_{b,c}) = \{A \in Z(M_2(\mathbb{F}))\colon E_{b,c}A^t = 0\}$.
Note that $E_{b,c}E_a \neq 0$ implies $(E_a)^t = F^a \notin \mathcal{N}(E_{b,c})$. Similarly, $E_{b,c}E^a \neq 0$ also implies $(E^a)^t = F_a \notin \mathcal{N}(E_{b,c})$. Next, $E_{b,c}(E_a)^t = 0$ holds if and only if $a = -cb^{-1}$. Also, $E_{b,c}(E^a)^t = 0$ holds if and only if $a = -bc^{-1}$. Additionally, $E_{b,c}(E_{a_k,a_l})^t = 0$ holds if and only if $ba_k + a_lc = 0$ i.e., $a_l = -bc^{-1}a_k$. Therefore $\mathcal{N}(E_{b,c}) = \{E_{-cb^{-1}}, E^{-bc^{-1}}, E_{a_k,-a_kbc^{-1}}\colon a_k \in \mathbb{F} \setminus \{0,1\}\}$. Hence $|\mathcal{N}(E_{a_i,a_j})| = n$.

Thus, if $p = 4r + 3$, then $I^*(R)$ is n-regular graph. □

In the following result, we establish that if $p = 4r + 3$, then $I^*(R)$ is both disconnected and a regular bipartite graph.

Proposition 3.5. *Let $R = M_2(\mathbb{F})$ and $p = 4r + 3$. Then $I^*(R)$ has $\dfrac{n+1}{2}$ connected components, and each component is $K_{n,n}$.*

Proof. Define the sets: $X = \{E_0, E^0, F_a, F^a\}$ and $Y = \{E_b, E^b, E_{a,b}\colon a, b \in \mathbb{F} \setminus \{0\}, a \neq 1\}$. From the neighborhoods defined in Lemma 3.4, Figure 2 and Figure 3, observe that no vertex in X is connected by a path with a vertex in Y. Thus X and Y forms two disconnected components in $I^*(R)$. Hence $I^*(R)$ is disconnected. Additionally, the vertices in X forms complete bipartite graph with the partition of vertices $\{E^0, F_a\}$ and $\{E_0, F^a\}$, see Figure 3. Moreover, $U = \{E^{a^{-1}}, E_a, E_{b,ba^{-1}}, E^a, E_{a^{-1}}, E_{b,-ba}\}$ and



(a) Component X (b) Component Y

FIGURE 3. The components X and Y of $I^*(R)$.

$V = \{E_{-a^{-1}}, E^{-a}, E_{b,ba}, E^{-a^{-1}}, E_{-a}, E_{b,-ba^{-1}}\}$ gives a partition of vertex set Y, such that no vertices within U are adjacent, and no vertices within V are adjacent. (Refer to Figure 3 for visualization). Thus Y is bipartite. By Lemma 3.4, $I^*(R)$ is regular with valency n. Hence each component of $I^*(R)$ is $K_{n,n}$.

Next, by Lemma 2.2, there are $n(n+1)$ non-trivial idempotent elements in $I^*(R)$. Since the components of $I^*(R)$ are $K_{n,n}$. Therefore each connected component of $I^*(R)$ contains $2n$ vertices. Hence there are $\dfrac{n(n+1)}{2n} = \dfrac{n+1}{2}$ connected components in $I^*(R)$. Thus, if $p = 4r+3$, then $I^*(R)$ has $\dfrac{n+1}{2}$ connected components, and each component is $K_{n,n}$. □

Spectral properties of k-regular graphs are used in network synchronization, graph partitioning, and expander graph analysis. Regular graphs often have symmetric spectra, making them useful in designing efficient algorithms for spectral clustering and optimization. Let λ be an eigenvalue with multiplicity s, then we denote it as $\lambda^{(s)}$. Recall the following:

Suppose G is a simple undirected regular graph with valency k. Then:
(i) The largest eigenvalue of the adjacency matrix of G is equal to k.
(ii) The multiplicity of an eigenvalue k is equal to the number of connected components in G.
(iii) The adjacency spectrum of K_n is a multiset: $\{(n-1)^{(1)}, (-1)^{(n-1)}\}$.
(iv) The adjacency spectrum of $K_{n,n}$ is a multiset: $\{n^{(1)}, (0)^{(2n-2)}, -n^{(1)}\}$.
(v) The Laplacian spectrum of K_n is a multiset: $\{0^{(1)}, (n)^{(n-1)}\}$.
(vi) The Laplacian spectrum of $K_{n,n}$ is a multiset: $\{0^{(1)}, (n)^{(2n-2)}, (2n)^{(1)}\}$.
Now, we apply this to analyze the spectrum of $I^*(R)$.

Proposition 3.6. *Let $R = M_2(\mathbb{F})$ and $p = 4r+3$. The adjacency spectrum of $I^*(R)$ is a multiset:*

$$\left\{(n)^{(\frac{n+1}{2})}, (0)^{(n^2-1)}, (-n)^{(\frac{n+1}{2})}\right\},$$

and the Laplacian spectrum of $I^*(R)$ is a multiset:

$$\left\{0^{(\frac{n+1}{2})}, (2n)^{(\frac{n+1}{2})}, n^{(n^2-1)}\right\}.$$

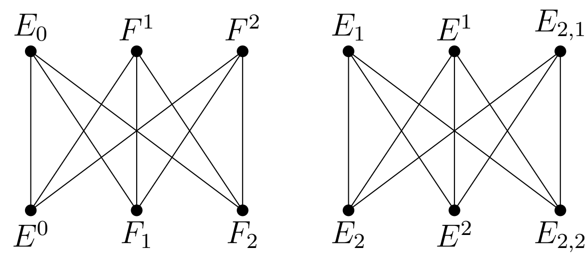


FIGURE 4. $I^*(M_2(\mathbb{Z}_3))$.

Proof. The graph $I^*(R)$ has $\dfrac{n+1}{2}$ connected components, and each component is $K_{n,n}$ (see Proposition 3.5). Thus n is the largest eigenvalue of $I^*(R)$, and its multiplicity is $\dfrac{n+1}{2}$. The eigenvalues of $K_{n,n}$ are given by: $(n)^{(1)}, 0^{(2n-2)}, (-n)^{(1)}$. Thus the adjacency spectrum of $I^*(R)$ is a multiset:

$$\left\{(n)^{(\frac{n+1}{2})}, (0)^{(n^2-1)}, (-n)^{(\frac{n+1}{2})}\right\}.$$

Furthermore, the Laplacian eigenvalues of $K_{n,n}$ are $0^{(1)}, (2n)^{(1)}, (n)^{(2n-2)}$. Hence the Laplacian spectrum of $I^*(R)$ is a multiset:

$$\left\{(0)^{(\frac{n+1}{2})}, (2n)^{(\frac{n+1}{2})}, (n)^{(n^2-1)}\right\}.$$

□

Using the above result, we compute the energy of $I^*(R)$.

Corollary 3.7. *Let $R = M_2(\mathbb{F})$ and $p = 4r + 3$. Then the energy of $I^*(R)$ is given by: $\mathcal{E}(I^*(R)) = n(n+1)$.*

Proof. Since the energy of a graph is given by the sum of the absolute values of its adjacency eigenvalues. Hence $\mathcal{E}(I^*(R)) = \left(\frac{n+1}{2}\right)n + \left(\frac{n+1}{2}\right)n = n(n+1)$. □

As an illustration consider the example given below:

Example 3.8. Let $R = M_2(\mathbb{Z}_3)$. The idempotent elements in R are listed below:
$E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}, E^1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$
$E^2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, F^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, F^2 = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, F_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},$
$E_{2,1} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, E_{2,2} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$
The neighborhoods of the vertices are given below:
$\mathcal{N}(E_0) = \{E^0, F_1, F_2\}$, $\mathcal{N}(E^0) = \{E_0, F^1, F^2\}$, $\mathcal{N}(E_1) = \{E_{2,1}, E_2, E^2\}$,
$\mathcal{N}(E_2) = \{E_{2,2}, E^1, E_1\}$, $\mathcal{N}(E^1) = \{E_{2,1}, E^2, E_2\}$, $\mathcal{N}(E^2) = \{E_{2,2}, E^1, E_1\}$,
$\mathcal{N}(F^1) = \{E^0, F_1, F_2\}$, $\mathcal{N}(F^2) = \{E^0, F_1, F_2\}$, $\mathcal{N}(F_1) = \{E_0, F^1, F^2\}$,
$\mathcal{N}(F_2) = \{E_0, F^1, F^2\}$, $\mathcal{N}(E_{2,2}) = \{E_{2,1}, E_2, E^2\}$, $\mathcal{N}(E_{2,1}) = \{E_{2,2}, E_1, E^1\}$.
The graph $I^*(R)$ is depicted in Figure 4.

Here, $p = 3$ and $p = 4r + 3$. Observe that $I^*(R)$ is $p = 3$-regular graph with $\frac{n+1}{2} = 2$ connected components, each of which is $K_{3,3}$. The adjacency spectrum of $K_{3,3}$ is $\{-3^{(1)}, 0^{(4)}, 3^{(1)}\}$. Therefore the adjacency spectrum of $I^*(R)$ is $\{-3^{(2)}, 0^{(8)}, 3^{(2)}\}$. The energy is $\mathcal{E}(I^*(R)) = 12$. The Laplacian eigenvalues of $K_{3,3}$ are $0^{(1)}, (3)^{(4)}, (6)^{(1)}$. Hence the Laplacian spectrum of $I^*(R)$ is a multiset: $\{0^{(2)}, (3)^{(8)}, (6)^{(2)}\}$.

Let $p \neq 4r + 3$, the following result demonstrate that $I^*(R)$ is not regular.

Lemma 3.9. *Let $R = M_2(\mathbb{F})$ and $p \neq 4r + 3$. Then $I^*(R)$ is not regular.*

Proof. Suppose $p \neq 4r + 3$. Then there exists an element $a \in \mathbb{F}$ with $a^2 = -1$. The neighborhood of E_0 is $\mathcal{N}(E_0) = \{E^0, F_a : a \in \mathbb{F} \setminus \{0\}\}$, which means the size of $\mathcal{N}(E_0)$ is n. For $a \in \mathbb{F} \setminus \{0\}$, the neighborhood of E_a is given by: $\mathcal{N}(E_a) = \{A \in M_2(\mathbb{F}) : E_a A^t = 0,\ \textit{i.e.},\ A(E_a)^t = AF^a = 0\}$.
If $E_{b,c} \in \mathcal{N}(E_a)$, then $E_{b,c}F^a = 0$, which leads to the equation $ba + c = 0$ implying $c = -ba$. Observe that $E^{-a}F^a = 0$, and $E_{-a^{-1}}F^a = 0$. Therefore $E_{-a^{-1}} \in \mathcal{N}(E_a)$ if $E_{-a^{-1}} \neq E_a$ *i.e.*, if $a^2 \neq -1$ in $\mathbb{F}$. Thus, if $a = -a^{-1}$ in $\mathbb{F}$, then $E_{-a^{-1}} \notin \mathcal{N}(E_a)$. Therefore $\mathcal{N}(E_a) = \{E_{-a^{-1}}, E^{-a}, E_{b,-ba} : b \in \mathbb{F} \setminus \{0, 1\}$, for $a^2 \neq -1$. Hence $|\mathcal{N}(E_a)| = n$. And if $a^2 = -1$ in $\mathbb{F}$, then $\mathcal{N}(E_a) = \{E^{-a}, E_{b,-ba} : b \in \mathbb{F} \setminus \{0, 1\}\}$. That is for $a^2 = -1$ in $\mathbb{F}$, $|\mathcal{N}(E_a)| = n - 1$. Therefore the degree of E_0 is n as shown by the neighborhood $\mathcal{N}(E_0)$, whereas the degree of the vertex E_a is $n - 1$ as shown by the neighborhood $\mathcal{N}(E_a)$ when $a^2 = -1$. Since the degrees of E_0 and E_a are different, $I^*(R)$ is not regular. □

Next, we discuss a necessary and sufficient condition for regularity of $I^*(R)$.

Theorem 3.10. *Let $R = M_2(\mathbb{F})$. Then $I^*(R)$ is regular if and only if $p = 4r + 3$.*

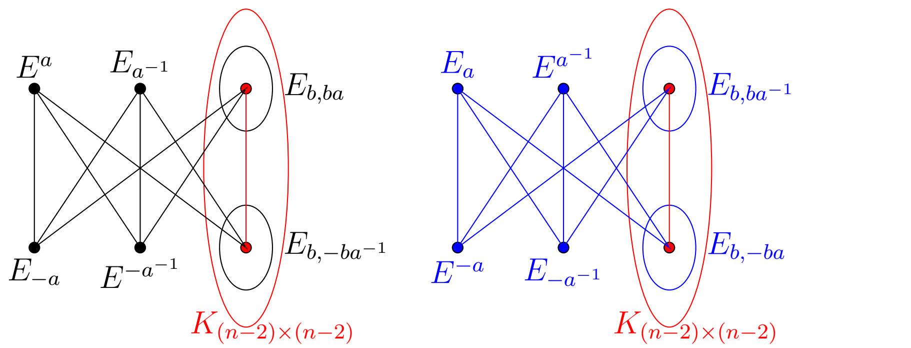
Proof. The proof is obtained using Lemma 3.4 and Lemma 3.9. □

Suppose $p \neq 4r + 3$, then $\mathbb{F}$ contains an element a with $a^2 = -1$, the structure of $I^*(R)$ can be analyzed by examining the neighborhoods of its vertices. The vertices corresponding to the elements E_0 and E^a (with $a^2 = -1$) have different degrees, leading to a non-regular graph, as discussed in the proof above. This characterization shows how the presence of elements $a \in \mathbb{F}$ with $a^2 = -1$ disrupts the regularity of the graph $I^*(R)$, and when such elements do not exist in $\mathbb{F}$, the graph $I^*(R)$ is regular.

Let $p \neq 4r + 3$. In the next result we describes the structure of $I^*(R)$.

Proposition 3.11. *Let $R = M_2(\mathbb{F})$ and $p \neq 4r + 3$. Then the components corresponding to E^a, E_a are complete bipartite graph $K_{n,n}$ for $a^2 \neq -1$, and complete graph K_n for $a^2 = -1$.*

Proof. Since $p \neq 4r + 3$, there exists an element $a \in \mathbb{F}$ with $a^2 = -1$.
Case (1): Let $a \in \mathbb{F} \setminus \{0\}$ with $a^2 \neq -1$. Then:
(i) Consider the neighborhoods of the elements adjacent to E^a.
$\mathcal{N}(E^a) = \{E_{-a}, E^{-a^{-1}}, E_{b,-ba^{-1}} : b \in \mathbb{F} \setminus \{0, 1\}\} = \mathcal{N}(E_{a^{-1}}) = \mathcal{N}(E_{b,ba})$.
Next, $\mathcal{N}(E^{-a^{-1}}) = \{E_{a^{-1}}, E^a, E_{b,ba} : b \in \mathbb{F} \setminus \{0, 1\}\} = \mathcal{N}(E_{-a}) = \mathcal{N}(E_{b,-ba^{-1}})$.



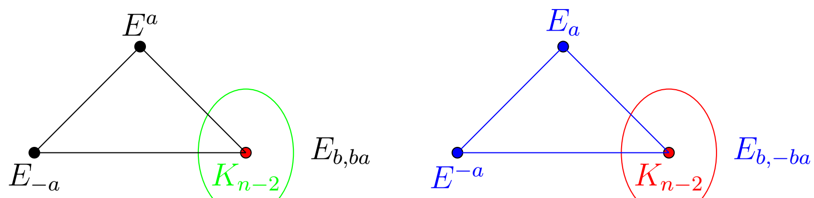
(a) Component corresponding to E^a (b) Component corresponding to E_a

Figure 5. The components of $I^*(R)$ corresponding to the vertices E^a and E_a for $a^2 \neq -1$.

Let $A = \{E_{b,ba} \colon b \in \mathbb{F} \setminus \{0,1\}\}$ and $B = \{E_{b,-ba^{-1}} \colon b \in \mathbb{F} \setminus \{0,1\}\}$. Note that $E_{b,ba}(E_{c,ca})^t \neq 0$, and $E_{b,ba}(E_{c,-ca^{-1}})^t = 0$. Therefore all vertices in A are adjacent to all vertices in B, and no vertices within A are adjacent. Also, no vertices within B are adjacent. Thus, the component corresponding to E^a forms a complete bipartite graph $K_{n,n}$ in $I^*(R)$ as shown in Figure 5 (a).

(ii) Now, we find the component of $I^*(R)$ corresponding to the vertex E_a.
$\mathcal{N}(E_a) = \{E_{-a^{-1}}, E^{-a}, E_{b,-ba} \colon b \in \mathbb{F} \setminus \{0,1\}\} = \mathcal{N}(E^{a^{-1}}) = \mathcal{N}(E_{b,ba^{-1}})$.
$\mathcal{N}(E^{-a}) = \{E^{a^{-1}}, E_a, E_{b,ba^{-1}} \colon b \in \mathbb{F} \setminus \{0,1\}\} = \mathcal{N}(E_{-a^{-1}}) = \mathcal{N}(E_{b,-ba})$.
Let $A_1 = \{E_{b,-ba} \colon b \in \mathbb{F} \setminus \{0,1\}\}$ and $B_1 = \{E_{b,ba^{-1}} \colon b \in \mathbb{F} \setminus \{0,1\}\}$. Note that $E_{b,-ba}(E_{c,-ca})^t \neq 0$, and $E_{b,-ba}(E_{c,ca^{-1}})^t = 0$. Similar to above part $\{A_1, B_1\}$, forms the partition of vertex set of the component corresponding to E_a. Hence it forms a complete bipartite graph $K_{n,n}$ in $I^*(R)$ as shown in Figure 5 (b).
Case (2): Let $a \in \mathbb{F} \setminus \{0\}$ such that $a^2 = -1$. Then:
(i) $\mathcal{N}(E^a) = \{E_{-a}, E_{b,ba} \colon b \in \mathbb{F} \setminus \{0,1\}\}$. Note that $|\mathcal{N}(E^a)| = n-1$.
Next, consider the neighborhoods of the elements adjacent to E^a.
$\mathcal{N}(E_{-a}) = \{E^a, E_{b,ba} \colon b \in \mathbb{F} \setminus \{0,1\}\}$. Therefore $|\mathcal{N}(E_{-a})| = n-1$.
$\mathcal{N}(E_{b,ba}) = \{E^a, E_{-a}, E_{c,ca} \colon c \in \mathbb{F} \setminus \{0,1\}, c \neq b\}$. Therefore $|\mathcal{N}(E_{b,ba})| = n-1$.
This structure leads to the component corresponding to E^a forming a complete graph K_n in $I^*(R)$ (see Figure 6 (a)).
(ii) $\mathcal{N}(E_a) = \{E^{-a}, E_{b,-ba} \colon b \in \mathbb{F} \setminus \{0,1\}\}$. Note that $|\mathcal{N}(E_a)| = n-1$.
Consider the neighborhoods of the elements adjacent to E_a.
$\mathcal{N}(E^{-a}) = \{E_a, E_{b,-ba} \colon b \in \mathbb{F} \setminus \{0,1\}\}$. Therefore $|\mathcal{N}(E^{-a})| = n-1$.
$\mathcal{N}(E_{b,-ba}) = \{E_a, E^{-a}, E_{c,-ca} \colon c \in \mathbb{F} \setminus \{0,1\}, c \neq b\}$. Therefore $|\mathcal{N}(E_{b,-ba})| = n-1$. Hence the component corresponding to the vertex E_a for $a^2 = -1$ forms complete graph K_n in $I^*(R)$ (see Figure 6 (b)). $\square$

Let $p = 4r+1$. The following result determines the adjacency and Laplacian spectrum of $I^*(R)$.



(a) Component corresponding to E^a (b) Component corresponding to E_a

FIGURE 6. The components of $I^*(R)$ corresponding to the vertices E^a and E_a for $a^2 = -1$.

Proposition 3.12. *Let $R = M_2(\mathbb{F})$ and $p = 4r + 1$. The adjacency spectrum of $I^*(R)$ is a multiset:*

$$\left\{(-1)^{2(n-1)}, (n-1)^{(2)}, (n)^{(\frac{n-1}{2})}, (-n)^{(\frac{n-1}{2})}, (0)^{(n-1)^2}\right\},$$

and the Laplacian spectrum is a multiset:

$$\left\{(n)^{(n^2-1)}, 0^{(\frac{n+3}{2})}, (2n)^{(\frac{n-1}{2})}\right\}.$$

Furthermore, the energy is given by: $\mathcal{E}(I^*(R)) = (n-1)(n+4)$.

Proof. By Lemma 1, there are $n(n+1)$ non-trivial idempotent elements in $I^*(R)$. According to Proposition 3.11, the components of $I^*(R)$ are either bipartite or complete graphs. Specifically: The components E_a and E^a corresponding to an element $a^2 = -1$ forms complete graph K_n. Hence the eigenvalues corresponding to this component are $(n-1)^{(1)}$, and $(-1)^{(n-1)}$. Also, note that the component associated to $a \in \mathbb{F}$ and $-a \in \mathbb{F}$ are same for $a^2 = -1$. By Remark 3.3, there are only 2 elements in $\mathbb{F}$ with $a^2 = -1$. Therefore there are exactly two components of K_n. Thus, the eigenvalues corresponding to these two K_n components are $(-1)^{(2(n-1))}$ and $(n-1)^{(2)}$. The remaining components are regular bipartite graphs $K_{n,n}$. Total number of vertices in bipartite components are $n^2 + n - 2n = n^2 - n$. Therefore there are $\dfrac{n^2-n}{2n} = \dfrac{n-1}{2}$ bipartite components in $I^*(R)$. The eigenvalues of $K_{n,n}$ are $(0)^{(2n-2)}, (n)^{(1)}, (-n)^{(1)}$. Therefore the eigenvalues of $\frac{n-1}{2}$ bipartite components $K_{n,n}$ are $(0)^{((n-1)^2)}, (n)^{(\frac{n-1}{2})}, (-n)^{(\frac{n-1}{2})}$. Hence the adjacency spectrum of $I^*(R)$ is a multiset

$$\left\{(-1)^{(2(n-1))}, (n-1)^{(2)}, (n)^{(\frac{n-1}{2})}, (-n)^{(\frac{n-1}{2})}, (0)^{((n-1)^2)}\right\}.$$

Furthermore, the energy is given by:

$$\mathcal{E}(I^*(R)) = 2(n-1) + 2(n-1) + n\left(\frac{n-1}{2}\right) + n\left(\frac{n-1}{2}\right) = (n-1)(n+4).$$

Next, the Laplacian eigenvalues corresponding to K_n are $0^{(1)}, n^{(n-1)}$. Hence the Laplacian eigenvalues corresponding to 2 components of K_n are $0^{(2)}, n^{(2(n-1))}$. The Laplacian eigenvalues corresponding to $K_{n,n}$ are $0^{(1)}, (n)^{(2n-2)}, (2n)^{(1)}$. Since there are $\frac{n-1}{2}$ complete bipartite components, hence the Laplacian eigenvalues corresponding to these components are $0^{(\frac{n-1}{2})}, n^{(n-1)^2}, (2n)^{(\frac{n-1}{2})}$. Thus, the Laplacian spectrum of $I^*(R)$ is a multiset:

$$\left\{n^{(n^2-1)}, 0^{(\frac{n+3}{2})}, (2n)^{(\frac{n-1}{2})}\right\}.$$

□

Now, we determine the spectrum and energy of $I^*(R)$ for $p = 2$.

Proposition 3.13. *Let $R = M_2(\mathbb{F})$ and $p = 2$. The adjacency spectrum of $I^*(R)$ is given by a multiset:*

$$\left\{(-1)^{(n-1)}, (n-1)^{(1)}, (0)^{(n(n-1))}, (n)^{(\frac{n}{2})}, (-n)^{(\frac{n}{2})}\right\},$$

and the Laplacian spectrum is a multiset:

$$\left\{n^{(n^2-1)}, 0^{(\frac{n+2}{2})}, (2n)^{(\frac{n}{2})}\right\}.$$

Furthermore, the energy is given by: $\mathcal{E}(I^*(R)) = n^2 + 2n - 2.$.

Proof. There are $n(n+1)$ non-trivial idempotent elements in $I^*(R)$ (see Lemma 1). According to Proposition 3.11, the components of $I^*(R)$ are either bipartite or complete graphs. In this case when $p = 2$, there is only one element $a \in \mathbb{F}$ such that $a^2 = -1$ (see Remark 3.3). Therefore there is exactly one component K_n. Its eigenvalues are $(n-1)^{(1)}, (-1)^{(n-1)}$. Total number of vertices in bipartite components is $n^2 + n - n = n^2$. Thus, the number of bipartite components in $I^*(R)$ is $\dfrac{n^2}{2n} = \dfrac{n}{2}$. The eigenvalues of adjacency matrix of $K_{n,n}$ are $(n^{(1)}, (-n)^{(1)}, 0^{(2n-2)})$. Therefore the eigenvalues of $\frac{n}{2}$ bipartite components $K_{n,n}$ are $(n)^{(\frac{n}{2})}, (-n)^{(\frac{n}{2})}, (0)^{(n(n-1))}$. Hence the adjacency spectrum of $I^*(R)$ is a multiset

$$\left\{(-1)^{(n-1)}, (n-1)^{(1)}, (0)^{(n(n-1))}, (n)^{(\frac{n}{2})}, (-n)^{(\frac{n}{2})}\right\}.$$

The energy is: $\mathcal{E}(I^*(R)) = n^2 + 2n - 2$.

Next, the Laplacian eigenvalues corresponding to K_n are $0^{(1)}, (n)^{(n-1)}$. The Laplacian eigenvalues corresponding to $K_{n,n}$ are $0^{(1)}, (n)^{(2n-2)}, (2n)^{(1)}$. Since there are $\frac{n}{2}$ complete bipartite components, hence the Laplacian eigenvalues corresponding to these components are $0^{(\frac{n}{2})}, (n)^{(n(n-1))}, (2n)^{(\frac{n}{2})}$. Thus, the Laplacian spectrum of $I^*(R)$ is a multiset: $\left\{(n)^{(n^2-1)}, 0^{(\frac{n+2}{2})}, (2n)^{(\frac{n}{2})}\right\}$. □

Remark 3.14. Let $R = M_2(\mathbb{F})$, where $\mathbb{F} \neq \mathbb{Z}_2$. Then $I^*(R)$ is not planar as it contains $K_{3,3}$ as a subgraph.

We end this section by showing that Beck's conjecture holds for $I^*(R)$.

Proposition 3.15. *Let $R = M_2(\mathbb{F})$. Then the chromatic number and clique number of $I^*(R)$ are:*

$$\chi(I^*(R)) = \omega(I^*(R)) = \begin{cases} 2, & \text{if } p = 4r+3 \\ n, & \text{if } p \neq 4r+3. \end{cases}$$

Proof. Suppose that $p = 4r + 3$. By Proposition 3.5, each component of $I^*(R)$ forms a bipartite graph. For a bipartite graph $\chi(G) = \omega(G) = 2$. Hence $\chi(I^*(R)) = \omega(I^*(R)) = 2$. Suppose $p \neq 4r + 3$, then there exists $a \in \mathbb{F}$ with $a^2 = -1$. By Proposition 3.11, the components of $I^*(R)$ include K_n and $K_{n,n}$. The subgraph K_n gives the largest complete subgraph of $I^*(R)$, so $\omega(I^*(R)) = n$. The graph K_n requires n colors, since it is a complete graph, and we need one color for each vertex. For the bipartite components $K_{n,n}$, any two of the n available colors can be used to color the vertices, so they do not affect the total chromatic number. Therefore $\chi(I^*(R)) = n$. Hence, in both cases, we have $\chi(I^*(R)) = \omega(I^*(R))$, and the values are given by the above cases depending on whether $p = 4r + 3$ or not. Thus,

$$\chi(I^*(R)) = \omega(I^*(R)) = \begin{cases} 2, & \text{if } p = 4r + 3 \\ n, & \text{if } p \neq 4r + 3. \end{cases}$$

□

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