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ENTROPY SOLUTIONS FOR SOME NONLINEAR AND NONCOERCIVE PARABOLIC PROBLEMS IN THE ANISOTROPIC SOBOLEV SPACES

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ABSTRACT. In this paper, we studied the existence of solutions for a class parabolic problem involving a non-coercive Leray-Lions operator with a Hardy potential. Using an approximate method and some a priori estimation techniques, we show the existence of entropy solutions for the studied problem within the framework of anisotropic Sobolev spaces.

1. INTRODUCTION

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$. For $T > 0$, the cylinder $\Omega \times (0,T)$ has denoted by Q_T, furthermore $\partial\Omega \times (0,T)$ is the lateral surface denoted by Σ_T. In this paper, we have studied the following nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + A\omega + \nu|\omega|^{s-1}\omega = \lambda \dfrac{|\omega|^{p_0-2}\omega}{|x|^{p_0}} - \displaystyle\sum_{i=1}^{N} D^i f_i(x,t,\omega) & \text{in } Q_T, \\ \omega = 0 & \text{on } \Sigma_T, \\ \omega(x,0) = \omega_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\omega_0 \in L^1(\Omega)$, and $f_i(x,t,\tau)$ verifying some growth condition. The non-coercive Leray-Lions operator A acted from $L^{\vec{p}}(0,T;W^{1,\vec{p}}(\Omega))$ into its dual $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$ is given by

$$A\omega = -\sum_{i=1}^{N} D^i a_i(x,t,\omega,\nabla\omega).$$

In the specific case where $\nu = 0$ and $f_i \equiv 0$ for $i = 1, \cdots, N$, in [10], Baras and Goldstein studied the linear version of problem (1.1), i.e., with $\sum_{i=1}^{N} D^i a_i(x,t,\omega,\nabla\omega) = \Delta\omega$ and $p_0 = 2$. They established the existence of solutions for $\lambda \leq \Lambda_{N,2} = \left(\frac{N-2}{2}\right)^2$ and the initial datum ω_0. However, for $\lambda > \Lambda_{N,2}$, they proved non-existence of

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local solutions for any $\omega_0 > 0$. For the nonlinear case, in [9], Azorero and Alonso demonstrated the existence of solutions to problem (1.1) for $\lambda \leq \lambda_N = \left(\frac{N-p}{p}\right)^p$. Conversely, they showed non-existence of local solutions for $\lambda > \lambda_N$ and any positive initial data ω_0. See also [5], and for related elliptic problems, we refer the reader to [2, 32].

In [15], Bocardo et al. have studied problem (1.1) with $A\omega = -\text{div}(|\nabla\omega|^{p-2}\nabla\omega)$, and the right-hand side replaced by f is belongs to $L^1(Q_T)$. They have shown the existence and regularity of solutions for problem (1.1). In [16], Bocardo et al. proved the existence results for the nonlinear parabolic problem (1.1) with the right-hand side replaced by μ belongs $M(Q_T)$, the space of bounded Borel measure on Q_T. For a framework in the setting of anisotropic Sobolev spaces, we refer to [1, 3, 7, 20, 29, 31, 34, 36] for more details.

Nonlinear parabolic equations have garnered significant attention in mathematical analysis due to their wide-ranging applications, including the study of non-homogeneous materials that exhibit varying behavior in different spatial directions. Examples of such materials include electrorheological, thermoelectric fluids, and image processing [4, 19, 33].

A substantial body of literature has been dedicated to studying elliptic and parabolic equations with principal operators exhibiting Leray-Lions-type degenerate coercivity and classical divergence. Key research areas include the existence, uniqueness, and regularity of solutions for such equations. For a comprehensive overview [21, 5, 6, 8, 11, 13, 14, 17, 18, 22, 24, 25, 27, 30].

The primary objective of this work is to investigate the existence of entropy solutions for the nonlinear parabolic problem (1.1). The singular nature of the Hardy potential presents significant challenges in obtaining solutions. To address this, we leverage the regularizing properties of the lower-order terms to mitigate the non-existence effects induced by the Hardy potential. Similar difficulties were encountered in previous studies, such as the work of Baras and Goldstein [10], which examined the behavior of solutions about a specific parameter. Another relevant research [9] explored the non-existence of solutions for the p-heat equation due to a singular term, requiring a particular condition on a parameter to ensure existence.

This paper is organized as follows: Section 2 provides a review of essential definitions and properties of anisotropic Sobolev spaces. Section 3 presents crucial lemmas used to prove the main result. Section 4 introduces the definition of entropy solutions for our parabolic equation and establishes the main theorem.

2. Preliminaries

Let Ω be an open bounded domain in $I\!R^N$ $(N \geq 2)$ with boundary $\partial\Omega$. Consider a vector of real exponents $\vec{p} = (p_0, p_1, \dots, p_N)$, where $1 < p_i < \infty$ for $i = 0, 1, \dots, N$.

We introduce the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as

$$W^{1,\vec{p}}(\Omega) = \left\{\omega \in L^{p_0}(\Omega) \quad \text{such that} \quad D^i\omega \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \dots, N\right\},$$

where
$$p_0 = \max\{p_1, p_2, \dots, p_N\},\ D^0\omega = \omega \quad \text{and} \quad D^i\omega = \frac{\partial \omega}{\partial x_i}.$$
Moreover, we denote
$$\underline{p} = \min\{p_1, p_2, \dots, p_N\}.$$
The space $W^{1,\vec{p}}(\Omega)$ is equipped with the norm
$$\|\omega\|_{1,\vec{p}} = \sum_{i=0}^{N} \|D^i\omega\|_{p_i}. \tag{2.1}$$
We set $W_0^{1,\vec{p}}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ for the norm (2.1).
The Sobolev spaces $W^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$ are separable and reflexive Banach space.

Lemma 2.1. *Let Ω be a bounded open set in $\mathbb{R}^N$. Then the following embedding is compact.*

- *if $\overline{p} < N$ then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ for any $r \in [1, \underline{p}^*[$, where $\frac{1}{\underline{p}^*} = \frac{1}{\underline{p}} - \frac{1}{N}$.*
- *if $\overline{p} = N$ then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ for any $r \in [1, +\infty[$,*
- *if $\overline{p} > N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$.*

The dual of the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ is denoted by $W^{-1,\vec{p}'}(\Omega)$, We cite [23] for more information on anisotropic Sobolev spaces.
Now, we present the definition of the anisotropic parabolic space $L^{\vec{p}}(0,T; W^{1,\vec{p}}(\Omega))$ as
$$L^{\vec{p}}(0,T; W^{1,\vec{p}}(\Omega)) = \left\{ \omega \text{ measurable function } / \sum_{i=0}^{N} \int_0^T \left\|D^i\omega\right\|_{L^{p_i}(\Omega)}^{p_i} dt < \infty \right\},$$
endowed with the norm
$$\|\omega\|_{L^{\vec{p}}(0,T;W^{1,\vec{p}}(\Omega))} = \sum_{i=0}^{N} \left\|D^i\omega\right\|_{L^{p_i}(Q_T)}.$$
The space $L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$ is introduced by
$$L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega)) = \left\{\omega \in L^{\vec{p}}(0,T; W^{1,\vec{p}}(\Omega)) / \omega = 0 \text{ on } \partial\Omega \times [0,T]\right\}.$$
The dual of the anisotropic Sobolev space $L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$ is denoted by $L^{\vec{p}'}(0,T; W^{-1,\vec{p}'}(\Omega))$.
Note that $L^{\vec{p}}(0,T; W^{1,\vec{p}}(\Omega))$ and $L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$ are separable and reflexive Banach spaces.

Lemma 2.2. *(see [35]) Let B_0, B and B_1 be a Banach spaces with $B_0 \subset B \subset B_1$. Let us set*
$$Y = \left\{\omega \ :\ \omega \in L^{p_0}(0,T; B_0) \quad \textit{and} \quad \omega' \in L^{p_1}(0,T; B_1)\right\}$$
where $p_0 > 1$ and $p_1 > 1$ are reals numbers.
Assuming that the embedding $B_0 \hookrightarrow\hookrightarrow B$ is compact, then
$$Y \hookrightarrow\hookrightarrow L^{p_0}(0,T; B)$$

and this imbedding is compact.

Remark 2.3. Let $\underline{p} > \dfrac{2N}{N+2}$, we set

$$B_0 = W_0^{1,\vec{p}}(\Omega), \qquad B = L^2(\Omega) \qquad \text{and} \qquad B_1 = W^{-1,\vec{p}'}(\Omega),$$

with $p_0 = \underline{p}$ and $p_1 = p_0'$. In view of the Lemma 2.2 we obtain

$$\{\omega \,:\, \omega \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \text{and} \quad \omega' \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))\} \subseteq Y \hookrightarrow\hookrightarrow L^1(Q_T). \tag{2.2}$$

Moreover, in view of [12], we have

$$\{\omega \,:\, \omega \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \text{and} \quad \omega' \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))\} \subseteq C([0,T];L^1(\Omega)). \tag{2.3}$$

3. Essential assumptions

Let $\Omega \subset I\!R^N (N \geq 2)$ be open and bounded containing the origin with boundary $\partial\Omega$. Denote

$$Q_T = \Omega \times (0,T) \qquad \text{and} \qquad \Sigma_T = \partial\Omega \times (0,T).$$

Let us consider the nonlinear and noncoercive parabolic problem with Hardy potential in anisotropic Sobolev space

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + A\omega + \nu|\omega|^{s-1}\omega = \lambda \dfrac{|\omega|^{p_0-2}\omega}{|x|^{p_0}} - \displaystyle\sum_{i=1}^N D^i f_i(x,t,\omega) & \text{in } Q_T, \\ \omega = \ 0 & \text{on } \Sigma_T, \\ \omega(x,0) = \ \omega_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

with $\omega_0 \in L^1(\Omega)$, $\nu > 0$ and $\lambda > 0$. The non-coercive Leray-Lions operator A is a defined by

$$A\omega = \sum_{i=1}^N D^i a_i(x,t,\omega,\nabla\omega),$$

where $a_i : Q_T \times I\!R \times I\!R^N \longmapsto I\!R^N$ are Carathéodory functions verifying the following conditions

I) there exists a positive functions $K_i \in L^{p_i'}(Q_T)$ and $\beta > 0$ such that for a.e. $(x,t) \in Q_T$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$
$$|a_i(x,t,s,\xi)| \leq \beta(K_i(x,t) + |s|^{p_i-1} + |\xi_i|^{p_i-1}), \tag{3.2}$$

II) for a.e. $(x,t) \in Q_T$ and for any $\xi_i \neq \xi_i'$
$$(a_i(x,t,s,\xi) - a_i(x,t,s,\xi'))(\xi_i - \xi_i') > 0, \tag{3.3}$$

III) there exists a constant $b_0 > 0$ for a.e. $(x,t) \in Q_T$ and all $(s,\xi) \in I\!R \times I\!R^N$ such that
$$a_i(x,t,s,\xi)\xi_i \geq \frac{b_0}{(1+|s|)^\delta}|\xi_i|^{p_i}, \tag{3.4}$$
where $0 < \delta < \min\{p_0 - 1,\ s(\underline{p}-1)\}$, such that $s > \max\left(\frac{N(p_0-1)}{N-p_0}, \frac{1}{p_0-1}\right)$, and let $\beta > 0$.

The Carathéodory functions $f_i : Q_T \times I\!R \longmapsto I\!R$ satisfies only the growth condition

$$|f_i(x,t,s)| \leq c_i(x,t) + d_i(x,t)|s|^{\sigma_i}, \tag{3.5}$$

where $c_i(x,t),\ d_i(x,t)$ respectively belongs to $L^{m_i}(Q_T)$ and $L^{r_i}(Q_T)$, with $m_i > \frac{sp_i}{(s+1)(p_i-1)-\delta}$, $r_i > \frac{sp_i}{(s+1)(p_i-1)-\sigma_i\, p_i-\delta}$, and $0 < \sigma_i < \frac{s(p_i-1)-\delta}{p_i}$.

Lemma 3.1. *(see. [20]) Let $(\omega_n)_n \subset L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ be a sequence such that $\dfrac{\partial \omega_n}{\partial t} \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$ and satisfying the following convergences:*

(i) *$\omega_n \rightharpoonup \omega$ weakly in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$,*
(ii)

$$\sum_{i=1}^{N} \int_{Q_T} \Big(a_i(x,t,\omega_n,\nabla \omega_n) - a_i(x,t,\omega_n,\nabla \omega)\Big)(D^i\omega_n - D^i\omega)\, dx\, dt$$
$$+ \int_{Q_T} \Big(|\omega_n|^{p_0-2}\omega_n - |\omega|^{p_0-2}\omega\Big)(\omega_n - \omega)\, dx\, dt \longrightarrow 0 \qquad as \quad n \to \infty, \tag{3.6}$$

and assuming that conditions (3.2)–(3.4) hold, then $\omega_n \to \omega$ strongly in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ for a subsequence.

Proposition 3.2. *Let $\omega \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ we set for all $\mu \geq 0$*

$$\omega_\mu(x,t) = \mu \int_{-\infty}^{t} \omega(x,s)\exp(\mu(s-t))\chi_{(0,T)}(s)\ ds.$$

Then, the following assertions hold

(i) *If $\omega \in L^{p_0}(Q_T)$, then ω_μ is measurable in Q_T, with $\dfrac{\partial \omega_\mu}{\partial t} = \mu(\omega - \omega_\mu)$ and*
$$\int_{Q_T} |\omega_\mu|^{p_0} dxdt \leq \int_{Q_T} |\omega|^{p_0} dxdt$$
(ii) *If $\omega \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, then $\omega_\mu \to \omega$ strongly in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ as $\mu \to +\infty$.*
(iii) *If $\omega_n \to \omega$ strongly in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, then $(\omega_n)_\mu \to \omega_\mu$ strongly in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.*
(iv) *$|(T_k(\omega))_\mu| \leq k$ for all $\omega \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.*

Proof of Proposition 3.2 is similar to those in the classical space $L^p(0,T;W_0^{1,p}(\Omega))$.

4. Main results

First, we define an entropy solution for our noncoercive parabolic problem with Hardy potential in anisotropic Sobolev space. Let $T_k(\sigma) = \max(-k, \min(\sigma,k))$, and

$$\varphi_k(\sigma) = \int_0^{\sigma} T_k(s)\, ds = \begin{cases} \frac{\sigma^2}{2} & \text{if} \quad |\sigma| \leq k, \\ k|\sigma| - \frac{k^2}{2} & \text{if} \quad |\sigma| > k, \end{cases}$$

Definition 4.1. A measurable function ω is called entropy solution of the noncoercive parabolic equation (3.1) if $T_k(\omega) \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ for all $k>0$, such that ω satisfies the following equality

$$\begin{aligned}
&\int_\Omega \varphi_k(\omega(T)-\psi(T))\,dx - \int_\Omega \varphi_k(\omega_0-\psi(0))\,dx + \int_{Q_T} \frac{\partial\psi}{\partial t} T_k(\omega-\psi)\,dx\,dt \\
&+\sum_{i=1}^N \int_{Q_T} a_i(x,t,\omega,\nabla\omega) D^i T_k(\omega-\psi)\,dx\,dt + \nu \int_{Q_T} |\omega|^{s-1}\omega\; T_k(\omega-\psi)\,dx\,dt \\
&\le \lambda \int_{Q_T} \frac{|\omega|^{p_0-2}\omega}{|x|^{p_0}}\; T_k(\omega-\psi)\,dx\,dt + \sum_{i=1}^N \int_{Q_T} f_i(x,t,\omega) D^i T_k(\omega-\psi)\,dx\,dt,
\end{aligned} \tag{4.1}$$

for all $\psi \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \cap L^\infty(Q_T)$ with $\dfrac{\partial\psi}{\partial t} \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T)$.

Remark 4.2. In view of Young's inequality, we have

$$\Big|\int_{Q_T} \frac{|\omega|^{p_0-2}\omega}{|x|^{p_0}} T_k(\omega-\psi)dxdt\Big| \le k(\frac{p_0-1}{s}) \int_{Q_T} |\omega|^s dxdt + kT(\frac{s-p_0+1}{s}) \int_\Omega \frac{1}{|x|^{\frac{p_0 s}{s-p_0+1}}} dx,$$

since $s > \dfrac{N(p_0-1)}{N-p_0}$ then $\dfrac{1}{|x|^{p_0}} \in L^{\frac{s}{s-p_0+1}}(\Omega)$, it follows that $\dfrac{|\omega|^{p_0-2}\omega}{|x|^{p_0}} T_k(\omega-\psi) \in L^1(Q_T)$ hold true.

Theorem 4.3. *Under the assumptions* (3.2)−(3.4) *and* (3.5), *the non-coercive anisotropic parabolic problem* (3.1) *has at least one entropy solution, such that*

$$|\omega|^s \in L^1(Q_T) \qquad \textit{for all} \quad s > \frac{N(p_0-1)}{N-p_0}.$$

Proof of the Theorem 4.3.

Step 1 : Approximate problems.
Let $(\omega_{0,n})$ be a sequence in $C_0^\infty(\Omega)$ such that $\omega_{0,n} \to \omega_0$ in strongly $L^1(\Omega)$ with $|\omega_{0,n}| \le |\omega_0|$. We consider the sequence of approximate problems

$$\begin{cases}
(\omega_n)_t + A_n\omega_n + \nu |T_n(\omega_n)|^{s-1} T_n(\omega_n) = \lambda \dfrac{|T_n(\omega_n)|^{p_0-2} T_n(\omega_n)}{|x|^{p_0} + \frac{1}{n}} - \displaystyle\sum_{i=1}^N D^i f_i(x,t,T_n(\omega_n)) & \text{in} \quad Q_T, \\
\omega_n(x,t) = 0 & \text{on} \quad \Sigma_T, \\
\omega_n(x,0) = \omega_{0,n} & \text{in} \quad \Omega,
\end{cases} \tag{4.2}$$

where

$$A_n\omega_n = -\sum_{i=1}^N D^i a_i(x,t,T_n(\omega_n),\nabla\omega_n) + \frac{1}{n}|\omega_n|^{p_0-2}\omega_n.$$

It is easy to prove from (3.4) that the operator A_n is coercive. Indeed, let $\omega \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, we have

$$\begin{aligned}
\langle A_n\omega,\omega\rangle &= \sum_{i=1}^{N}\int_{Q_T} a_i(x,t,T_n(\omega),\nabla\omega)D^i\omega\,dx\,dt + \frac{1}{n}\int_{Q_T}|\omega|^{p_0}\,dx\,dt.\\
&\geq \frac{b_0}{(1+n)^{\delta}}\sum_{i=1}^{N}\int_{Q_T}|D^i\omega|^{p_i}\,dx\,dt + \frac{1}{n}\int_{Q_T}|\omega|^{p_0}\,dx\,dt.\\
&\geq \alpha_n\|\omega\|^{\underline{p}}_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} - \frac{Nb_0}{(1+n)^{\delta}} - \frac{1}{n},
\end{aligned} \tag{4.3}$$

with $\alpha_n = \min\left\{\dfrac{b_0}{(1+n)^{\delta}},\ \dfrac{1}{n}\right\}$.

For all $\omega, v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, we define the operator $G_n : L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \longmapsto L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$ by

$$\begin{aligned}
\langle G_n\omega,v\rangle &= \nu\int_{Q_T}|T_n(\omega)|^{s-1}T_n(\omega)v\,dx\,dt - \lambda\int_{Q_T}\frac{|T_n(\omega)|^{p_0-2}T_n(\omega)}{|x|^{p_0}+\frac{1}{n}}v\,dx\,dt\\
&\quad -\sum_{i=1}^{N}\int_{Q_T} f_i(x,t,T_n(\omega))D^iv\,dx\,dt.
\end{aligned}$$

Let $\omega, v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, we can show from (3.5) that $f_i(x,t,T_n(\omega)) \in L^{p_i'}(Q_T)$, and in view of Hölder inequality we have

$$\begin{aligned}
|G_n\omega,v\rangle| &\leq \nu\int_{Q_T}|T_n(\omega)|^{s}|v|\,dx\,dt + \lambda\,n\int_{Q_T}|T_n(\omega)|^{p_0-1}|v|\,dx\,dt\\
&\quad +\sum_{i=1}^{N}\int_{Q_T}|f_i(x,t,T_n(\omega))||D^iv|\,dx\,dt\\
&\leq (\lambda n^{p_0}+\nu n^{s})\,(\text{meas}(Q_T))^{\frac{1}{p_0'}}\|v\|_{L^{p_0}(Q_T)}\\
&\quad +\sum_{i=1}^{N}\|f_i(x,t,T_n(\omega))\|_{L^{p_i'}(Q_T)}\|D^iv\|_{L^{p_i'}(Q_T)}\\
&\leq C\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}.
\end{aligned} \tag{4.4}$$

Therefore, proceeding as in [20], there exists at least one weak solution $\omega_n \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ for the problem (4.2), (cf. [28]).

Step 2 : A priori estimates.

Let $\gamma > 1$ small enough, such that $m_i > \dfrac{sp_i}{(s+\gamma)(p_i-1)-\delta}$ and $r_i > \frac{sp_i}{(s+\gamma)(p_i-1)-\sigma_i\,p_i-\delta}$ for $i = 1,\ldots,N$.

Employing the test function $\psi(\omega_n) = \Big(1 - \dfrac{1}{(1+|T_n(\omega_n)|)^{\gamma-1}}\Big)\text{sign}(\omega_n)$within the framework of the approximate problem (4.2) yields

$$\begin{aligned}&\int_0^T \langle (\omega_n)_t, \psi(\omega_n)\rangle dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(\omega_n),\nabla\omega_n) D^i\psi(\omega_n)\, dx\, dt\\ &\quad +\nu\int_{Q_T} |T_n(\omega_n)|^s|\psi(\omega_n)|\, dx\, dt + \frac{1}{n}\int_{Q_T}|\omega_n|^{p_0-1}|\psi(\omega_n)|\, dx\, dt\\ &= \lambda\int_{Q_T}\frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|\psi(\omega_n)|\, dx\, dt + \sum_{i=1}^N\int_{Q_T} f_i(x,t,T_n(\omega_n))D^i\psi(\omega_n)\, dx\, dt.\end{aligned} \tag{4.5}$$

It's clear that $|\psi(\omega_n)| \leq 1$, and using (3.4) and (3.5) we get

$$\begin{aligned}&\int_0^T \langle (\omega_n)_t, \psi(\omega_n)\rangle\, dt + b_0(\gamma-1)\sum_{i=1}^N\int_{Q_T}\frac{|D^iT_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}}\, dx\, dt\\ &\quad +\nu\int_{Q_T}|T_n(\omega_n)|^s|\psi(\omega_n)|\, dx\, dt \leq \lambda\int_{Q_T}\frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\, dx\, dt\\ &\quad +(\gamma-1)\sum_{i=1}^N\int_{Q_T}\frac{|c_i(x,t)||D^iT_n(\omega_n)|}{(1+|T_n(\omega_n)|)^{\gamma}}\, dx\, dt\\ &\quad +(\gamma-1)\sum_{i=1}^N\int_{Q_T}\frac{|d_i(x,t)||T_n(\omega_n)|^{\sigma_i}|D^iT_n(\omega_n)|}{(1+|T_n(\omega_n)|)^{\gamma}}\, dx\, dt.\end{aligned} \tag{4.6}$$

We remark that

$$\Big(1-\frac{1}{(1+|T_n(\omega_n)|)^{\gamma-1}}\Big) \geq \frac{1}{2} \quad \text{for} \quad |T_n(\omega_n)| \geq R = 2^{\frac{1}{\gamma-1}}-1,$$

hence

$$\begin{aligned}\frac{\nu}{2}\int_{Q_T}|T_n(\omega_n)|^s\, dx\, dt &\leq \frac{\nu}{2}TR^s|\Omega| + \frac{\nu}{2}\int_{\{|T_n(\omega_n)|\geq R\}}|T_n(\omega_n)|^s\, dx\, dt\\ &\leq \frac{\nu}{2}TR^s|\Omega| + \nu\int_{Q_T}|T_n(\omega_n)|^s|\psi(\omega_n)|\, dx\, dt.\end{aligned} \tag{4.7}$$

Moreover, since $s > p_0-1$, we arrive by using Young's inequality that

$$\lambda\int_{Q_T}\frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}}\, dx\, dt \leq \frac{\nu}{4}\int_{Q_T}|T_n(\omega_n)|^s\, dx\, dt + C_1T\int_\Omega \frac{1}{|x|^{\frac{p_0 s}{s-p_0+1}}}\, dx, \tag{4.8}$$

Hence, according to (4.6) – (4.8), we conclude that

$$\begin{aligned}&\int_0^T \langle (\omega_n)_t, \psi(\omega_n)\rangle\, dt + b_0(\gamma-1)\sum_{i=1}^N\int_{Q_T}\frac{|D^iT_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}}\, dx\, dt + \frac{\nu}{4}\int_{Q_T}|T_n(\omega_n)|^s\, dx\, dt\\ &\leq \frac{\nu}{2}TR^s|\Omega| + C_1T\int_\Omega\frac{1}{|x|^{\frac{p_0 s}{s-p_0+1}}}\, dx + (\gamma-1)\sum_{i=1}^N\int_{Q_T}\frac{|c_i(x,t)||D^iT_n(\omega_n)|}{(1+|T_n(\omega_n)|)^{\gamma}}\, dx\, dt\\ &+(\gamma-1)\sum_{i=1}^N\int_{Q_T}\frac{|d_i(x,t)||T_n(\omega_n)|^{\sigma_i}|D^iT_n(\omega_n)|}{(1+|T_n(\omega_n)|)^{\gamma}}\, dx\, dt.\end{aligned} \tag{4.9}$$

Thanks to Young's inequality, we have

$$\begin{aligned}
&(\gamma-1)\sum_{i=1}^{N}\int_{Q_T}\frac{|c_i(x,t)||D^iT_n(\omega_n)|}{(1+|T_n(\omega_n)|)^{\gamma}}\,dx\,dt\\
&\le C_2\sum_{i=1}^{N}\int_{Q_T}\frac{|c_i(x,t)|^{p_i'}(1+|T_n(\omega_n)|)^{\frac{\delta}{p_i-1}}}{(1+|T_n(\omega_n)|)^{\gamma}}\,dx\,dt\\
&\quad+\frac{b_0(\gamma-1)}{4}\sum_{i=1}^{N}\int_{Q_T}\frac{|D^iT_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}}\,dx\,dt\\
&\le C_3\sum_{i=1}^{N}\int_{Q_T}|c_i(x,t)|^{\frac{sp_i}{(s+\gamma)(p_i-1)-\delta}}\,dx\,dt+\frac{\nu}{16}\int_{Q_T}|T_n(\omega_n)|^{s}\,dx\,dt\\
&\quad+\frac{b_0(\gamma-1)}{4}\sum_{i=1}^{N}\int_{Q_T}\frac{|D^iT_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}}\,dx\,dt+C_4.
\end{aligned}\tag{4.10}$$

Similarly, we get

$$\begin{aligned}
&(\gamma-1)\sum_{i=1}^{N}\int_{Q_T}\frac{|d_i(x,t)||T_n(\omega_n)|^{\sigma_i}|D^iT_n(\omega_n)|}{(1+|T_n(\omega_n)|)^{\gamma}}\,dx\,dt\\
&\le C_5\sum_{i=1}^{N}\int_{Q_T}\frac{|d_i(x,t)|^{p_i'}|T_n(\omega_n)|^{\sigma_ip_i'}(1+|T_n(\omega_n)|)^{\frac{\delta}{p_i-1}}}{(1+|T_n(\omega_n)|)^{\gamma}}\,dx\,dt\\
&\quad+\frac{b_0(\gamma-1)}{4}\sum_{i=1}^{N}\int_{Q_T}\frac{|D^iT_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}}\,dx\,dt\\
&\le C_6\sum_{i=1}^{N}\int_{Q_T}|d_i(x,t)|^{\frac{sp_i}{(s+\gamma)(p_i-1)-\sigma_ip_i-\delta}}\,dx\,dt+\frac{\nu}{16}\int_{Q_T}|T_n(\omega_n)|^{s}\,dx\,dt\\
&\quad+\frac{b_0(\gamma-1)}{4}\sum_{i=1}^{N}\int_{Q_T}\frac{|D^iT_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}}\,dx\,dt.
\end{aligned}\tag{4.11}$$

It follows that

$$\begin{aligned}
&\int_0^T\langle(\omega_n)_t,\psi(\omega_n)\rangle\,dt+\frac{b_0(\gamma-1)}{2}\sum_{i=1}^{N}\int_{Q_T}\frac{|D^iT_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}}\,dx\,dt+\frac{\nu}{8}\int_{Q_T}|T_n(\omega_n)|^{s}\,dx\,dt\\
&\le\frac{\nu}{2}TR^s|\Omega|+C_1T\int_{\Omega}\frac{1}{|x|^{\frac{p_0s}{s-p_0+1}}}\,dx+C_3\sum_{i=1}^{N}\int_{Q_T}|c_i(x,t)|^{\frac{sp_i}{(s+\gamma)(p_i-1)-\delta}}\,dx\,dt\\
&+C_6\sum_{i=1}^{N}\int_{Q_T}|d_i(x,t)|^{\frac{sp_i}{(s+\gamma)(p_i-1)-\sigma_ip_i-\delta}}\,dx\,dt+C_4.
\end{aligned}\tag{4.12}$$

We set

$$\Psi(s)=\begin{cases}|s|+\dfrac{1}{\gamma-2}\dfrac{1}{(1+|s|)^{\gamma-2}}+\dfrac{1}{2-\gamma} & \text{for}\quad \gamma\in]1,2[\cup]2,\infty[,\\ |s|-\log(1+|s|) & \text{for}\quad \gamma=2,\end{cases}$$

then $\psi(s) = (\Psi(s))'$. Taking $\gamma \in]1,2[\cup]2,\infty[$ we obtain

$$\begin{aligned}\int_0^T \langle (\omega_n)_t, \psi(\omega_n)\rangle \, dt &= \int_\Omega \int_0^T \frac{\partial \Psi(\omega_n)}{\partial t} \, dt \, dx \\ &= \int_\Omega \Psi(\omega_n(T)) \, dx - \int_\Omega \Psi(\omega_{0,n}) \, dx.\end{aligned}$$

Since $\Psi(\omega_n(T)) \geq 0$, it follows that

$$\begin{aligned}&\frac{b_0(\gamma-1)}{2} \sum_{i=1}^N \int_{Q_T} \frac{|D^i T_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}} \, dx \, dt + \frac{\nu}{8} \int_{Q_T} |T_n(\omega_n)|^s \, dx \, dt \\ &\leq \frac{\nu}{2} T R^s |\Omega| + C_1 T \int_\Omega \frac{1}{|x|^{\frac{p_0 s}{s-p_0+1}}} \, dx + C_3 \sum_{i=1}^N \int_{Q_T} |c_i(x,t)|^{\frac{sp_i}{(s+\gamma)(p_i-1)-\delta}} \, dx \, dt \\ &+ C_6 \sum_{i=1}^N \int_{Q_T} |d_i(x,t)|^{\frac{sp_i}{(s+\gamma)(p_i-1)-\sigma_i p_i-\delta}} \, dx \, dt + \int_\Omega \Psi(\omega_{0,n}) \, dx + C_4.\end{aligned} \tag{4.13}$$

Under the assumption $s > \frac{N(p_0-1)}{N-p_0}$ we have $\frac{1}{|x|^{\frac{p_0 s}{s-p_0+1}}} \in L^1(\Omega)$.
In the case of $1 < \gamma < 2$, we have $\Psi(\omega_{0,n}) \leq |\omega_{0,n}|$ then

$$\begin{aligned}&\frac{b_0(\gamma-1)}{2} \sum_{i=1}^N \int_{Q_T} \frac{|D^i T_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}} \, dx \, dt + \frac{\nu}{8} \int_{Q_T} |T_n(\omega_n)|^s \, dx \, dt \\ &\leq C_7 + \int_\Omega |\omega_{0,n}| \, dx \\ &\leq C_8.\end{aligned} \tag{4.14}$$

We can also directly obtain the same result from (4.13) when $\gamma > 2$.
one other hand for $\gamma = 2$, we have

$$\begin{aligned}\int_0^T \langle (\omega_n)_t, \psi(\omega_n)\rangle \, dt &= \int_\Omega |\omega_n(T)| - \log(1+|\omega_n(T)|) \, dx - \int_\Omega |\omega_{0,n}| - \log(1+|\omega_{0,n}|) \, dx \\ &\geq - \int_\Omega |\omega_0| \, dx.\end{aligned}$$

Yields from (4.12), that

$$\frac{b_0(\gamma-1)}{2} \sum_{i=1}^N \int_{Q_T} \frac{|D^i T_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}} \, dx \, dt + \frac{\nu}{8} \int_{Q_T} |T_n(\omega_n)|^s \, dx \, dt \leq C_9 + \int_\Omega |\omega_0| \, dx. \tag{4.15}$$

Therefore from (4.14) and (4.15), yields that: for any $\gamma \in]1,\infty[$, there exists a constant C_{10} independent of n, such that

$$\sum_{i=1}^N \int_{Q_T} \frac{|D^i T_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}} \, dx \, dt + \int_{Q_T} |T_n(\omega_n)|^s \, dx \, dt \leq C_{10} \quad \text{for all} \quad \gamma > 1. \tag{4.16}$$

Let $n > k > 0$, we conclude that

$$\begin{aligned}
&\|T_k(\omega_n)\|^{\underline{p}}_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \leq \sum_{i=1}^{N}\int_{Q_T}|D^iT_k(\omega_n)|^{p_i}\,dx\,dt + \int_{Q_T}|T_k(\omega_n)|^{p_0}\,dx\,dt + N + 1\\
&= \sum_{i=1}^{N}\int_{\{|\omega_n|\leq k\}}|D^iT_n(\omega_n)|^{p_i}\,dx\,dt + \int_{Q_T}|T_k(\omega_n)|^{p_0}\,dx\,dt + N + 1\\
&\leq (1+k)^{\gamma+\delta}\sum_{i=1}^{N}\int_{Q_T}\frac{|D^iT_n(\omega_n)|^{p_i}}{(1+|T_n(\omega_n)|)^{\gamma+\delta}}\,dx\,dt + \int_{Q_T}|T_k(\omega_n)|^{p_0}\,dx\,dt + N + 1\\
&\leq C_{10}k^{\gamma+\delta} + C_{11}k^{p_0}.
\end{aligned} \tag{4.17}$$

Step 3 : Weak convergence of truncations.
Thanks to (4.16) we have

$$\begin{aligned}
k^s\text{meas}\{|\omega_n| > k\} &= \int_{\{|\omega_n|>k\}}|T_k(\omega_n)|^s\,dx\,dt\\
&\leq \int_{Q_T}|T_n(\omega_n)|^s\,dx\,dt\\
&\leq C_{10}.
\end{aligned}$$

Therefore, we obtain

$$\limsup_{n\to\infty}\text{meas}\{|\omega_n| > k\} \leq \frac{C_{10}}{k^s} \longrightarrow 0 \quad \text{as} \quad k \to +\infty. \tag{4.18}$$

Therefore, similarly as in [20], we show that $(\omega_n)_n$ is a Cauchy sequence in measure, therefore, the sequence converges almost everywhere, for a subsequence, to some measurable function ω. Thanks to (4.17), we deduce that

$$T_k(\omega_n) \rightharpoonup T_k(\omega) \qquad \text{weakly in} \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)), \tag{4.19}$$

and in view of the Lebesgue dominated convergence theorem, we have

$$T_k(\omega_n) \to T_k(\omega) \quad \text{strongly in} \quad L^{p_0}(Q_T). \tag{4.20}$$

Step 4: The equi-integrability of the sequences $(|T_n(\omega_n)|^r))_n$ in $L^1(Q_T)$.

In this step, we will show that the sequence $(|T_n(\omega_n)|^r)_n$ converges strongly to $|\omega|^r$ in $L^1(Q_T)$ for any $0 < r < s$.
Let $\eta > 0$ such that $s - \eta > 0$, it's clear that $|T_n(\omega_n)|^{s-\eta} \to |\omega|^{s-\eta}$ a.e. in Q_T. In view of Vitali's theorem, it's sufficient to prove that the sequence $(|T_n(\omega_n)|^{s-\eta})_n$ is uniformly equi-integrable.
Let $0 < h < n$, for any measurable subset $E \subset Q_T$ we have

$$\int_E |T_n(\omega_n)|^{s-\eta}\,dx\,dt \leq \int_E |T_h(\omega_n)|^{s-\eta}\,dx + \int_{\{|\omega_n|>h\}}|T_n(\omega_n)|^{s-\eta}\,dx. \tag{4.21}$$

Thanks to (4.16) we have

$$\int_{Q_T}|T_n(\omega_n)|^s dx\,dt \leq C_{10}, \tag{4.22}$$

which implies that

$$h^{\eta}\int_{\{|\omega_n|>h\}}|T_n(\omega_n)|^{s-\eta}\,dx\,dt \;\;\leq \int_{\{|\omega_n|>h\}}|T_n(\omega_n)|^{s}\,dx \leq C_{10}. \tag{4.23}$$

It follows that

$$\int_{\{|\omega_n|>h\}}|T_n(\omega_n)|^{s-\eta}dx\,dt \leq \frac{C_{15}}{h^{\eta}} \longrightarrow 0 \quad \text{as} \quad h\to\infty.$$

Then, for any $\varepsilon>0$, there exists $h_0(\varepsilon)>0$ such that

$$\int_{\{|\omega_n|>h\}}|T_n(\omega_n)|^{s-\eta}dx\,dt \leq \frac{\varepsilon}{2} \quad \text{for any} \quad h\geq h_0(\varepsilon). \tag{4.24}$$

On the other hand, there exists $\beta(\varepsilon,h)>0$ small enough such that: for any $E\subset Q_T$ with $\text{meas}(E)\leq\beta(\varepsilon,h)$ we have

$$\int_E |T_h(\omega_n)|^{s-\eta}dx\,dt \leq \frac{\varepsilon}{2}. \tag{4.25}$$

By combining (4.21), (4.24) and (4.25), we deduce that

$$\int_E |T_n(\omega_n)|^{s-\eta}dx\,dt \leq \varepsilon, \qquad \text{with} \;\; E\subseteq Q_T \quad \text{such that} \quad \text{meas}(E)\leq\beta(\varepsilon). \tag{4.26}$$

As a result, the sequence $(|T_n(\omega_n)|^{s-\eta})_n$ is uniformly equi-integrable, and in view of Vitali's theorem, we conclude that

$$|T_n(\omega_n)|^{r} \longrightarrow |\omega|^{r} \qquad \text{strongly in} \quad L^1(Q_T) \quad \text{for any} \quad 0<r<s. \tag{4.27}$$

Step 5 : The strongly convergence in $L^1(Q_T)$.
In this step, we will show that

$$|T_n(\omega_n)|^{s-1}T_n(\omega_n) \longrightarrow |\omega|^{s-1}\omega \quad \text{strongly in} \quad L^1(Q_T), \tag{4.28}$$

$$\frac{|T_n(\omega_n)|^{p_0-2}T_n(\omega_n)}{|x|^{p_0}+\frac{1}{n}} \longrightarrow \frac{|\omega|^{p_0-2}\omega}{|x|^{p_0}} \quad \text{strongly in} \quad L^1(Q_T), \tag{4.29}$$

and

$$\frac{1}{n}|\omega_n|^{p_0-2}\omega_n \longrightarrow 0 \quad \text{strongly in} \quad L^1(Q_T). \tag{4.30}$$

First of all, we have

$$|T_n(\omega_n)|^{s-1}T_n(\omega_n) \longrightarrow |\omega|^{s-1}\omega \quad \text{a.e. in } Q_T,$$

$$\frac{|T_n(\omega_n)|^{p_0-2}T_n(\omega_n)}{|x|^{p_0}+\frac{1}{n}} \longrightarrow \frac{|\omega|^{p_0-2}\omega}{|x|^{p_0}} \quad \text{a.e. in } Q_T,$$

and

$$\frac{1}{n}|\omega_n|^{p_0-2}\omega_n \longrightarrow 0 \quad \text{a.e. in } Q_T.$$

Now, we will show that the three sequences $(|T_n(\omega_n)|^{s-1}T_n(\omega_n))_n$, $(\dfrac{|T_n(\omega_n)|^{p_0-2}T_n(\omega_n)}{|x|^{p_0}+\frac{1}{n}})_n$ and $(\dfrac{1}{n}|\omega_n|^{p_0-2}\omega_n)_n$ are uniformly equi-integrable.
Employing the test function $T_{h+1}(\omega_n)-T_h(\omega_n)$ within the framework of the

approximate problem (4.2), and since $T_{h+1}(\omega_n)-T_h(\omega_n)$ have the same sign as ω_n yields

$$\int_0^T\langle\frac{\partial\omega_n}{\partial t},T_{h+1}(\omega_n)-T_h(\omega_n)\rangle dt+\sum_{i=1}^N\int_{\{h<|\omega_n|\le h+1\}}a_i(x,t,\omega_n,\nabla\omega_n)D^i\omega_n\,dx\,dt$$
$$+\nu\int_{\{h<|\omega_n|\}}|T_n(\omega_n)|^s|T_{h+1}(\omega_n)-T_h(\omega_n)|\,dx\,dt+\frac{1}{n}\int_{\{h<|\omega_n|\}}|\omega_n|^{p_0-1}|T_{h+1}(\omega_n)$$
$$-T_h(\omega_n)|\,dx\,dt=\lambda\int_{\{h<|\omega_n|\}}\frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|T_{h+1}(\omega_n)-T_h(\omega_n)|\,dx\,dt$$
$$+\sum_{i=1}^N\int_{\{h<|\omega_n|\le h+1\}}f_i(x,t,\omega_n)D^iT_{h+1}(\omega_n)\,dx\,dt.$$

We have

$$\int_0^T\langle\frac{\partial\omega_n}{\partial t},T_{h+1}(\omega_n)-T_h(\omega_n)\rangle\,dt=\int_\Omega\int_0^T\frac{\partial\varphi_{h+1}(\omega_n)}{\partial t}\,dt\,dx-\int_\Omega\int_0^T\frac{\partial\varphi_h(\omega_n)}{\partial t}\,dt\,dx$$
$$=\int_\Omega\varphi_{h+1}(\omega_n(T))-\varphi_{h+1}(\omega_{0,n})\,dx-\int_\Omega\varphi_h(\omega_n(T))-\varphi_h(\omega_{0,n})\,dx,$$

with

$$\begin{aligned}\int_\Omega\varphi_{h+1}(\omega_n(T))\,dx-\int_\Omega\varphi_h(\omega_n(T))\,dx\;&=\int_{\{h<|\omega_n(T)|\le h+1\}}(\frac{\omega_n^2(T)}{2}-h|\omega_n(T)|+\frac{h^2}{2})\,dx\\&\quad+\int_{\{h+1<|\omega_n(T)|\}}(|\omega_n(T)|-h-\frac{1}{2})\,dx\ge 0.\end{aligned}\tag{4.31}$$

It follows that

$$\begin{aligned}&\sum_{i=1}^N\int_{\{h<|\omega_n|\le h+1\}}a_i(x,t,\omega_n,\nabla\omega_n)D^i\omega_n\,dx\,dt+\nu\int_{\{h<|\omega_n|\}}|T_n(\omega_n)|^s|T_{h+1}(\omega_n)-T_h(\omega_n)|\,dx\,dt\\&\quad+\frac{1}{n}\int_{\{h<|\omega_n|\}}|\omega_n|^{p_0-1}|T_{h+1}(\omega_n)-T_h(\omega_n)|\,dx\,dt\\&\le\lambda\int_{\{h<|\omega_n|\}}\frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}}|T_{h+1}(\omega_n)-T_h(\omega_n)|\,dx\,dt+\sum_{i=1}^N\int_{\{h<|\omega_n|\le h+1\}}|c_i(x,t)||D^i\omega_n|\,dx\,dt\\&\quad+\sum_{i=1}^N\int_{\{h<|\omega_n|\le h+1\}}|d_i(x,t)||\omega_n|^{\sigma_i}|D^i\omega_n|\,dx\,dt+\int_\Omega\varphi_{h+1}(\omega_{0,n})\,dx-\int_\Omega\varphi_h(\omega_{0,n})\,dx.\end{aligned}\tag{4.32}$$

Thanks to Young's inequality, we have

$$\begin{aligned}\lambda\int_{\{h<|\omega_n|\}}\frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}}|T_{h+1}(\omega_n)-T_h(\omega_n)|\,dx\,dt\;&\le\frac{\nu}{3}\int_{\{h<|\omega_n|\}}|T_n(\omega_n)|^s|T_{h+1}(\omega_n)-T_h(\omega_n)|\,dx\,dt\\&\quad+C_{12}\int_{\{h<|\omega_n|\}}\frac{|T_{h+1}(\omega_n)-T_h(\omega_n)|}{|x|^{\frac{p_0s}{s-p_0+1}}}\,dx\,dt.\end{aligned}\tag{4.33}$$

For the second term on the right-hand side of (4.32), Let $\varepsilon > 0$ small enough such that $s-\varepsilon > \frac{\delta}{\underline{p}-1}$, by using Young inequality we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}|c_i(x,t)||D^i\omega_n|\,dx\,dt\\
&\leq C_{13}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}|c_i(x,t)|^{p_i'}(1+|\omega_n|)^{\frac{\delta}{p_i-1}}\,dx\,dt+\frac{b_0}{4}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}\frac{|D^i\omega_n|^{p_i}}{(1+|\omega_n|)^{\delta}}\,dx\,dt\\
&\leq C_{14}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}|c_i(x,t)|^{\frac{(s-\varepsilon)p_i}{(s-\varepsilon)(p_i-1)-\delta}}\,dx\,dt+\int_{\{h<|\omega_n|\leq h+1\}}|T_n(\omega_n)|^{s-\varepsilon}\,dx\,dt\\
&\quad+\frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}a_i(x,t,\omega_n,\nabla\omega_n)D^i\omega_n\,dx\,dt.
\end{aligned}\tag{4.34}$$

Concerning the third term on the right-hand side of (4.32). Similarly to (4.34) we can write

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}|d_i(x,t)||\omega_n|^{\sigma_i}|D^i\omega_n|\,dx\,dt\\
&\leq C_{15}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}|d_i(x,t)|^{p_i'}|\omega_n|^{\sigma_ip_i'}(1+|\omega_n|)^{\frac{\delta}{p_i-1}}\,dx\,dt+\frac{b_0}{4}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}\frac{|D^i\omega_n|^{p_i}}{(1+|\omega_n|)^{\delta}}\,dx\,dt\\
&\leq C_{16}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}|d_i(x,t)|^{\frac{(s-\varepsilon)p_i}{(s-\varepsilon)(p_i-1)-\sigma_ip_i-\delta}}\,dx\,dt+\int_{\{h<|\omega_n|\leq h+1\}}|T_n(\omega_n)|^{s-\varepsilon}\,dx\,dt\\
&\quad+\frac{1}{4}\int_{\{h<|\omega_n|\leq h+1\}}a_i(x,t,\omega_n,\nabla\omega_n)D^i\omega_n\,dx\,dt.
\end{aligned}\tag{4.35}$$

By combining (4.32) and (4.33) – (4.35) we get

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}a_i(x,t,\omega_n,\nabla\omega_n)D^i\omega_n\,dx\,dt+\frac{\nu}{3}\int_{\{h+1<|\omega_n|\}}|T_n(\omega_n)|^{s}\,dx\,dt\\
&\quad+\lambda\int_{\{h+1<|\omega_n|\}}\frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}}\,dx\,dt+\frac{1}{n}\int_{\{h+1<|\omega_n|\}}|\omega_n|^{p_0-1}\,dx\,dt\\
&\leq 2C_{12}\int_{\{h<|\omega_n|\}}\frac{1}{|x|^{\frac{p_0s}{s-p_0+1}}}\,dx\,dt+C_{14}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}|c_i(x,t)|^{\frac{(s-\varepsilon)p_i}{(s-\varepsilon)(p_i-1)-\delta}}\,dx\,dt\\
&\quad+\int_{\{h<|\omega_n|\leq h+1\}}|T_n(\omega_n)|^{s-\varepsilon}\,dx\,dt+C_{16}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\leq h+1\}}|d_i(x,t)|^{\frac{(s-\varepsilon)p_i}{(s-\varepsilon)(p_i-1)-\sigma_ip_i-\delta}}\,dx\,dt\\
&\quad+\int_{\Omega}\varphi_{h+1}(\omega_{0,n})\,dx-\int_{\Omega}\varphi_h(\omega_{0,n})\,dx.
\end{aligned}\tag{4.36}$$

We have meas$\{|\omega_n|>h\}\to 0$ as $h\to\infty$, then

$$\int_{\{h<|\omega_n|\}}\frac{1}{|x|^{\frac{p_0s}{s-p_0+1}}}\,dx\,dt\longrightarrow 0\qquad\text{as}\quad h\to\infty.\tag{4.37}$$

Also, since $\omega_0 \in L^1(\Omega)$, it follows that

$$\begin{aligned}
&\int_\Omega \varphi_{h+1}(\omega_{0,n})\,dx - \int_\Omega \varphi_h(\omega_{0,n})\,dx \\
&= \int_{\{h<|\omega_{0,n}|\le h+1\}} (\frac{|\omega_{0,n}|^2}{2} - h|\omega_{0,n}| + \frac{h^2}{2})\,dx + \int_{\{h+1<|\omega_{0,n}|\}} (|\omega_{0,n}| - h - \frac{1}{2})\,dx \\
&\le \int_{\{h<|\omega_{0,n}|\le h+1\}} \frac{1}{2}\,dx + \int_{\{h+1<|\omega_{0,n}|\}} (|\omega_0| - h - \frac{1}{2})\,dx \longrightarrow 0 \quad \text{as} \quad h \to \infty,
\end{aligned} \tag{4.38}$$

According to (4.27), we obtain

$$\int_{\{h<|\omega_n|\le h+1\}} |T_n(\omega_n)|^{s-\varepsilon}\,dx\,dt \to 0 \quad \text{as} \quad n, h \to \infty. \tag{4.39}$$

Choosing $\varepsilon > 0$ such that $\frac{(s-\varepsilon)p_i}{(s-\varepsilon)(p_i-1)-\delta} < m_i$, and $\frac{(s-\varepsilon)p_i}{(s-\varepsilon)(p_i-1)-\sigma_i p_i-\delta} < r_i$, we obtain

$$\begin{aligned}
&C_{14}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\le h+1\}} |c_i(x,t)|^{\frac{(s-\varepsilon)p_i}{(s-\varepsilon)(p_i-1)-\delta}}\,dx\,dt \\
&+C_{16}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\le h+1\}} |d_i(x,t)|^{\frac{(s-\varepsilon)p_i}{(s-\varepsilon)(p_i-1)-\sigma_i p_i-\delta}}\,dx\,dt \to 0 \quad \text{as} \quad n, h \to \infty.
\end{aligned} \tag{4.40}$$

By combining (4.36) and (4.37) – (4.40), we deduce that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\le h+1\}} a_i(x,t,\omega_n,\nabla\omega_n)D^i\omega_n\,dx\,dt = 0, \tag{4.41}$$

$$\int_{\{h+1<|\omega_n|\}} |T_n(\omega_n)|^s\,dx\,dt + \int_{\{h+1<|\omega_n|\}} \frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}}\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n, h \to \infty, \tag{4.42}$$

and

$$\frac{1}{n}\int_{\{h+1<|\omega_n|\}} |\omega_n|^{p_0-1}\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n, h \to \infty. \tag{4.43}$$

Moreover, in view of (4.34) – (4.35) and (4.41), we deduce that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{h<|\omega_n|\le h+1\}} f_i(x,t,\omega_n)D^i\omega_n\,dx\,dt = 0. \tag{4.44}$$

Then, for any value of $\eta > 0$, there exists $h(\eta) > 0$ such that

$$\int_{\{h(\eta)<|\omega_n|\}} |T_n(\omega_n)|^s\,dx\,dt + \int_{\{h(\eta)<|\omega_n|\}} \frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}}\,dx\,dt + \frac{1}{n}\int_{\{h(\eta)<|\omega_n|\}} |\omega_n|^{p_0-1}\,dx\,dt \le \frac{\eta}{2}. \tag{4.45}$$

On the one hand : for any measurable set E in Q_T we have

$$\begin{aligned}
&\int_E |T_n(\omega_n)|^s \,dx\,dt + \int_E \frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}}\,dx\,dt + \frac{1}{n}\int_E |\omega_n|^{p_0-1}\,dx\,dt \\
&\le \int_E |T_{h(\eta)}(\omega_n)|^s \,dx\,dt + \int_E \frac{|T_{h(\eta)}(\omega_n)|^{p_0-1}}{|x|^{p_0}}\,dx\,dt + \frac{1}{n}\int_E |T_{h(\eta)}(\omega_n)|^{p_0-1}\,dx\,dt \\
&+ \int_{\{h(\eta)<|\omega_n|\}} |T_n(\omega_n)|^s \,dx\,dt + \int_{\{h(\eta)<|\omega_n|\}} \frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}}\,dx\,dt + \frac{1}{n}\int_{\{h(\eta)<|\omega_n|\}} |\omega_n|^{p_0-1}\,dx\,dt.
\end{aligned} \tag{4.46}$$

On the other hand, for any $\eta > 0$ there exists $\beta(\eta) > 0$ such that : for any $E \subset Q_T$ with $\text{meas}(E) \le \beta(\eta)$ we have

$$\int_E |T_{h(\eta)}(\omega_n)|^s \,dx\,dt + \int_E \frac{|T_{h(\eta)}(\omega_n)|^{p_0-1}}{|x|^{p_0}}\,dx\,dt + \frac{1}{n}\int_E |T_{h(\eta)}(\omega_n)|^{p_0-1}\,dx\,dt \le \frac{\eta}{2}. \tag{4.47}$$

By combining (4.45) − (4.47) we conclude that

$$\int_E |T_n(\omega_n)|^s dxdt + \int_E \frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}} dxdt + \frac{1}{n}\int_E |\omega_n|^{p_0-1} dxdt \le \eta \quad \text{with} \quad \text{meas}(E) \le \beta(\eta). \tag{4.48}$$

In view of Vitali's theorem, we conclude that (4.28) − (4.30) hold true.

Step 6 : Convergence of the gradient.
We will use $\varepsilon_j(n)$ to represent various functions that tends to 0 as n approaches infinity, respectively for $\varepsilon_j(h)$, $\varepsilon_j(n,h)$, and $\varepsilon_j(n,\mu,h)$.
Let $S_h(\cdot) \in C^2(I\!R)$ be an increasing function defined as

$$\begin{cases} S_h(\sigma) = \sigma \qquad \text{if} \quad |\sigma| \le h, \\ \text{supp}(S'_h(\cdot)) \subset [-h-1, h+1], \\ \text{supp}(S''_h(\cdot)) \subset [-h-1,-h] \cup [h, h+1]. \end{cases}$$

Employing the test function $S'_h(\omega_n)(T_k(\omega_n) - (T_k(\omega))_\mu)$ within the framework of the approximate problem (4.2), we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t S_h(\omega_n)(T_k(\omega_n) - T_k(\omega))\,dx\,dt + \sum_{i=1}^N \int_{Q_T} (T_k(\omega_n) - T_k(\omega)) S''_h(\omega_n) a_i(x,t,T_n(\omega_n),\nabla\omega_n) D^i\omega_n\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} S'_h(\omega_n) a_i(x,t,T_n(\omega_n),\nabla\omega_n)(D^i T_k(\omega_n) - D^i(T_k(\omega))_\mu)\,dx\,dt \\
&\quad + \nu \int_{Q_T} |T_n(\omega_n)|^{s-1} T_n(\omega_n) S'_h(\omega_n)(T_k(\omega_n) - T_k(\omega))\,dx\,dt \\
&\quad + \frac{1}{n}\int_{Q_T} |\omega_n|^{p_0-2}\omega_n S'_h(\omega_n)(T_k(\omega_n) - T_k(\omega))\,dx\,dt \\
&= \lambda \int_{Q_T} \frac{|T_n(\omega_n)|^{p_0-2} T_n(\omega_n)}{|x|^{p_0} + \frac{1}{n}} S'_h(\omega_n)(T_k(\omega_n) - T_k(\omega))\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} f_i(x,t,T_n(\omega_n)) S'_h(\omega_n)(D^i T_k(\omega_n) - D^i(T_k(\omega))_\mu)\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} S''_h(\omega_n) f_i(x,t,T_n(\omega_n)) D^i\omega_n (T_k(\omega_n) - (T_k(\omega))_\mu)\,dx\,dt.
\end{aligned} \tag{4.49}$$

According to definition of $S_h(\cdot)$, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t S_h(\omega_n)(T_k(\omega_n) - (T_k(\omega))_\mu)\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} S_h'(\omega_n) a_i(x,t,T_n(\omega_n),\nabla\omega_n)(D^i T_k(\omega_n) - D^i(T_k(\omega))_\mu)\,dx\,dt \\
&\le 2k\|S_h''(\cdot)\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{\{h<|\omega_n|\le h+1\}} a_i(x,t,T_n(\omega_n),\nabla\omega_n) D^i\omega_n\,dx\,dt \\
&\qquad + \nu\|S_h'(\cdot)\|_{L^\infty(I\!R)} \int_{Q_T} |T_n(\omega_n)|^s |T_k(\omega_n) - (T_k(\omega))_\mu|\,dx\,dt \\
&\qquad + \frac{1}{n}\|S_h'(\cdot)\|_{L^\infty(I\!R)} \int_{Q_T} |\omega_n|^{p_0-1} |T_k(\omega_n) - (T_k(\omega))_\mu|\,dx\,dt \\
&\qquad + \lambda\|S_h'(\cdot)\|_{L^\infty(I\!R)} \int_{Q_T} \frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}} |T_k(\omega_n) - (T_k(\omega))_\mu|\,dx\,dt \\
&\qquad + \|S_h'(\cdot)\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{Q_T} |f_i(x,t,T_{h+1}(\omega_n))||D^i T_k(\omega_n) - D^i(T_k(\omega))_\mu|\,dx\,dt \\
&\qquad + 2k\|S_h''(\cdot)\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{\{h<|\omega_n|\le h+1\}} |f_i(x,t,\omega_n)||D^i\omega_n|\,dx\,dt.
\end{aligned} \tag{4.50}$$

Thanks to (4.41), we have

$$2k\|S_h''(\cdot)\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{\{h<|\omega_n|\le h+1\}} a_i(x,t,\omega_n,\nabla\omega_n) D^i\omega_n\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n,h\to\infty. \tag{4.51}$$

Remark that $T_k(\omega_n) - (T_k(\omega))_\mu \rightharpoonup 0$ weak$-*$ in $L^\infty(Q_T)$, and in view of (4.28) $-$ (4.30) we obtain

$$\begin{aligned}
&\nu\|S_h'(\cdot)\|_{L^\infty(I\!R)} \int_{Q_T} |T_n(\omega_n)|^s |T_k(\omega_n) - (T_k(\omega))_\mu|\,dx\,dt \\
&\quad + \frac{1}{n}\|S_h'(\cdot)\|_{L^\infty(I\!R)} \int_{Q_T} |\omega_n|^{p_0-1} |T_k(\omega_n) - (T_k(\omega))_\mu|\,dx\,dt \\
&\quad + \lambda\|S_h'(\cdot)\|_{L^\infty(I\!R)} \int_{Q_T} \frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}} |T_k(\omega_n) - (T_k(\omega))_\mu|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n,\mu\to\infty.
\end{aligned} \tag{4.52}$$

From (3.5) and Lebesgue dominated convergence theorem yields that

$$f_i(x,t,T_{h+1}(\omega_n)) \longrightarrow f_i(x,t,T_{h+1}(\omega)) \quad \text{strongly in} \quad L^{p_i'}(Q_T) \quad \text{as} \quad n\to\infty,$$

and since $D^i T_k(\omega_n) - D^i(T_k(\omega))_\mu \rightharpoonup 0$ weakly in $L^{p_i}(Q_T)$, it follows that

$$\|S_h'(\cdot)\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{Q_T} |f_i(x,t,T_{h+1}(\omega_n))|\ |D^i T_k(\omega_n) - D^i(T_k(\omega))_\mu|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n,\mu\to\infty. \tag{4.53}$$

Moreover, using (4.64), we have

$$2k\|S_h''(\cdot)\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{\{h<|\omega_n|<h+1\}} |f_i(x,t,\omega_n)||D^i\omega_n|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n,h\to\infty. \tag{4.54}$$

Putting equations (4.50) and (4.54) together, we see that

$$
\begin{aligned}
&\int_0^T \int_\Omega \partial_t S_h(\omega_n)(T_k(\omega_n) - (T_k(\omega))_\mu)\,dx\,dt \\
&+\sum_{i=1}^N \int_{Q_T} S'_h(\omega_n)a_i(x,t,T_n(\omega_n),\nabla\omega_n)(D^iT_k(\omega_n) - D^i(T_k(\omega))_\mu)\,dx\,dt \le \varepsilon_1(n,\mu,h).
\end{aligned} \tag{4.55}
$$

We can show that (see [20], step 4, page 17).

$$
\int_0^T \int_\Omega \partial_t S_h(\omega_n)(T_k(\omega_n) - (T_k(\omega))_\mu)\,dx\,dt \ge \varepsilon_2(n,\mu). \tag{4.56}
$$

Since $S'_h(\cdot) > 0$ and $S'_h(\omega_n) = 1$ on the set $\{|\omega_n| \le k\}$, then

$$
\begin{aligned}
&\sum_{i=1}^N \int_{Q_T} S'_h(\omega_n)a_i(x,t,T_n(\omega_n),\nabla\omega_n)(D^iT_k(\omega_n) - D^i(T_k(\omega))_\mu)\,dx\,dt \\
&= \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(\omega_n),\nabla T_k(\omega_n))(D^iT_k(\omega_n) - D^i(T_k(\omega))_\mu)\,dx\,dt \\
&\quad - \sum_{i=1}^N \int_{\{|\omega_n|>k\}} S'_h(\omega_n)a_i(x,t,T_{h+1}(\omega_n),\nabla\omega_n)D^i(T_k(\omega))_\mu\,dx\,dt \\
&\ge \sum_{i=1}^N \int_{Q_T} (a_i(x,t,T_k(\omega_n),\nabla T_k(\omega_n)) - a_i(x,t,T_k(\omega_n),\nabla T_k(\omega)))(D^iT_k(\omega_n) - D^iT_k(\omega))\,dx\,dt \\
&\quad - \sum_{i=1}^N \int_{Q_T} |a_i(x,t,T_k(\omega_n),\nabla T_k(\omega))||D^iT_k(\omega_n) - D^iT_k(\omega)|\,dx\,dt \\
&\quad - \sum_{i=1}^N \int_{Q_T} |a_i(x,t,T_k(\omega_n),\nabla T_k(\omega_n))||D^iT_k(\omega) - (D^iT_k(\omega))_\mu|\,dx\,dt \\
&\quad - \|S'_h(\cdot)\|_{L^\infty(I\!R)} \sum_{i=1}^N \int_{\{|\omega_n|>k\}} |a_i(x,t,T_{h+1}(\omega_n),\nabla T_{h+1}(\omega_n))||D^i(T_k(\omega))_\mu)|\,dx\,dt.
\end{aligned} \tag{4.57}
$$

By applying the same reasoning as in the referenced work [20], we can demonstrate that: the last third terms on the right-hand side of equation (4.57) approaches zero as both n, μ and h increase infinitely. Consequently

$$
\begin{aligned}
&\sum_{i=1}^N \int_{Q_T} S'_h(\omega_n)a_i(x,t,T_n(\omega_n),\nabla\omega_n)(D^iT_k(\omega_n) - D^i(T_k(\omega))_\mu)\,dx\,dt \\
&\ge \sum_{i=1}^N \int_{Q_T} (a_i(x,t,T_k(\omega_n),\nabla T_k(\omega_n)) - a_i(x,t,T_k(\omega_n),\nabla T_k(\omega))) \\
&\qquad\qquad \times(D^iT_k(\omega_n) - D^iT_k(\omega))\,dx\,dt + \varepsilon_4(n,\mu,h).
\end{aligned} \tag{4.58}
$$

Putting (4.55) – (4.56) and (4.58) together, we deduce that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{Q_T}(a_i(x,t,T_k(\omega_n),\nabla T_k(\omega_n))-a_i(x,t,T_k(\omega_n),\nabla T_k(\omega)))\cdot(D^iT_k(\omega_n)-D^iT_k(\omega))\,dx\,dt\\
&\leq \int_0^T\int_\Omega \partial_t S_h(\omega_n)(T_k(\omega_n)-(T_k(\omega))_\mu)\,dx\,dt\\
&\quad+\sum_{i=1}^{N}\int_{Q_T} S_h'(\omega_n)a_i(x,t,\omega_n,\nabla\omega_n)\cdot(D^iT_k(\omega_n)-D^i(T_k(\omega))_\mu)\,dx\,dt+\varepsilon_5(n,\mu,h)\\
&\leq \varepsilon_6(n,\mu,h).
\end{aligned}\tag{4.59}$$

By letting n,μ and h tend to infinity, and thanks to (4.20), we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{Q_T}\Big(a_i(x,t,T_k(\omega_n),\nabla T_k(\omega_n))-a_i(x,t,T_k(\omega_n),\nabla T_k(\omega))\Big)\cdot(D^iT_k(\omega_n)-D^iT_k(\omega))\,dx\,dt\\
&\quad+\int_{Q_T}\Big(|T_k(\omega_n)|^{p_0-2}T_k(\omega_n)-|T_k(\omega)|^{p_0-2}T_k(\omega)\Big)(T_k(\omega_n)-T_k(\omega))\,dx\,dt\to 0 \quad\text{as}\quad n\to\infty.
\end{aligned}\tag{4.60}$$

In view of Lemma 3.1, we deduce that

$$T_k(\omega_n)\to T_k(\omega)\quad\text{in}\quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))\quad\text{then}\quad \nabla\omega_n\to\nabla\omega\quad\text{a.e. in}\quad Q_T.\tag{4.61}$$

By using Lemma Fatou's and (4.41) yields that

$$\lim_{h\to\infty}\sum_{i=1}^{N}\int_{\{h<|\omega|\leq h+1\}} a_i(x,t,\omega,\nabla\omega)D^i\omega\,dx\,dt=0.\tag{4.62}$$

Similarly, we have

$$\lim_{h\to\infty}\sum_{i=1}^{N}\int_{\{h<|\omega|\leq h+1\}} |f_i(x,t,\omega)||D^i\omega|\,dx\,dt=0.\tag{4.63}$$

Step 7 : The convergence of ω_n *in* $C([0,T];L^1(\Omega))$.

Let $h\geq 1$ and $0<s\leq T$, by taking $T_1(\omega_n-(T_h(\omega))_\mu)\chi_{[0,s]}(t)$ as a test function for the approximate problem (4.2), we obtain

$$\begin{aligned}
&\int_\Omega\int_0^s \frac{\partial\omega_n}{\partial t}T_1(\omega_n-(T_h(\omega))_\mu)\,dt\,dx+\sum_{i=1}^{N}\int_{Q_s} a_i(x,t,T_n(\omega_n),\nabla\omega_n)D^iT_1(\omega_n-(T_h(\omega))_\mu)\,dx\,dt\\
&\quad+\nu\int_{Q_s}|T_n(\omega_n)|^{s-1}\,T_n(\omega_n)T_1(\omega_n-(T_h(\omega))_\mu)\,dx\,dt+\frac{1}{n}\int_{Q_s}|\omega_n|^{p_0-2}\omega_nT_1(\omega_n-(T_h(\omega))_\mu)\,dx\,dt\\
&=\lambda\int_{Q_s}\frac{|T_n(\omega_n)|^{p_0-2}T_n(\omega_n)}{|x|^{p_0}+\frac{1}{n}}T_1(\omega_n-(T_h(\omega))_\mu)\,dx\,dt+\sum_{i=1}^{N}\int_{Q_s} f_i(x,t,T_n(\omega_n))D^iT_1(\omega_n-(T_h(\omega))_\mu)\,dx\,dt.
\end{aligned}\tag{4.64}$$

The initial term on the left-hand side of equation (4.64) may be written as

$$\begin{aligned}\int_\Omega\int_0^s \frac{\partial \omega_n}{\partial t} T_1(\omega_n-(T_h(\omega))_\mu)\,dt\,dx &= \int_\Omega\int_0^s \frac{\partial}{\partial t}(\omega_n-(T_h(\omega))_\mu) T_1(\omega_n-(T_h(\omega))_\mu)\,dt\,dx\\ &\quad+\int_\Omega\int_0^s \frac{\partial (T_h(\omega))_\mu}{\partial t} T_1(\omega_n-(T_h(\omega))_\mu)\,dt\,dx\\ &= \int_\Omega \varphi_1(\omega_n(s)-(T_h(\omega(s)))_\mu)\,dx-\int_\Omega \varphi_1(\omega_{0,n}-T_h(\omega_0))\,dx\\ &\quad+\mu\int_\Omega\int_0^s (T_h(\omega)-(T_h(\omega))_\mu) T_1(\omega_n-(T_h(\omega))_\mu)\,dt\,dx,\end{aligned} \tag{4.65}$$

It's clear that $T_h(\omega)-(T_h(\omega))_\mu$ and $T_1(\omega-(T_h(\omega))_\mu)$ have same sign, then for every $s\in[0,T]$, we have

$$\begin{aligned}&\int_\Omega\int_0^s (T_h(\omega)-(T_h(\omega))_\mu) T_1(\omega_n-(T_h(\omega))_\mu)\,dt\,dx\\ &\qquad\to\int_\Omega\int_0^s (T_h(\omega)-(T_h(\omega))_\mu) T_1(\omega-(T_h(\omega))_\mu)\,dt\,dx\geq 0,\end{aligned} \tag{4.66}$$

and

$$\int_\Omega \varphi_1(\omega_{0,n}-T_h(\omega_0))\,dx \to \int_\Omega \varphi_1(\omega_0-T_h(\omega_0))\,dx \quad \text{as} \quad n\to\infty. \tag{4.67}$$

If we look at the second term of the left side of equation (4.64), we see that

$$\begin{aligned}&\sum_{i=1}^N\int_{Q_s} a_i(x,t,T_n(\omega_n),\nabla\omega_n) D^i T_1(\omega_n-(T_h(\omega))_\mu)\,dx\,dt\\ &=\sum_{i=1}^N\int_{Q_s}\big(a_i(x,t,T_{h+1}(\omega_n),\nabla T_{h+1}(\omega_n))-a_i(x,t,T_{h+1}(\omega_n),\nabla (T_h(\omega))_\mu)\big)\\ &\qquad\qquad\times(D^iT_{h+1}(\omega_n)-D^i(T_h(\omega))_\mu)\,dx\,dt\\ &\quad+\sum_{i=1}^N\int_{Q_s} a_i(x,t,T_{h+1}(\omega_n),\nabla (T_h(\omega))_\mu)(D^iT_{h+1}(\omega_n)-D^i(T_h(\omega))_\mu)\,dx\,dt.\end{aligned} \tag{4.68}$$

Given assumptions (3.3), the initial term on the right-hand side of equation (4.68) is non-negative. With respect to the final term, given the strong convergence of $a_i(x,t,T_{h+1}(\omega_n),\nabla(T_h(\omega))_\mu)$ to $a_i(x,t,T_{h+1}(\omega),\nabla T_h(\omega))$ in $L^{p_i'}(Q_T)$ and the weak convergence of $D^iT_{h+1}(\omega_n)$ to $D^iT_{h+1}(\omega)$ in $L^{p_i}(Q_T)$, we obtain

$$\begin{aligned}\varepsilon_1(n,\mu)&=\sum_{i=1}^N\int_{Q_s} a_i(x,t,T_{h+1}(\omega_n),\nabla (T_h(\omega))_\mu)(D^iT_{h+1}(\omega_n)-D^i(T_h(\omega))_\mu)\,dx\,dt\\ &\to\sum_{i=1}^N\int_{Q_s} a_i(x,t,T_{h+1}(\omega),\nabla T_h(\omega))(D^iT_{h+1}(\omega)-D^iT_h(\omega))\,dx\,dt \quad \text{as} \quad n,\mu\to\infty\\ &=\sum_{i=1}^N\int_{\{h<|\omega|<h+1\}} a_i(x,t,T_{h+1}(\omega),0)D^i\omega\,dx\,dt=0.\end{aligned} \tag{4.69}$$

For the third and fourth terms on the left side of equation (4.64), due to (4.28)−(4.30), and the weak-* convergence of $T_1(\omega_n - (T_h(\omega))_\mu)$ to $T_1(\omega - T_h(\omega))$ in $L^\infty(Q_T)$, we have

$$\begin{aligned}
&\int_0^s \int_\Omega |T_n(\omega_n)|^{s-1}\, T_n(\omega_n) T_1(\omega_n - (T_h(\omega))_\mu)\, dx\, dt + \frac{1}{n}\int_{Q_s} |\omega_n|^{p_0-2}\omega_n T_1(\omega_n - (T_h(\omega))_\mu)\, dx\, dt \\
&\longrightarrow \int_0^s \int_\Omega |\omega|^{s-1}\omega\, T_1(\omega - T_h(\omega))\, dx\, dt \quad \text{as} \quad n,\mu \to \infty \\
&\le \int_{\{|\omega|>h\}} |\omega|^s\, dx\, dt \longrightarrow 0 \quad \text{as} \quad h \to \infty.
\end{aligned} \tag{4.70}$$

In view of (4.29) we obtain

$$\begin{aligned}
&\int_0^s \int_\Omega \frac{|T_n(\omega_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} |T_1(\omega_n - (T_h(\omega))_\mu)|\, dx\, dt \\
&\longrightarrow \int_0^s \int_\Omega \frac{|\omega|^{p_0-1}}{|x|^{p_0}} |T_1(\omega - T_h(\omega))|\, dx\, dt \quad \text{as} \quad n,\mu \to \infty \\
&\le \int_{\{|u|>h\}} \frac{|\omega|^{p_0-1}}{|x|^{p_0}}\, dx\, dt \longrightarrow 0 \quad \text{as} \quad h \to \infty.
\end{aligned} \tag{4.71}$$

With respect to last term on the right-hand side of equation (4.64), we have $f_i(x,t,T_{h+1}(\omega_n)) \to f_i(x,t,T_{h+1}(\omega))$ strongly in $L^{p_i'}(Q_T)$, and since $D^i T_{h+1}(\omega_n) \rightharpoonup D^i T_{h+1}(\omega)$ weakly in $L^{p_i}(Q_T)$, we get

$$\begin{aligned}
&\sum_{i=1}^N \int_0^s \int_\Omega f_i(x,t,T_n(\omega_n)) D^i T_1(\omega_n - (T_h(\omega))_\mu)\, dx\, dt \\
&\quad = \sum_{i=1}^N \int_0^s \int_\Omega f_i(x,t,T_{h+1}(\omega_n))(D^i T_{h+1}(\omega_n) - D^i (T_h(\omega))_\mu)\, dx\, dt \\
&\longrightarrow \sum_{i=1}^N \int_0^s \int_\Omega f_i(x,t,T_{h+1}(\omega))(D^i T_{h+1}(\omega) - D^i T_h(\omega))\, dx\, dt \quad \text{as} \quad n,\mu \to \infty \\
&= \sum_{i=1}^N \int_{\{h<|\omega|<h+1\}} f_i(x,t,\omega) D^i\omega\, dx\, dt \longrightarrow 0 \quad \text{as} \quad h \to \infty.
\end{aligned} \tag{4.72}$$

Putting (4.64) − (4.72), together, yield that

$$\int_\Omega \varphi_1(\omega_n - (T_h(\omega))_\mu)\, dx \;\; \le \varepsilon_5(n,\mu,h) + \int_\Omega \varphi_1(\omega_0 - T_h(\omega_0))\, dx. \tag{4.73}$$

Since

$$\int_\Omega \varphi_1(\omega_0 - T_h(\omega_0))\, dx \longrightarrow 0 \quad \text{as} \quad h \to \infty, \tag{4.74}$$

which implies that by combining (4.73) − (4.74), we conclude that

$$\int_\Omega \varphi_1(\omega_n(s) - (T_h(\omega(s)))_\mu)\, dx \le \varepsilon_6(n,\mu,h) \longrightarrow 0 \quad \text{as} \quad n,\mu \text{ and } h \to \infty. \tag{4.75}$$

On the other hand, we have

$$\int_\Omega \varphi_1(\frac{\omega_n(s)-\omega_m(s)}{2})\,dx \le \frac{1}{2}\Big(\int_\Omega \varphi_1(\omega_n(s)-(T_h(\omega(s)))_\mu)\,dx + \int_\Omega \varphi_1(\omega_m(s)-(T_h(\omega(s)))_\mu)\,dx\Big)$$
$$\longrightarrow 0 \quad \text{as} \quad n,m\to\infty. \tag{4.76}$$

We have

$$\int_{\{|\omega_n-\omega_m|\le 2\}} \Big|\frac{\omega_n(s)-\omega_m(s)}{2}\Big|^2 dx + \int_{\{|\omega_n-\omega_m|>2\}} \Big|\frac{\omega_n(s)-\omega_m(s)}{2}\Big| dx \le 2\int_\Omega \varphi_1(\frac{\omega_n(s)-\omega_m(s)}{2})dx, \tag{4.77}$$

and

$$\begin{aligned}\int_\Omega |\omega_n(s)-\omega_m(s)|\,dx &= \int_{\{|\omega_n-\omega_m|\le 2\}} |\omega_n(s)-\omega_m(s)|\,dx + \int_{\{|\omega_n-\omega_m|>2\}} |\omega_n(s)-\omega_m(s)|\,dx \\ &\le \Big(\int_{\{|\omega_n-\omega_m|\le 2\}} |\omega_n(s)-\omega_m(s)|^2\,dx\Big)^{\frac{1}{2}} \cdot (\text{meas}(\Omega))^{\frac{1}{2}} \\ &\quad + \int_{\{|\omega_n-\omega_m|>2\}} |\omega_n(s)-\omega_m(s)|\,dx.\end{aligned} \tag{4.78}$$

In view of (4.75) – (4.77), we deduce that

$$\int_\Omega |\omega_n(s)-\omega_m(s)|\,dx \longrightarrow 0 \quad \text{as} \quad m,n\to\infty. \tag{4.79}$$

Hence (ω_n) is a Cauchy sequence in $C([0,T];L^1(\Omega))$, thus $\omega\in C([0,T];L^1(\Omega))$ and for $0\le s\le T$ we have $\omega_n(s)\to\omega(s)$ strongly in $L^1(\Omega)$.

Step 8 : Passage to the limit.

Let $\psi\in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))\cap L^\infty(Q_T)$ with $\dfrac{\partial\psi}{\partial t}\in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))+L^1(Q_T)$, and $M=k+\|\psi\|_{L^\infty(Q_T)}$ with $k>0$.

Plugging $T_k(\omega_n-\psi)$ into equation (4.2), we obtain

$$\begin{aligned}&\int_0^T \langle\frac{\partial\omega_n}{\partial t}, T_k(\omega_n-\psi)\rangle\,dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(\omega_n),\nabla\omega_n)D^iT_k(\omega_n-\psi)\,dx\,dt \\ &+\nu\int_{Q_T} |T_n(\omega_n)|^{s-1}T_n(\omega_n)T_k(\omega_n-\psi)\,dx\,dt + \frac{1}{n}\int_{Q_T} |\omega_n|^{p_0-2}\omega_n\,T_k(\omega_n-\psi)\,dx\,dt \\ &= \lambda\int_{Q_T} \frac{|T_n(\omega_n)|^{p_0-2}T_n(\omega_n)}{|x|^{p_0}+\frac{1}{n}}T_k(\omega_n-\psi)\,dx\,dt + \sum_{i=1}^N \int_{Q_T} f_i(x,t,T_n(\omega_n))D^iT_k(\omega_n-\psi)\,dx\,dt.\end{aligned} \tag{4.80}$$

If $|\omega_n| > M$ then $|\omega_n - \psi| \geq |\omega_n| - \|\psi\|_\infty > k$, therefore $\{|\omega_n - \psi| \leq k\} \subseteq \{|\omega_n| \leq M\}$, which implies that

$$\begin{aligned}
&\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(\omega_n),\nabla\omega_n) D^i T_k(\omega_n-\psi)\,dx\,dt\\
&= \sum_{i=1}^N \int_{\{|\omega_n-\psi|\leq k\}} a_i(x,t,T_M(\omega_n),\nabla T_M(\omega_n))(D^i T_M(\omega_n) - D^i\psi)\,dx\,dt\\
&= \sum_{i=1}^N \int_{\{|\omega_n-\psi|\leq k\}} (a_i(x,t,T_M(\omega_n),\nabla T_M(\omega_n)) - a_i(x,t,T_M(\omega_n),\nabla\psi))(D^i T_M(\omega_n) - D^i\psi)\,dx\,dt\\
&\quad + \sum_{i=1}^N \int_{\{|\omega_n-\psi|\leq k\}} a_i(x,t,T_M(\omega_n),\nabla\psi)(D^i T_M(\omega_n) - D^i\psi)\,dx\,dt.
\end{aligned}$$

Since $D^i T_M(\omega_n) \to D^i T_M(\omega)$ strongly in $L^{p_i}(Q_T)$, and in view of Fatou's Lemma we obtain

$$\begin{aligned}
&\liminf_{n\to+\infty} \sum_{i=1}^N \int_{Q_T} a_i(x,t,\omega_n,\nabla\omega_n) D^i T_k(\omega_n-\psi)\,dx\,dt\\
&\geq \sum_{i=1}^N \int_{\{|\omega-\psi|\leq k\}} (a_i(x,t,T_M(\omega),\nabla T_M(\omega)) - a_i(x,t,T_M(\omega),\nabla\psi))(D^i T_M(\omega) - D^i\psi)\,dx\,dt\\
&\qquad + \sum_{i=1}^N \int_{\{|\omega-\psi|\leq k\}} a_i(x,t,T_M(\omega),\nabla\psi)(D^i T_M(\omega) - D^i\psi)\,dx\,dt\\
&= \sum_{i=1}^N \int_{Q_T} a_i(x,t,\omega,\nabla\omega) D^i T_k(\omega-\psi)\,dx\,dt.
\end{aligned} \tag{4.81}$$

With respect to the initial term on the left-hand side of equation (4.80), we observe that

$$\begin{aligned}
\int_0^T \langle \frac{\partial\omega_n}{\partial t}, T_k(\omega_n-\psi)\rangle\,dt &= \int_0^T \langle \frac{\partial(\omega_n-\psi)}{\partial t}, T_k(\omega_n-\psi)\rangle\,dt + \int_0^T \langle \frac{\partial\psi}{\partial t}, T_k(\omega_n-\psi)\rangle\,dt\\
&= \int_\Omega \varphi_k(\omega_n(T)-\psi(T))\,dx - \int_\Omega \varphi_k(\omega_{0,n}-\psi(0))\,dx\\
&\quad + \int_{Q_T} \frac{\partial\psi}{\partial t} T_k(\omega_n-\psi)\,dx\,dt,
\end{aligned}$$

and since $\omega_n \to \omega$ in $C([0,T];L^1(\Omega))$ then $\omega_n(T) \to \omega(T)$ strongly in $L^1(\Omega)$, it follows that

$$\int_\Omega \varphi_k(\omega_{0,n}-\psi(0))\,dx \longrightarrow \int_\Omega \varphi_k(\omega_0-\psi(0))\,dx, \tag{4.82}$$

and

$$\int_\Omega \varphi_k(\omega_n(T)-\psi(T))\,dx \longrightarrow \int_\Omega \varphi_k(\omega(T)-\psi(T))\,dx. \tag{4.83}$$

We have $\dfrac{\partial\psi}{\partial t} \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T)$, with $T_k(\omega_n-\psi) \rightharpoonup T_k(\omega-\psi)$ strongly in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ and weak$-*$ in $L^\infty(Q_T)$, then

$$\int_{Q_T} \frac{\partial\psi}{\partial t} T_k(\omega_n-\psi)\,dx\,dt \longrightarrow \int_{Q_T} \frac{\partial\psi}{\partial t} T_k(\omega-\psi)\,dx\,dt, \tag{4.84}$$

Having in mind $(4.28)-(4.30)$, we deduce that

$$\int_{Q_T} |T_n(\omega_n)|^{s-1}T_n(\omega_n)T_k(\omega_n-\psi)\,dx\,dt \longrightarrow \int_{Q_T} |\omega|^{s-1}\omega\; T_k(\omega-\psi)\,dx\,dt, \tag{4.85}$$

$$\frac{1}{n}\int_{Q_T} |\omega_n|^{p_0-2}\omega_n\, T_k(\omega_n-\psi)\,dx\,dt \longrightarrow 0, \tag{4.86}$$

and

$$\int_{Q_T} \frac{|T_n(\omega_n)|^{p_0-2}T_n(\omega_n)}{|x|^{p_0}+\frac{1}{n}} T_k(\omega_n-\psi)\,dx\,dt \longrightarrow \int_{Q_T} \frac{|\omega|^{p_0-2}\omega}{|x|^{p_0}} T_k(\omega-\psi)\,dx\,dt. \tag{4.87}$$

If we look at the last term of the right-hand side of equation (4.80), we have $f_i(x,t,T_M(\omega_n)) \to f_i(x,t,T_M(\omega))$ strongly in $L^{p_i}(Q_T)$, and since $D^iT_M(\omega_n) \rightharpoonup D^iT_M(\omega)$ weakly in $L^{p_i}(Q_T)$, it follows that

$$\begin{aligned}
&\sum_{i=1}^N \int_{Q_T} f_i(x,t,T_n(\omega_n))D^iT_k(\omega_n-\psi)\,dx\,dt \\
&= \sum_{i=1}^N \int_{\{|\omega_n-\psi|\le k\}} f_i(x,t,T_M(\omega_n))(D^iT_M(\omega_n)-D^i\psi)\,dx\,dt. \\
&\longrightarrow \sum_{i=1}^N \int_{\{|\omega-\psi|\le k\}} f_i(x,t,T_M(\omega))(D^iT_M(\omega)-D^i\psi)\,dx\,dt \qquad \text{as} \quad n\to\infty \\
&= \sum_{i=1}^N \int_{Q_T} f_i(x,t,\omega)D^iT_k(\omega-\psi)\,dx\,dt.
\end{aligned} \tag{4.88}$$

By combining $(4.80)-(4.88)$, we deduce that

$$\begin{aligned}
&\int_\Omega \varphi_k(\omega(T)-\psi(T))\,dx - \int_\Omega \varphi_k(\omega_0-\psi(0))\,dx + \int_{Q_T} \frac{\partial\psi}{\partial t}T_k(\omega-\psi)\,dx\,dt \\
&+\sum_{i=1}^N \int_{Q_T} a_i(x,t,\omega,\nabla\omega)\cdot D^iT_k(\omega-\psi)\,dx\,dt + \nu\int_{Q_T} |\omega|^{s-1}\omega\; T_k(\omega-\psi)\,dx\,dt \\
&\le \lambda\int_{Q_T} \frac{|\omega|^{p_0-2}\omega}{|x|^{p_0}}\; T_k(\omega-\psi)\,dx\,dt + \sum_{i=1}^N \int_{Q_T} f_i(x,t,\omega)D^iT_k(\omega-\psi)\,dx\,dt,
\end{aligned}$$

which conclude the proof of theorem 4.3.

Conflict of interest.
All authors declare that they have no conflicts of interest.

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