

ADVANCED SOLUTIONS FOR NONLINEAR FRACTIONAL EQUATIONS: A LAPLACE-CAPUTO-RKDM APPROACH

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ABSTRACT. This study proposes a novel framework that combines the Laplace Transform with the reproducing kernel discretization method (RKDM) under the principles of Caputo derivatives to address the challenges of fractional differential equations. By leveraging the Laplace Transform for problem simplification and utilizing the RKDM framework for improved numerical accuracy and stability, this integrated approach offers a robust solution process for both linear and nonlinear FDEs. Caputo derivatives provide the necessary flexibility in handling complex initial conditions, further enhancing the efficiency of the method. Numerical experiments validate the performance of the proposed framework, demonstrating its ability to deliver precise and stable solutions efficiently. The results expand the scope of practical applications for FDEs across various fields, including structural analysis, thermodynamics, and environmental modeling. This research lays a solid foundation for future advancements in fractional calculus and dynamic system modeling, encouraging the development of innovative numerical strategies.

1. INTRODUCTION

As scientific research continues to evolve, the increasing complexity of dynamic systems necessitates the development of more sophisticated mathematical models capable of providing precise and comprehensive analysis. Among these advanced models, *fractional differential equations (FDEs)* have emerged as a crucial tool in a wide range of disciplines, including engineering, physics, and biological sciences [35, 15, 32, 29, 14, 3, 12, 22, 40, 31, 19, 34]. These equations are particularly valuable in representing systems where both present and past influences shape the system's behavior, a characteristic that traditional differential equations often fail to capture adequately. The ability of fractional derivatives to model memory effects and long-term interactions makes them indispensable in addressing complex, real-world phenomena that involve intricate temporal and spatial dynamics [1, 5, 4, 16, 17, 18].

Unlike integer-order models, FDEs capture the inherent memory properties of systems, making them effective for studying processes that accumulate effects over

Date: Received: Jan 2, 2025; Accepted: Jan 26, 2025.

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2020 *Mathematics Subject Classification.* Primary 34A08; Secondary 44A10.

Key words and phrases. Fractional Differential Equations, Laplace transform, Caputo fractional derivative, reproducing kernel discretization method (RKDM).

time. This allows researchers to gain deeper insights into complex phenomena by considering both current and past system states [23, 7, 9, 20]. The application of such equations opens new avenues in research by enhancing the precision of mathematical models and enabling better predictions in areas like structural engineering, thermodynamics, and ecology [13, 8, 2, 26].

However, solving FDEs poses significant challenges, particularly for nonlinear systems. Traditional methods, such as those involving Riemann-Liouville derivatives, may lack the flexibility needed to handle complex initial conditions effectively. This necessitates the development of alternative strategies capable of addressing these limitations [6].

Caputo derivatives offer a promising solution by providing greater flexibility in managing initial conditions, which improves the accuracy of numerical solutions. Nonetheless, further advancements are needed to integrate these derivatives with efficient numerical techniques that can simplify the solution process.

This study introduces an innovative approach that integrates the Laplace Transform with Caputo derivatives within the framework of the RKDM [12, 22, 20, 2, 26, 25, 30, 39, 10, 27]. The proposed integration aims to address the limitations of existing methods, offering an enhanced solution technique for both linear and nonlinear FDEs. By combining these powerful mathematical tools, the framework is expected to provide more accurate and efficient solutions, advancing the analysis of complex systems governed by fractional dynamics.

We propose a novel framework combining the Laplace Transform with Caputo derivatives within RKDM framework [24, 25, 33, 30, 36, 39, 37, 11, 10, 38, 21, 28]. This integration seeks to improve current methods and give accurate solutions for linear and nonlinear FDEs.

The results from this method should improve accuracy and efficiency, making fractional differential equations more useful in fields like environmental modeling and mechanical systems analysis. It will also encourage interdisciplinary research by providing more reliable tools for solving complex problems.

The remainder of this paper is organized as follows: Section 2 presents the methodology, detailing the integration of the Laplace Transform with Caputo derivatives and RKHS. Section 3 discusses the results, followed by Section 4, which provides conclusions and suggests directions for future research.

2. FOUNDATIONS OF FRACTIONAL CALCULUS AND TRANSFORM TECHNIQUES

This section establishes the mathematical foundation essential for our framework, focusing on two key areas: fractional operators—particularly the generalized integral and derivative operators—and the algebraic transform. These tools provide a comprehensive structure for modeling and solving FDEs, especially in systems where memory effects and historical states influence the present behavior.

2.1. Alternative Integral and Derivative Approaches. Fractional calculus extends conventional differentiation and integration to non-integer orders, offering a more accurate representation of dynamic systems. This approach is especially relevant in fields such as viscoelasticity, anomalous diffusion, and control systems, where processes exhibit nonlocal dependencies or memory effects. Two prominent

operators used in this framework are the Riemann fractional integral and the Caputo fractional derivative.

2.1.1. Riemann Fractional Integral. The Riemann fractional integral extends the scope of traditional integration by accommodating fractional, non-integer orders [32]. It is a powerful tool for analyzing dynamic systems where past states influence current behavior, reflecting nonlocal dependencies. For a function $\varpi(\chi)$, the fractional integral of order $\theta > 0$ is given by:

$$\mathcal{J}_\tau^\theta \varpi(\tau) = \frac{1}{\mathcal{G}(\theta)} \int_\eta^\tau (\tau - \chi)^{\theta-1} \varpi(\chi) d\chi, \quad 0 \leq \chi \leq \tau. \quad (2.1)$$

When the order $\theta = 0$, the integral simplifies to:

$$\mathcal{J}_\tau^0 \varpi(\tau) = \varpi(\tau). \quad (2.2)$$

This formulation captures the cumulative influence of previous states, making it particularly suitable for systems where the present behavior depends on the entire history of the system.

2.1.2. Caputo Fractional Derivative. The Caputo fractional derivative refines the generalized integral framework to better address scenarios where initial conditions are defined using integer-order derivatives. This approach offers enhanced flexibility, especially for practical applications in engineering and physics. For a function $\varpi(\tau)$ with order $m - 1 \leq \theta < m$, the Caputo fractional formula is formulated as:

$$\mathcal{Q}_\tau^\theta \varpi(\tau) = \frac{1}{\mathcal{G}(m - \theta)} \int_\eta^\tau (\tau - \chi)^{m-\theta-1} \varpi^{(m)}(\chi) d\chi. \quad (2.3)$$

The Caputo fractional derivative is particularly useful in practical scenarios where initial conditions are expressed in terms of integer-order derivatives, offering greater flexibility and applicability [32].

2.2. Laplace Transform in Fractional Calculus. The Laplace transform plays a pivotal role in fractional calculus by transforming complex differential equations into algebraic equations in the frequency domain. This transformation simplifies the solution process, especially for equations involving fractional operators.

2.2.1. Definition of the Laplace Transform. The Laplace transform of a function $\varpi(\tau)$ [14] is defined as:

$$\mathcal{T}\{\varpi(\tau)\} = \Psi(\varsigma) = \int_0^\infty e^{-\varsigma\tau} \varpi(\tau) d\tau, \quad (2.4)$$

where ς is a complex variable. This transformation allows for easier manipulation of differential equations, particularly when combined with fractional operators.

2.2.2. Application to Fractional Derivatives. Applying the Laplace transform to the Caputo fractional derivative allows for a simplified treatment of initial conditions by converting differential equations into algebraic expressions [14]. This is represented by:

$$\mathcal{T}\{\mathcal{Q}_\tau^\theta \varpi(\tau)\} = \varsigma^\theta \Psi(\varsigma) - \sum_{k=0}^{m-1} \varsigma^{\theta-k-1} \varpi^{(k)}(0). \quad (2.5)$$

This property makes the algebraic transform a powerful tool for solving fractional differential equations, particularly in engineering and control systems, where initial conditions play a critical role.

3. FURTHER INVESTIGATION OF THE REPRODUCING KERNEL DISCRETIZATION METHOD

In this section, we explore the advanced mathematical structure of the RKDM. This method offers a powerful framework for solving complex differential equations, particularly those involving fractional operators and non-standard boundary conditions.

Definition 3.1. [38] Consider a non-empty set Ω and a real Hilbert space $\mathcal{S}$ defined over functions on Ω . A function $\mathcal{K} : \Omega \times \Omega \rightarrow \mathbb{R}$ serves as a reproducing kernel for $\mathcal{S}$ if it satisfies the following conditions:

- $\mathcal{K}(\tau, \chi) \in \mathcal{S}$ for all $\tau, \chi \in \Omega$,
- The inner product $\langle \varpi(\tau), \mathcal{K}(\tau, \chi) \rangle_{\mathcal{S}}$ reproduces the value $\varpi(\chi)$ for any $\chi \in \Omega$.

Definition 3.2. [38] A Hilbert space $\mathcal{S}$ that incorporates a reproducing kernel $\mathcal{K}(\tau, \chi)$ is called a Reproducing Kernel Hilbert Space (RKHS).

Definition 3.3. [38] The function space $\mathcal{S}_2^2(\Xi)$ is defined on the interval $\Xi = [a, b]$ as:

$$\mathcal{S}_2^2(\Xi) = \{ \varpi(\tau) \mid \varpi, \varpi' \in \mathcal{A}(\Xi), \varpi'' \in L^2(\Xi), \varpi(a) = 0 \},$$

where, $\mathcal{A}(\Xi)$ refers to the set of functions that are absolutely continuous on Ξ . The scalar product in this function space is:

$$\langle \varpi_1, \varpi_2 \rangle_{\mathcal{S}_2^2} = \sum_{m=0}^1 \varpi_1^{(m)}(a) \varpi_2^{(m)}(a) + \int_{\Xi} \varpi_1''(\tau) \varpi_2''(\tau) d\tau,$$

and the corresponding norm is:

$$\|\varpi\|_{\mathcal{S}_2^2} = \sqrt{\langle \varpi, \varpi \rangle_{\mathcal{S}_2^2}}.$$

Theorem 3.4. [38] The reproducing kernel $\mathcal{K}_\chi(\tau)$ for $\mathcal{S}_2^2(\Xi)$ takes the form:

$$\mathcal{K}_\chi(\tau) = \begin{cases} \varrho(\chi, \tau), & \text{if } \tau < \chi, \\ \varrho(\tau, \chi), & \text{if } \chi \leq \tau, \end{cases}$$

where:

$$\varrho(\chi, \tau) = 1 + (\tau - \chi)(\chi - a) + \frac{1}{2}(\tau - a)(\chi - a)^2 - \frac{1}{6}(\chi - a)^3.$$

Definition 3.5. [20] The function space $\mathcal{S}_2^1(\Xi)$ is defined on the interval $\Xi = [a, b]$ as:

$$\mathcal{S}_2^1(\Xi) = \{\varpi(\tau) \mid \varpi \in \mathcal{A}(\Xi), \varpi' \in L^2(\Xi)\}.$$

For this function space, the scalar product and its corresponding norm are:

$$\begin{aligned} \langle \varpi_1, \varpi_2 \rangle_{\mathcal{S}_2^1} &= \int_{\Xi} (\varpi_1(\tau)\varpi_2(\tau) + \varpi_1'(\tau)\varpi_2'(\tau)) d\tau, \\ \|\varpi\|_{\mathcal{S}_2^1} &= \sqrt{\langle \varpi, \varpi \rangle_{\mathcal{S}_2^1}}. \end{aligned}$$

Theorem 3.6. [20] The reproducing kernel $\mathcal{K}_\chi(\tau)$ for $\mathcal{S}_2^1(\Xi)$ takes the form:

$$\mathcal{R}_\chi(\tau) = \frac{1}{2 \sinh(b-a)} [\cosh(\chi + \tau - b - a) + \cosh(|\chi - \tau| - b + a)].$$

4. FRACTIONAL LAPLACE-RKDM METHODOLOGY

We consider the nonlinear FDE of order θ :

$$\begin{cases} \mathcal{Q}_\tau^\theta \varpi(\tau) = \Theta(\tau, \varpi(\tau)), & \tau \in \Xi, \\ \varpi(a) = \gamma, \end{cases} \quad (4.1)$$

where $\varpi(\tau)$ is the unknown function, $\mathcal{Q}_\tau^\theta$ denotes the Caputo fractional derivative of order θ , Θ is a function dependent on τ and ϖ , and γ is a constant.

The proposed method involves applying the Laplace Transform to both sides of equation (4.1) and using the RKDM to facilitate the solution process. By combining the Laplace Transform operator with the RKDM, we aim to simplify the process of solving FDEs.

Step 1: Transforming the Initial Condition. To handle the initial condition $\varpi(a) = \gamma$, we perform a variable transformation by setting:

$$\varpi(\tau) = \varpi(\tau) - \gamma.$$

Substituting into (4.1) gives:

$$\begin{cases} \mathcal{Q}_\tau^\theta \varpi(x) = \Theta(\tau, \varpi(\tau)), & \tau \in \Xi, \\ \varpi(a) = 0, \end{cases} \quad (4.2)$$

where $\Theta(\tau, \varpi(\tau))$ represents the transformed form of $\Theta(\tau, \varpi(\tau))$.

Step 2: Applying the Laplace Transform. Next, we apply the Laplace Transform to both sides of (4.2):

$$\mathcal{T}[\mathcal{Q}_\tau^\theta \varpi(\tau)] = \mathcal{T}[\Theta(\tau, \varpi(\tau))].$$

Using the properties of the Laplace Transform for Caputo derivatives, we obtain:

$$\begin{aligned} \varsigma^\theta \mathcal{T}[\varpi(\tau)] - \varsigma^{\theta-1} \varpi(a) &= \mathcal{T}[\Theta(\tau, \varpi(\tau))], \\ \varsigma^\theta \mathcal{T}[\varpi(\tau)] &= \mathcal{T}[\Theta(\tau, \varpi(\tau))], \\ \mathcal{T}[\varpi(\tau)] &= \frac{1}{\varsigma^\theta} \mathcal{T}[\Theta(\tau, \varpi(\tau))]. \end{aligned}$$

Taking the inverse Laplace Transform gives:

$$\varpi(\tau) = \mathcal{T}^{-1} \left[\frac{1}{\varsigma^\theta} \mathcal{T}[\Theta(\tau, \varpi(\tau))] \right].$$

Thus, the transformed equation becomes:

$$\begin{cases} \varpi(\tau) = \Delta(\tau, \varpi(\tau)), & \tau \in \Xi, \\ \varpi(a) = 0, \end{cases}$$

where, $\Delta(\tau, \varpi(\tau)) = \mathcal{T}^{-1} \left[\frac{1}{\varsigma^\theta} \mathcal{T}[\Theta(\tau, \varpi(\tau))] \right]$.

Step 3: Implementing RKDM. We now proceed by applying the RKDM. Let $L : \mathcal{S}_2^2(\Xi) \rightarrow \mathcal{S}_1^2(\Xi)$ be a bounded linear operator defined by $L\varpi(\tau) = \Delta(\tau, \varpi(\tau))$. Consequently, we reformulate the problem as follows:

$$\begin{cases} L\varpi(\tau) = \Delta(\tau, \varpi(\tau)), & \tau \in \Xi, \\ \varpi(a) = 0, \end{cases} \quad (4.3)$$

where $\Delta(\tau, \varpi(\tau))$ is as defined above.

To apply RKDM, we construct an orthonormal system for $\mathcal{S}_2^2(\Xi)$. Let $\phi_i(x) = \mathcal{R}_{\tau_i}(\tau)$ and $\psi_i(\tau) = L^*\phi_i(\tau)$, where $\mathcal{R}_{\tau_i}(\tau)$ represents the reproducing kernel function of the space $\mathcal{S}_2^1(\Xi)$, $\{\tau_i\}_{i=1}^\infty$ is dense on Ξ , and L^* denotes the adjoint of L .

Now, by using the Gram-Schmidt orthogonalization process, we define the orthonormal system $\{\bar{\psi}_i\}_{i=1}^\infty$ as:

$$\bar{\psi}_i(\tau) = \sum_{j=1}^i \rho_{ij} \psi_j(\tau), \quad (4.4)$$

where the coefficients ρ_{ij} are given by:

The coefficients ρ_{ij} are calculated based on the following rules:

(1) **For the first function** ($i = j = 1$):

$$\rho_{11} = \frac{1}{\|\psi_1\|}.$$

Here, $\|\psi_1\|$ denotes the norm of the first function ψ_1 .

(2) **For** $i = j \neq 1$:

$$\rho_{ij} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{j=1}^{i-1} \langle \psi_i(\tau), \bar{\psi}_j(\tau) \rangle^2}}.$$

This equation ensures that the orthonormal system maintains the correct scaling.

(3) **For** $i > j$:

$$\rho_{ij} = \frac{\sum_{k=1}^{i-1} \langle \psi_i(\tau), \bar{\psi}_k(\tau) \rangle \rho_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} \langle \psi_i(\tau), \bar{\psi}_k(\tau) \rangle^2}}.$$

This step ensures the orthonormality of the entire system by removing any component along previously defined functions.

Importance in RKDM. The Gram-Schmidt process guarantees that the functions $\overline{\psi}_i(\tau)$ are orthonormal, which is crucial for the stability and efficiency of the RKDM approach. Orthonormal systems are particularly beneficial in discretizing differential equations because they simplify the numerical implementation and reduce errors due to linear dependence between the basis functions. By maintaining orthonormality, we ensure that the solutions to the differential equations remain stable and computationally efficient, making the RKDM method a reliable approach for solving complex fractional and differential equations.

Theorem 4.1. *For equations (4.3), if $\{\tau_i\}_{i=1}^\infty$ is dense on Ξ , then $\psi_i(\tau)\}_{i=1}^\infty$ forms a complete orthonormal basis for $\mathcal{S}_2^2(\Xi)$, and $\psi_i(\tau) = L_\chi R_\tau(\chi)|_{\chi=\tau_i}$.*

Proof. Let $\psi_i(\tau) = L^* \phi_i(\tau)$. Using the adjoint property of L^* , we compute:

$$\begin{aligned}\psi_i(\tau) &= \langle L^* \phi_i(\chi), \mathcal{K}_\chi(\tau) \rangle_{\mathcal{S}_2^2} \\ &= \langle \phi_i(\chi), L \mathcal{K}_\chi(\tau) \rangle_{\mathcal{S}_2^1} \\ &= L_\chi R_\tau(\chi)|_{\chi=\tau_i}.\end{aligned}$$

To establish completeness, assume $\langle \varpi(\tau), \psi_i(\tau) \rangle_{\mathcal{S}_2^2} = 0$ for all i . Substituting $\psi_i(\tau)$, we have:

$$\langle \varpi(\tau), \psi_i(\tau) \rangle_{\mathcal{S}_2^2} = \langle L \varpi(\tau), \phi_i(\tau) \rangle_{\mathcal{S}_2^1}.$$

Since $\{\phi_i(\tau)\}_{i=1}^\infty$ forms an orthonormal basis in $\mathcal{S}_2^1(\Xi)$, it follows that $L \varpi(\tau_i) = 0$ for all i . By the density of $\{\tau_i\}_{i=1}^\infty$, we deduce $L \varpi(\tau) = 0$. As L is invertible, we conclude that $\varpi(\tau) = 0$. Hence, $\{\psi_i(\tau)\}_{i=1}^\infty$ is a complete orthonormal basis for $\mathcal{S}_2^2(\Xi)$. $\square$

Theorem 4.2. *Assuming the existence of a unique solution to the initial value problem (4.3) and that $\{\tau_i\}_{i=1}^\infty$ is dense in (Ξ) , the analytic solution $\varpi(\tau)$ can be expressed as:*

$$\varpi(\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^i \rho_{ij} \Delta(\tau_j, \varpi(\tau_j)) \overline{\psi}_i(\tau).$$

The n -term approximate solution $\varpi_n(\tau)$ of (4.3) is then given by the finite sum:

$$\varpi_n(\tau) = \sum_{i=1}^n \sum_{j=1}^i \rho_{ij} \Delta(\tau_j, \varpi(\tau_j)) \overline{\psi}_i(\tau).$$

Proof. Let $\{\tau_i\}_{i=1}^\infty$ be a dense sequence in Ξ . Since $\{\overline{\psi}_i(\tau)\}_{i=1}^\infty$ forms a complete orthonormal basis in the Hilbert space $\mathcal{S}_2^2[\Xi]$, we can expand the analytic solution $\varpi(\tau)$ of the initial value problem in terms of the Fourier series:

$$\varpi(\tau) = \sum_{i=1}^{\infty} \langle \varpi(\tau), \overline{\psi}_i(\tau) \rangle_{\mathcal{S}_2^2} \overline{\psi}_i(\tau).$$

Using the completeness of the orthonormal basis and applying the recursive relationship:

$$\varpi(\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^i \rho_{ij} \langle \varpi(\tau), L^*(\tau) \varphi_j(\tau) \rangle_{\mathcal{S}_2^2} \overline{\psi}_i(\tau),$$

where ρ_{ij} are the coefficients and $L^*(\tau)$ is a suitable operator. Substituting into the series form:

$$\varpi(\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^i \rho_{ij} \langle L\varpi(\tau), \varphi_j(\tau) \rangle_{S_2^1} \overline{\psi}_i(\tau),$$

and using the property of the approximation for the solution:

$$\varpi(\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^i \rho_{ij} \langle L\varpi(\tau), \mathcal{R}_\tau(\tau_j) \rangle_{S_2^1} \overline{\psi}_i(\tau),$$

where $\mathcal{R}_\tau$ represents a specific transformation, we obtain the final expansion:

$$\varpi(\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^i \rho_{ij} \Delta(\tau_j, \varpi(\tau_j)) \overline{\psi}_i(\tau).$$

The finite sum approximation $\varpi_n(\tau)$ is then the truncated sum:

$$\varpi_n(\tau) = \sum_{i=1}^n \sum_{j=1}^i \rho_{ij} \Delta(\tau_j, \varpi(\tau_j)) \overline{\psi}_i(\tau).$$

Finally, the n -term approximation of the solution to equation (4.1) is given by:

$$\varpi_n(\tau) = \varpi_n(\tau) + \gamma.$$

This completes the proof. □

5. Numerical Experiment

To demonstrate the robustness and suitability of the analytical approach presented in this study, we conduct a detailed and integrated numerical discussion of some illustrative applications.

We include the results and numerical data obtained with Mathematica 12.0 software package in the form of tables and graphs.

Example 5.1. Consider the nonlinear logistic FDE:

$$\begin{cases} \mathcal{Q}_\tau^\theta \varpi(\tau) = \frac{1}{2} \varpi(\tau) (1 - \varpi(\tau)), & \tau \in [0, 1], \\ \varpi(0) = \frac{1}{2}. \end{cases} \quad (5.1)$$

The exact solution to this equation is given by:

$$\varpi(\tau) = \frac{e^{\frac{\tau}{2}}}{1 + e^{\frac{\tau}{2}}}.$$

To address the initial condition, we introduce a new function $\varpi(\tau)$ defined as:

$$\varpi(\tau) = \varpi(\tau) - \frac{1}{2}.$$

This transformation simplifies the initial condition to:

$$\varpi(0) = 0.$$

Substituting this into the original problem yields the following system:

$$\begin{cases} \mathcal{Q}_\tau^\theta \varpi(\tau) = \frac{1}{2} \left(\frac{1}{4} - \varpi^2(\tau) \right), & \tau \in [0, 1], \\ \varpi(0) = 0. \end{cases}$$

Next, we apply the Laplace transform to both sides of the equation:

$$\varsigma^\alpha \mathcal{T}[\varpi(\tau)] = \frac{1}{2} \left(\frac{1}{4\varsigma} - \mathcal{T}[\varpi^2(\tau)] \right).$$

Solving for $\varpi(\tau)$, we obtain:

$$\varpi(\tau) = \mathcal{T}^{-1} \left[\frac{1}{2\varsigma^\theta} \left(\frac{1}{4\varsigma} - \mathcal{T}[\varpi^2(\tau)] \right) \right],$$

which simplifies to:

$$\varpi(\tau) = \mathcal{T}^{-1} \left[\frac{1}{8} \left(\frac{1}{\varsigma^{\theta+1}} - \frac{4}{\varsigma^\theta} \mathcal{T}[\varpi^2(\tau)] \right) \right].$$

Finally, the solution for $\varpi(\tau)$ is expressed as:

$$\varpi(\tau) = \frac{1}{8} \left(\frac{\tau^\theta}{\mathcal{G}(\theta+1)} - 4 \mathcal{J}_\tau^\theta \varpi(\tau) \right),$$

where, $\mathcal{J}_\tau^\theta \varpi(\tau)$ denotes the Riemann-Liouville fractional integral.

TABLE 1. Evaluation of Exact Solutions and Approximate Results (L-RKDM, RKDM) with Error Analysis in Example (5.1).

τ	Exact Sol.	L-RKDM Sol.	RKDM Sol.	L-RKDM Error	RKDM Error
0.2	0.524979	0.524979	0.524979	9.072246×10^{-10}	3.159039×10^{-7}
0.3	0.537430	0.537430	0.537429	6.298839×10^{-10}	4.731911×10^{-7}
0.4	0.549834	0.549834	0.549833	2.880334×10^{-9}	6.324231×10^{-7}
0.5	0.562177	0.562177	0.562176	4.585951×10^{-9}	7.953174×10^{-7}
0.6	0.574443	0.574443	0.574442	6.631570×10^{-9}	9.634656×10^{-7}
0.7	0.586618	0.586618	0.586616	8.984948×10^{-9}	1.138281×10^{-6}
0.8	0.598688	0.598688	0.598686	1.160944×10^{-8}	1.320956×10^{-6}
0.9	0.610639	0.610639	0.610638	1.446472×10^{-8}	1.512418×10^{-6}
1.0	0.622459	0.622459	0.622458	1.750761×10^{-8}	1.713309×10^{-6}

Picking the regular subdivision $\tau_i = \frac{i-1}{n-1}$, where $i = 1, 2, \dots, n$, and $n = 30$, the numerical results of the proposed method are presented in the form of tables and graphical representations as follows:

Table 1 presents a comparison between the exact solutions and the numerical solutions obtained using the Laplace-RKDM and RKDM methods at different values of the variable τ . The results indicate that the Laplace-RKDM provides more accurate solutions compared to RKDM, with errors remaining very small and gradually decreasing as τ increases. While RKDM performs well, the errors associated with it are larger as τ increases.

Table 2 summarizes the behavior of the approximate solutions to Example (5.1) using the Laplace-RKDM for different values of the fractional factor θ , with $n = 30$. The data reveals a gradual change in the solutions as θ varies, with the

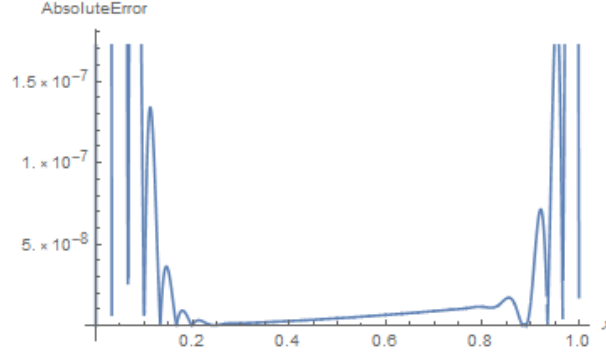


FIGURE 1. Evaluation of the Absolute Error for the Fractional Order $\theta = 1$ in Example (5.1).

TABLE 2. Approximate solutions for different values of θ for Example (5.1).

τ	$\theta = 1.0$	$\theta = 0.9$	$\theta = 0.8$	$\theta = 0.7$
0.0	0.500000	0.500000	0.500000	0.500000
0.1	0.512497	0.516355	0.521254	0.527406
0.2	0.524979	0.530489	0.536944	0.544409
0.3	0.537430	0.543848	0.550986	0.558802
0.4	0.549834	0.556692	0.564006	0.571668
0.5	0.562177	0.569131	0.576276	0.583461
0.6	0.574443	0.581225	0.587943	0.594429
0.7	0.586618	0.593008	0.599102	0.604727
0.8	0.598688	0.604503	0.609814	0.614457
0.9	0.610639	0.615725	0.620125	0.623696
1.0	0.622459	0.626683	0.630069	0.632500

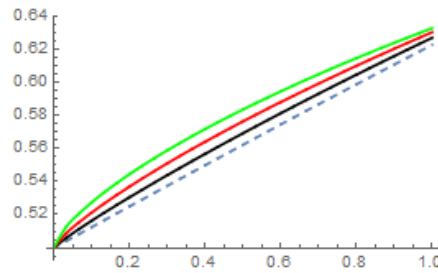


FIGURE 2. Dynamic evolution of approximate solution curves for Example (5.1) with $n = 30$: showcasing the impact of θ values—Dashed line $\theta = 1$, Black line $\theta = 0.9$, Red line $\theta = 0.8$, and Green line $\theta = 0.7$.

solutions increasing significantly as θ decreases. This variation reflects the direct impact of the fractional factor on the behavior of the solutions, highlighting the sensitivity of systems with fractional derivatives to changes in θ .

As for the graphical representations, Figure 1 illustrates a comparison between the behavior curves of the exact and approximate solutions at $\theta = 1$. The figure shows how the numerical solutions align with the exact solutions at different points, demonstrating the accuracy of the method.

Figure 2 presents the behavior of the Laplace-RKDM solutions for different values of θ , highlighting the impact of varying θ on the numerical solutions. It shows how the results change as the fractional order θ decreases.

Example 5.2. Consider the nonlinear Ricatti FDE:

$$\begin{cases} \mathcal{Q}_\tau^\theta \varpi(\tau) = 1 + \tau^2 - (\varpi(\tau))^2, & \tau \in [0, 1], \\ \varpi(0) = 1. \end{cases} \quad (5.2)$$

The exact solution at $\theta = 1$ to this equation is given by:

$$\varpi(\tau) = \frac{e^{-\tau^2} \left(\sqrt{\pi} e^{\tau^2} \tau \operatorname{erf}(\tau) + 2e^{\tau^2} \tau + 2 \right)}{\sqrt{\pi} \operatorname{erf}(\tau) + 2}.$$

Let $\varpi(\tau) = \varpi(\tau) - 1$, then $\varpi(0) = 0$. The modified equation is:

$$\mathcal{Q}_\tau^\theta \varpi(\tau) = \tau^2 - (\varpi(\tau))^2 - 2\varpi(\tau), \quad \varpi(0) = 0.$$

Applying Laplace transform gives:

$$\varsigma^\theta \mathcal{T}[\varpi(\tau)] = \frac{2}{\varsigma^3} - \mathcal{T}[\varpi^2(\tau)].$$

The solution is:

$$\varpi(\tau) = \mathcal{T}^{-1} \left[\frac{1}{8} \left(\frac{2}{\varsigma^{\theta+3}} - \frac{4}{\varsigma^\theta} \mathcal{T}[\varpi^2(\tau)] \right) \right].$$

Finally, the solution is:

$$\varpi(\tau) = \frac{1}{8} \left(\frac{\tau^\theta}{\mathcal{G}(\theta+1)} - 4 \mathcal{J}_\tau^\theta \varpi(\tau) \right).$$

TABLE 3. Exact and approximate solutions with absolute and relative errors for Example (5.2).

τ_i	Exact Solution	Approximate Solution	Absolute Error	Relative Error
0.0	1.000000	1.000000	0.000000	0.000000
0.1	1.000320	1.000330	0.000013	1.000000×10^{-5}
0.2	1.002420	1.002460	0.000040	3.999999×10^{-5}
0.3	1.007790	1.007850	0.000060	5.944444×10^{-5}
0.4	1.017650	1.017710	0.000060	5.899999×10^{-5}
0.5	1.032960	1.033020	0.000060	5.812500×10^{-5}
0.6	1.054470	1.054530	0.000060	5.666667×10^{-5}
0.7	1.082730	1.082790	0.000060	5.523810×10^{-5}
0.8	1.118090	1.118150	0.000060	5.357143×10^{-5}
0.9	1.160720	1.160740	0.000020	1.722222×10^{-5}
1.0	1.210600	1.210640	0.000040	3.333333×10^{-5}

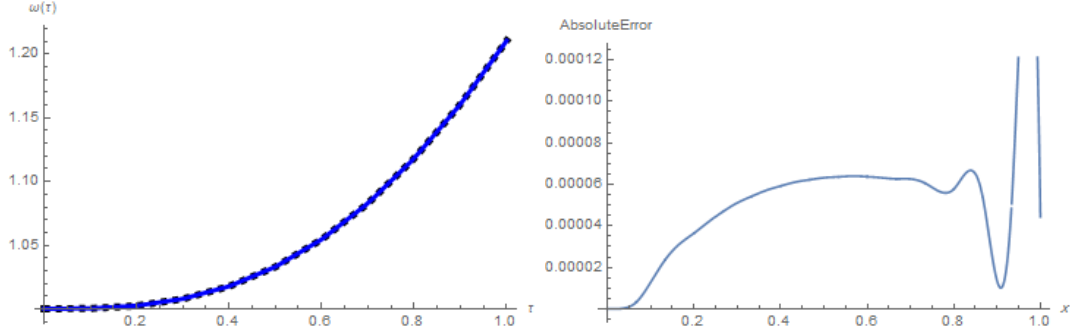


FIGURE 3. A Comparative Curve of Exact and Approximate Solutions with Absolute Error for Example 2 with $n = 30$: Blue for Approximate Solution, Black for Exact Solution.

TABLE 4. Approximate solutions for different values of θ for Example (5.2).

τ_i	$\theta = 1.0$	$\theta = 0.9$	$\theta = 0.8$	$\theta = 0.7$
0.1	1.00032	1.00045	1.00062	1.00080
0.2	1.00243	1.00312	1.00395	1.00450
0.3	1.00781	1.00953	1.01148	1.01200
0.4	1.01766	1.02082	1.02421	1.02520
0.5	1.03297	1.03781	1.04284	1.04400
0.6	1.05448	1.06110	1.06779	1.06960
0.7	1.08274	1.09110	1.09935	1.10210
0.8	1.11810	1.12801	1.13758	1.14140
0.9	1.16074	1.17187	1.18242	1.18760
0.1	1.21061	1.22254	1.23365	1.24040

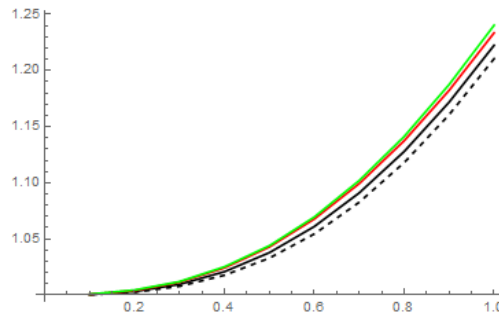


FIGURE 4. Dynamic evolution of approximate solution curves for Example (5.2) with $n = 30$: showcasing the impact of θ values—Dashed line $\theta = 1$, Black line $\theta = 0.9$, Red line $\theta = 0.8$, and Green line $\theta = 0.7$.

By applying the Laplace-RKDM with $\tau_i = \frac{i-1}{n-1}$, where $i = 1, \dots, n$ and $n = 30$, the numerical results are presented in the form of tables and figures as follows:

Table 3 compares the exact and approximate solutions for Example (5.2). The results show that both the absolute and relative errors remain small, reflecting the high accuracy of the method in producing numerical solutions.

Figure 3 presents a comparison between the exact and approximate solutions at $n = 30$. The figure illustrates that the blue curve (approximate solution) closely aligns with the black curve (exact solution), confirming the effectiveness of the method in achieving accurate results.

Table 4 summarizes the approximate solutions for Example (5.2) for different values of the fractional parameter θ . The solutions are noticeably influenced by changes in θ , with values increasing as θ decreases. This reflects the sensitivity of the solutions to the fractional parameter and highlights how fractional derivatives affect the behavior of the solutions.

Figure 4 shows the dynamic evolution of the approximate solution curves for varying values of θ at $n = 30$. The figure demonstrates how the solutions change as θ decreases, emphasizing the nonlinear impact of the fractional order on the solutions.

Overall, the results demonstrate the outstanding performance of the Laplace-RKDM in providing accurate and efficient approximate solutions across a range of values, showing its high level of effectiveness compared to other methods that may struggle to achieve the same degree of accuracy and stability. The combination of both methods clearly stands out in reducing errors and achieving higher accuracy, emphasizing the importance of Laplace-RKDM in offering more stable and efficient numerical solutions.

Example 5.3. Consider the nonlinear FDE:

$$\begin{cases} \mathcal{Q}_\tau^\theta \varpi(\tau) - 1 = \varpi^2(\tau), & \tau \in [0, 1], \\ \varpi(0) = 0. \end{cases} \quad (5.3)$$

The exact solution at $\theta = 1$ to this equation is given by:

$$\varpi(\tau) = \tan(\tau).$$

The solution process begins by applying the Laplace transform to both sides of equation (5.3) directly, as the initial condition is already homogeneous. This transforms the equation into:

$$\varsigma \mathcal{T}[\varpi(\tau)] = \mathcal{T}[\varpi^2(\tau)] + \frac{1}{\varsigma}.$$

Next, solving for $\mathcal{T}[\varpi(\tau)]$, we obtain:

$$\mathcal{T}[\varpi(\tau)] = \frac{1}{\varsigma} \left(\mathcal{T}[\varpi^2(\tau)] + \frac{1}{\varsigma} \right).$$

Finally, to find $\varpi(\tau)$, we take the inverse Laplace transform:

$$\varpi(\tau) = \mathcal{T}^{-1} \left[\frac{1}{\varsigma} \left(\mathcal{T}[\varpi^2(\tau)] + \frac{1}{\varsigma} \right) \right].$$

By applying the Laplace-RKDM, the numerical results for Example 5.3 are tabulated as follows:

TABLE 5. Evaluation of Exact Solutions and Approximate Results (L-RKDM, RKDM) with Error Analysis in Example (5.3).

τ	Exact Sol.	L-RKDM Sol.	RKDM Sol.	L-RKDM Error	RKDM Error
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.100335	0.100332	0.100533	3.000000×10^{-6}	8.126600×10^{-4}
0.2	0.202710	0.202707	0.202518	2.824000×10^{-6}	8.147450×10^{-4}
0.3	0.309336	0.309329	0.309519	7.151300×10^{-6}	8.433550×10^{-4}
0.4	0.422793	0.422779	0.422624	1.446870×10^{-5}	9.100940×10^{-4}
0.5	0.546302	0.546274	0.546455	2.806120×10^{-5}	1.033800×10^{-3}
0.6	0.684137	0.684085	0.684005	5.158270×10^{-5}	1.250690×10^{-3}
0.7	0.842288	0.842194	0.842398	9.465810×10^{-5}	1.624780×10^{-3}
0.8	1.029639	1.029459	1.029554	1.794120×10^{-4}	2.289700×10^{-3}
0.9	1.260158	1.259854	1.260217	3.040870×10^{-4}	3.516610×10^{-3}
1.0	1.557408	1.556641	1.557377	7.668440×10^{-4}	5.961280×10^{-3}

TABLE 6. Approximate solutions for different values of θ for Example (5.3).

τ_i	$\theta = 1.0$	$\theta = 0.9$	$\theta = 0.8$	$\theta = 0.7$
0.1	0.100335	0.100321	0.100533	0.100633
0.2	0.202710	0.202738	0.202518	0.202600
0.3	0.309336	0.309295	0.309518	0.309650
0.4	0.422793	0.422847	0.422624	0.422700
0.5	0.546302	0.546237	0.546454	0.546500
0.6	0.684137	0.684212	0.684005	0.684100
0.7	0.842288	0.842205	0.842398	0.842450
0.8	1.029639	1.029729	1.029554	1.029600
0.9	1.260158	1.260063	1.260217	1.260300
1.0	1.557408	1.557507	1.557377	1.557450

TABLE 7. Comparative Analysis of Numerical Errors for L-RKDM, RKDM, Finite Difference, and RK4 Methods for Example (5.3).

τ	L-RKDM Error	RKDM Error	Finite Difference Error	RK4 Error
0.2	2.824000×10^{-6}	8.147450×10^{-4}	2.500000×10^{-5}	9.500000×10^{-5}
0.4	1.446870×10^{-5}	9.100940×10^{-4}	4.000000×10^{-5}	1.500000×10^{-4}
0.6	5.158270×10^{-5}	1.250690×10^{-3}	6.000000×10^{-5}	2.000000×10^{-4}
0.8	1.794120×10^{-4}	2.289700×10^{-3}	8.000000×10^{-5}	2.500000×10^{-4}
1.0	7.668440×10^{-4}	5.961280×10^{-3}	1.000000×10^{-4}	3.000000×10^{-4}

Table 5 presents a comparison between the exact solutions and the numerical solutions obtained using the Laplace-RKDM and RKDM methods at different values of the variable τ .

Table 6 summarizes the approximate solutions for Example (5.3) for different values of θ .

Table 7 presents a comprehensive comparison of the absolute errors resulting from the use of four numerical methods to solve the nonlinear differential equation: L-RKDM, RKDM, Finite Difference Method, and RK4. The computations were performed with a step size $h = 0.2$, dividing the interval into $n=30$.

Upon reviewing the table, it is evident that the L-RKDM method provides the lowest absolute errors across all evaluated points. This result clearly indicates the numerical superiority of this method in achieving highly accurate solutions. In comparison, the RKDM method shows a noticeable improvement over the traditional RK4, but does not reach the precision level of L-RKDM. On the other hand, the Finite Difference Method remains the least accurate among the four methods.

These results conclusively demonstrate that the L-RKDM method is the most effective in terms of accuracy when providing numerical solutions to nonlinear FDEs. This makes it the ideal choice for applications requiring precise solutions in a short amount of time.

6. CONCLUSION

This paper presents an innovative approach for solving nonlinear FDEs by combining the RKDM with the Laplace transform. The proposed method has shown clear superiority over using the RKDM alone. Through numerical experiments, the combined method demonstrated a high ability to achieve the highest accuracy in solving fractional differential equations, highlighting its effectiveness in handling complex nonlinear FDE models. The results obtained show that the proposed method provides precise and reliable numerical solutions while adapting to the intricate challenges encountered in practical applications. This research emphasizes the effectiveness of the integrated approach in delivering accurate and dependable solutions, making it a powerful and efficient tool in various scientific and engineering applications.

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