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ON THE EXISTENCE OF PERIODIC SOLUTIONS UNDER THE LIGHT OF SEMI-FREDHOLM OPERATORS FOR SOME EVOLUTION EQUATIONS

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Abstract. In this work, we investigate the existence of periodic solutions for a class of evolution equations defined as $\dot{w}(t) = (\mathcal{L} + \mathcal{C}(t))w(t) + \mathcal{H}(t)$. Sufficient conditions on the family of operators $\{\mathcal{C}(t), t \geq 0\}$, $\mathcal{L}$ and $\mathcal{H}$ are proposed to derive the periodicity of solutions from bounded ones on the positive real half-line. The results are obtained by applying the theory of perturbation of semi-Fredholm operators. We suppose that $\mathcal{L}$ is a generally nondensely defined operator and satifies the Hille-Yosida condition. Finally, an example of the theoretical findings is presented with numerical simulations.

1. Introduction

We propose, through this paper, to discuss the existence of periodic solutions for the following abstract evolution equation:

$$\begin{cases} \dfrac{d}{dt}w(t) &= (\mathcal{L} + \mathcal{C}(t))w(t) + \mathcal{H}(t) \quad for \quad t \geq 0 \\ w(0) &= w_0, \end{cases} \tag{1.1}$$

such that $\mathcal{L}$ is a linear operator maps from its domain $D(\mathcal{L}) \subset \mathbb{E}$ to $\mathbb{E}$ which is nondensely defined on the Banach space $\mathbb{E}$. Consider the family $\{\mathcal{C}(t), t \geq 0\}$ of bounded linear operators defined from $\overline{D(\mathcal{L})}$ to $\mathbb{E}$, satisfying some boundedness and measurability conditions. Let $\mathcal{H}$ be a continuous and τ-periodic function defined from $\mathbb{R}^+$ to $\mathbb{E}$. Throughout this work, we assume that the following condition holds:

$(\mathbf{C_0})$: There are $K \geq 1$ and $\widehat{\mu} \in (-\infty, +\infty)$ with $\rho(\mathcal{L}) \supset (\widehat{\mu}; \infty)$, $\rho(\mathcal{L})$ means the resolvent set of the operator $\mathcal{L}$ and

$$|(\eta I - \mathcal{L})^{-m}| \quad \leq \frac{K}{(\eta - \widehat{\mu})^m} \qquad for \quad \eta > \widehat{\mu} \ \text{ and } \ m \in \mathbb{N}. \tag{1.2}$$

Recall that when the condition $(\mathbf{C_0})$ is fulfilled, we say that $\mathcal{L}$ is a Hille-Yosida operator. Throughout the rest of this work, $HY(\mathbb{E})$ signifies the set of all operators defined from $D(\mathcal{L})$ to $\mathbb{E}$ that verify condition $(\mathbf{C_0})$. Moreover, we assume

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the following hypothesis on the family of operators $\{\mathcal{C}(t), t \geq 0\}$:

$(\mathbf{C_1})$: For $y \in \overline{D(\mathcal{L})}$, $\mathcal{C}(.)y$ is a strongly measurable function. Moreover, there exists a function $c \in L^1_{loc}(\mathbb{R}^+)$, where the space $L^1_{loc}(\mathbb{R}^+)$ denotes the set of all locally integrable functions defined on $\mathbb{R}^+$ provided with the following norm

$$\|c\|_{Loc} = \sup_{t \in \mathbb{R}^+} \int_t^{t+1} c(\sigma)\, d\sigma \, \text{such that} \, \|\mathcal{C}(.)\| \leq c(.).$$

Examining the behavior of solutions of partial differential equations continues to be one of the most significant research areas. Several exploration and advancement have been made in this area. Many authors have pursued by different directions to reach this goal, see, for instance [2, 3, 11, 12, 14, 18] . Recall that among the most popular approaches to prove the existence of periodic solutions, the compactness characteristic of the famous Poincaré map, see, for instance [1, 5, 6, 9, 13]. Massera's theorem [15] is the most important outcome establishing the relationship between the periodicity and the boundedness of differential equations in finite dimension. Recently, in [5, 6], the authors discussed the periodicity problem for some finite retarded partial differential equations by using Chow and Hale fixed point theorem. In [9], the author proceeded by the Banach-Alaoglo theorem to ensure the existence of periodic solutions for a class of evolution equations. Moreover, using semigroup theory and the renowned Leray-Schauder fixed-point theorem, the authors in [13] establish the existence of periodic positive solutions.

In this work, the problem of the periodicity of solutions for this type of evolution equation is addressed using semi-Fredholm operator theory. Unlike other approaches, this method does not require the compactness of the C_0-semigroup generated by the part of $\mathcal{L}$ on $\overline{D(\mathcal{L})}$, which is denoted by $(S_0(t))_{t\geq 0}$. Here, we define $L(\mathbb{E})$ as the set of all bounded linear operators from $\mathbb{E}$ to $\mathbb{E}$ and $\mathcal{S}_+(\mathbb{E})$ denotes the set of all semi-Fredholm operators from $\mathbb{E}$ to $\mathbb{E}$. Recall that an operator $G \in \mathcal{S}_+(\mathbb{E})$ if $G \in L(\mathbb{E})$, $Im(G)$ is closed and $\dim \ker(G)$ is finite. Finally, we introduce a fundamental theorem that deals with the perturbation of semi-Fredholm operators by bounded linear operators:

Theorem 1.1. [17] *Let $G \in \mathcal{S}_+(\mathbb{E})$, which means that $dim \ker(G)$ is finite and there is a constant $\delta > 0$ such that for every $v \in \mathbb{E}$, $|\overline{v}| \leq \delta \|Gv\|$ with $\overline{v}$ belongs to the Banach quotient space $\mathbb{E}/\ker(G)$ provided with the following norm*

$$|\overline{v}| = dist(v, \ker(G)).$$

If $F \in L(\mathbb{E})$ and verifies the following estimation:

$$\|F\| < \frac{1}{2\delta(1 + \sqrt{dim \ker(G)})}.$$

Then, $G + F$ belongs to $\mathcal{S}_+(\mathbb{E})$ and

$$\dim \ker(G + F) \leq \dim \ker(G).$$

This means that $\mathcal{S}_+(\mathbb{E})$ is an open set in $L(\mathbb{E})$.

Inspired by the aforementioned theorem, we aim to investigate the τ-periodicity of solutions for the evolution equation (1.1). To achieve this, we provide suitable conditions on the family $\mathcal{C}(t), t \geq 0$, $\mathcal{L}$, and the function $\mathcal{H}$ to establish the connection between τ-periodicity and the existence of bounded solutions for the evolution equation (1.1) on $[0, +\infty)$. To accomplish our objective, we employ the fixed-point theorem of Chow and Hale applied to the Poincaré map associated with the evolution equation (1.1), which allows us to conclude the existence of τ-periodic solutions. This approach is grounded in the theory of perturbation of semi-Fredholm operators, summarized in Theorem 1.1.

In general, to apply Theorem 1.1, we provide sufficient conditions, such as an a priori estimate on the norm $|c|_{\mathrm{Loc}}$ and the semi-Fredholm type condition on $I - S_0(\tau)$, to establish that $I - \mathcal{W}(\tau)$ is semi-Fredholm. Here, $(\mathcal{W}(t))_{t\geq 0}$ denotes the semigroup solution of the evolution equation (1.1) with $\mathcal{H} = 0$. Moreover, to verify that $I - S_0(\tau) \in \mathcal{S}_+(\mathbb{E})$ when $(S_0(t))_{t\geq 0}$ does not satisfy the exponential stability condition, our investigation focuses on the case where the linear operator $\mathcal{L}$ is decomposed as $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, with $\mathcal{L}_1 \in HY(\mathbb{E})$ and $\mathcal{L}_2 \in L(\mathbb{E})$. We provide appropriate assumptions on $\mathcal{L}_1$ and $\mathcal{L}_2$ to derive the τ-periodicity of solutions for the evolution equation (1.1) from bounded solutions defined on $[0, +\infty)$. Recall that the case where $\mathcal{L}_2 = 0$ is equivalent to treating equation (1.1) with $\mathcal{L}$ generates an exponentially stable C_0-semigroup. Furthermore, the case where $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ is illustrated by an application supported by numerical simulations.

The sections of this work are organized as follows: Section 2 focuses on the definition of the integral solution of the evolution equation (1.1) and some useful results and estimations. In Section 3, we investigate the τ-periodicity of solutions for the evolution equation (1.1). Section 4 is devoted to studying τ-periodicity in the special case mentioned earlier. Finally, we validate the applicability of the derived theoretical results through a mathematical model supported by numerical simulations.

2. Preliminary Results and Estimations

Consider the following results on the integral solution of equation (1.1).

Definition 2.1. A continuous function w defined on $[0, +\infty)$ with values in $\mathbb{E}$, is considered an integral solution to Eq. (1.1) if it fulfills the following conditions:

(i) $\displaystyle\int_0^t w(\sigma)\, d\sigma \in D(\mathcal{L}) \quad for \quad t \geq 0,$

(ii) $\displaystyle w(t) = w_0 + \mathcal{L}\int_0^t v(\sigma)\, d\sigma + \int_0^t \left(\mathcal{C}(\sigma)(v(\sigma)) + \mathcal{H}(\sigma)\right) d\sigma \quad for \quad t \geq 0.$

As a consequence, if the integral solution w is continuous, then for any $t \geq 0$, $w(t) \in \overline{D(\mathcal{L})}$. Now, let $\mathcal{L}_0$ be the part of the operator $\mathcal{L}$ defined on $\overline{D(\mathcal{L})}$ by:

(i) $D(\mathcal{L}_0) = \{v \in D(\mathcal{L}) : \mathcal{L}v \in \overline{D(\mathcal{L})}\}$,
(ii) $\mathcal{L}_0 v = \mathcal{L}v$ for $v \in D(\mathcal{L}_0)$.

It is well known that $\mathcal{L}_0$ is the generator of a C_0-semigroup, denoted by $(S_0(t))_{t\geq 0}$ on $\overline{D(\mathcal{L})}$.

Let $\mu_0 \in \mathbb{R}$ and $K_0 \geq 1$ such that $|S_0(t)| \leq K_0 e^{\mu_0 t}$, $t \geq 0$.

Theorem 2.2. *If* $(\mathbf{C_0})$ *holds. Then, for* $w_0 \in \overline{D(\mathcal{L})}$, *Eq. (1.1) has a unique integral solution* $w : [0, +\infty) \to \mathbb{E}$ *satisfies*

$$\begin{cases} w(t) &= S_0(t)w_0 + \lim\limits_{\eta \to +\infty} \displaystyle\int_0^t S_0(t-\sigma)\eta(\eta I - \mathcal{L})^{-1} \left(\mathcal{C}(\sigma)w(\sigma) + \mathcal{H}(\sigma)\right) d\sigma, \qquad t \geq 0 \\ w(0) &= w_0. \end{cases} \tag{2.1}$$

Let $\mathcal{W}(t) \in L(\overline{D(\mathcal{L})})$ be such that

$$\mathcal{W}(t)w_0 = w(t, w_0, 0),$$

where $w(t, w_0, 0)$ denotes the integral solution of the equation given as follows:

$$\begin{cases} \dfrac{d}{dt}w(t) &= (\mathcal{L} + \mathcal{C}(t))w(t) \qquad for \quad t \geq 0 \\ w(0) &= w_0. \end{cases} \tag{2.2}$$

Then, one can decompose the operator $\mathcal{W}(t)$ as follows:

$$\mathcal{W}(t)w_0 = S_0(t)w_0 + \mathcal{R}(t)w_0,$$

where $\mathcal{R}(t) \in L(\overline{D(\mathcal{L})})$ and

$$\mathcal{R}(t)w_0 = \begin{cases} \lim\limits_{\eta \to +\infty} \displaystyle\int_0^t S_0(t-\sigma)\eta(\eta I - \mathcal{L})^{-1}\mathcal{C}(\sigma)w(\sigma, w_0)\, d\sigma & for \quad t \geq 0, \\ 0 & for \quad t = 0. \end{cases} \tag{2.3}$$

Proposition 2.3. *If the condition* $(\mathbf{C_1})$ *holds. Then,* $\mathcal{R}(t)$ *satisfies*

$$|\mathcal{R}(t)| \leq K_0^2 e^{\mu_0 t} K[t+1]\|c\|_{Loc} e^{K_0 K [t+1]\|c\|_{Loc}}.$$

Where [.] denotes the floor function.

Proof. Let $t \geq 0$. Then

$$\begin{aligned} \|\mathcal{R}(t)w_0\| &\leq \lim_{\eta \to +\infty} \frac{\eta K K_0}{\eta - \widehat{\mu}} \int_0^t e^{\mu_0(t-\sigma)} \|\mathcal{C}(\sigma)\| \|w(\sigma, w_0)\|\, d\sigma \\ &\leq K_0 K \int_0^t e^{\mu_0(t-\sigma)} \|\mathcal{C}(\sigma)\| \|w(\sigma, w_0)\|\, d\sigma. \end{aligned}$$

Since

$$\|w(\sigma, w_0)\| \leq K_0 e^{\mu_0 \sigma}\|w_0\| + \|\mathcal{R}(\sigma)w_0\|,$$

one has that,

$$\|\mathcal{R}(t)w_0\| \leq K_0^2 K e^{\mu_0 t} \int_0^t \|\mathcal{C}(\sigma)\|\, d\sigma \ \|w_0\| + K_0 K e^{\mu_0 t} \int_0^t e^{-\mu_0 \sigma} \|\mathcal{C}(\sigma)\| \|\mathcal{R}(\sigma)w_0\| d\sigma.$$

Moreover, from

$$\int_t^{t+1} |c(\sigma)|\, d\sigma \leq \|c\|_{Loc} \ \text{ for } t \in \mathbb{R}^+,$$

we get that

$$\int_0^t |c(\sigma)|\, d\sigma \leq [t+1]\|c\|_{Loc}.$$

Which gives that

$$\|\mathcal{R}(t)w_0\| \leq e^{\mu_0 t}K_0^2K[t+1]\|c\|_{Loc}\|w_0\| + K_0Ke^{\mu_0 t}\int_0^t \|\mathcal{C}(\sigma)\|e^{-\mu_0\sigma}\|\mathcal{R}(\sigma)w_0\|\,d\sigma$$

and

$$e^{-\mu_0 t}\|\mathcal{R}(t)w_0\| \leq K_0^2K[t+1]\|c\|_{Loc}\|w_0\| + K_0K\int_0^t \|\mathcal{C}(\sigma)\|e^{-\mu_0\sigma}\|\mathcal{R}(\sigma)w_0\|\,d\sigma.$$

Gronwall's Lemma, implies that

$$\begin{aligned} e^{-\mu_0 t}\|\mathcal{R}(t)w_0\| &\leq K_0^2K[t+1]\|c\|_{Loc}\|w_0\|e^{K_0K\int_0^t \|\mathcal{C}(\sigma)\|\,d\sigma} \\ &\leq K_0^2K[t+1]\|c\|_{Loc}\|w_0\|e^{K_0K\int_0^t |c(\sigma)|\,d\sigma} \\ &\leq K_0^2K[t+1]\|c\|_{Loc}e^{K_0K[t+1]\|c\|_{Loc}}\|w_0\|. \end{aligned}$$

Consequentely,

$$\|\mathcal{R}(t)w_0\| \leq K_0^2e^{\mu_0 t}K[t+1]\|c\|_{Loc}e^{K_0K[t+1]\|c\|_{Loc}}\|w_0\|.$$

$\square$

Proposition 2.4. *Suppose that* $\dim\ker(I - S_0(\xi)) < \infty$. *Let* n *be such that* $n := \dim\ker(I - S_0(\xi))$ *and a constant* $\delta > 0$ *such that for* $v \in \overline{D(\mathcal{L})}$, $|\overline{v}| \leq \delta\|(I - S_0(\xi))v\|$. *Then, for* $\xi > 0$, *there is a strictly positive constant* c^* *verifies*

$$c^* > \frac{1}{2K_0K[\xi+1]}\ln\left(1 + \frac{e^{-\mu_0\xi}}{2K_0\delta(1+\sqrt{n})}\right),$$

such that if $\|c\|_{Loc} \leq c^*$, *then, the bounded linear operator* $\mathcal{R}(\xi)$ *satisfies the following estimate*

$$|\mathcal{R}(\xi)| < \frac{1}{2\delta(1+\sqrt{n})}. \tag{2.4}$$

Proof. From Proposition 2.3, the estimation (2.4) is satisfied if

$$K_0^2e^{\mu_0\xi}K[\xi+1]\|c\|_{Loc}e^{K_0K[\xi+1]\|c\|_{Loc}} < \frac{1}{2\delta(1+\sqrt{n})},$$

which means that

$$\ln(\|c\|_{Loc}) + K_0K[\xi+1]\|c\|_{Loc} < \ln\left(\frac{e^{-\mu_0\xi}}{2\delta K_0^2K(1+\sqrt{n})[\xi+1]}\right).$$

Let $\varsigma = \ln\left(\frac{e^{-\mu_0\xi}}{2\delta K_0^2K(1+\sqrt{n})[\xi+1]}\right)$. Consider the following continuous function g defined from $\mathbb{R}_+^*$ to $\mathbb{R}$ by:

$$g(y) = \ln(y) + K_0K[\xi+1]y - \varsigma.$$

Then, $g'(y) = \dfrac{1}{y} + K_0K[\xi+1] > 0$ for all $y > 0$. Hence, g is an increasing function on $\mathbb{R}_+^*$. Moreover $g(y) \to +\infty$ as $y \to +\infty$ and $g(y) \to -\infty$ as $y \to 0^+$.

Therefore, by the Intermediate Value theorem, there exists a constant $c^* > 0$ with $g(c^*) = 0$, $g(y) < 0$ for $y \in (0, c^*)$ and $g(y) > 0$ for $y \in (c^*, +\infty)$. Now, let

$$b^* = \frac{1}{2K_0 K[\xi+1]} \ln\left(1 + \frac{e^{-\mu_0 \xi}}{2K_0 \delta(1+\sqrt{n})}\right). \tag{2.5}$$

Then,

$$K_0 e^{\mu_0 \xi}(e^{2K_0 K[\xi+1]b^*} - 1) = \frac{1}{2\delta(1+\sqrt{n})}. \tag{2.6}$$

Since $y < e^y - 1$ for $y > 0$, it follows that

$$K_0^2 e^{\mu_0 \xi} K[\xi+1] b^* e^{K_0 K[\xi+1]b^*} < K_0 e^{\mu_0 \xi}(e^{K_0 K[\xi+1]b^*} - 1) e^{K_0 K[\xi+1]b^*},$$

consequently

$$K_0^2 e^{\mu_0 \xi} K[\xi+1] b^* e^{K_0 K[\xi+1]b^*} < K_0 e^{\mu_0 \xi}(e^{2K_0 K[\xi+1]b^*} - 1).$$

Then, from equality (2.6), we get that

$$K_0^2 e^{\mu_0 \xi} K[\xi+1] b^* e^{K_0 K[\xi+1]b^*} < \frac{1}{2\delta(1+\sqrt{n})},$$

and

$$\ln b^* + K_0 K[\xi+1] b^* - \varsigma < 0.$$

Consequently, $g(b^*) < 0$ and then $b^* < c^*$, which proves our Proposition. $\square$

Remark 2.5. Since b^* is given explicitly, we use the fact $\|c\|_{Loc} < b^*$ instead of $\|c\|_{Loc} < c^*$.

Proposition 2.6. *If $I - S_0(\tau) \in \mathcal{S}_+(\overline{D(\mathcal{L})})$ and let δ be a strictly positive constant such that for $v \in \overline{D(\mathcal{L})}$*

$$|\overline{v}| \leq \delta \|(I - S_0(\tau))v\| \quad \textit{for} \quad v \in \overline{D(\mathcal{L})},$$

and $\dim\ker(I - S_0(\tau)) = n$. *If*

$$\|c\|_{Loc} < c^*.$$

Then,

$$I - \mathcal{W}(\tau) \in \mathcal{S}_+(\overline{D(\mathcal{L})}).$$

Proof. A proof of this fact can be obtained by combining Theorem 1.1 and Proposition 2.4. $\square$

3. Periodicity of solution of Eq. (1.1)

Next, we investigate the periodicity of solution of Eq. (1.1) when a bounded solution exists on $[0, +\infty)$. To achieve this objective, we assume that:

($\mathbf{C_2}$) $\mathcal{H}$ and $\mathcal{C}(.)$ are τ-periodic functions.

First, let $P_{\mathcal{K}}$ denote the set of all fixed points of the operator $\mathcal{K}$. Moreover, we introduce the following Chow and Hale's fixed point Theorem.

Theorem 3.1. *Let $\mathcal{F}$ be a linear affine map on $\mathbb{E}$ with $\mathcal{F}x = \overline{\mathcal{F}}x + y$ for $x \in \mathbb{E}$. If $Im(I - \overline{\mathcal{F}})$ is closed and $\{\mathcal{F}^k x_0, k \in \mathbb{N}\}$ is a bounded set in $\mathbb{E}$ for some $x_0 \in \mathbb{E}$. Then, $P_{\mathcal{F}} \neq \varnothing$.*

Corollary 3.2. *Let $\mathcal{F}$ be a linear affine map on $\mathbb{E}$ with $\mathcal{F}x = \overline{\mathcal{F}}x + y$ for $x \in \mathbb{E}$. If $I - \overline{\mathcal{F}} \in \mathcal{S}_+(\mathbb{E})$ and $\{\mathcal{F}^k x_0, k \in \mathbb{N}\}$ is a bounded set in $\mathbb{E}$ for some $x_0 \in \mathbb{E}$. Then, $P_{\mathcal{F}} \neq \varnothing$.*

Theorem 3.3. *Suppose that $(\mathbf{C_0})$, $(\mathbf{C_1})$ and $(\mathbf{C_2})$ hold. If $I - S_0(\tau) \in \mathcal{S}_+(\overline{D(\mathcal{L})})$ and $\|c\|_{Loc}$ satisfies the following estimate*

$$\|c\|_{Loc} < c^*. \tag{3.1}$$

Then, the boundedness of solutions on the interval $[0, +\infty)$ implies the τ-periodicity of solutions for Eq. (1.1).

Proof. Let $w(\tau, w_0, \mathcal{H})$ be the solution of Eq. (1.1). So, it suffices to demonstrate that the corresponding Poincaré map defined as:

$$\begin{array}{cccc} \mathcal{P}_\tau & : & \overline{D(\mathcal{L})} & \to & \overline{D(\mathcal{L})} \\ & & w_0 & \to & w(\tau, w_0, \mathcal{H}), \end{array}$$

possesses a fixed point. Moreover, we can decompose $\mathcal{P}_\tau$ as follows

$$\mathcal{P}_\tau w_0 = w(\tau, w_0, 0) + w(\tau, 0, \mathcal{H}),$$

with $w(\tau, w_0, 0)$ and $w(\tau, 0, \mathcal{H})$ represent the solutions of Eq. (1.1) with $\mathcal{H} = 0$ and $w_0 = 0$ respectively, which implies that $\mathcal{P}_\tau$ takes the form

$$\mathcal{P}_\tau w_0 = \mathcal{P}' w_0 + y_0,$$

with

$$\mathcal{P}' w_0 = w(\tau, w_0, 0) = \mathcal{W}(\tau) w_0 \text{ and } y_0 = w(\tau, 0, \mathcal{H}).$$

Consequently,

$$\mathcal{P}' = S_0(\tau) + \mathcal{R}(\tau).$$

Proposition 2.6 establishes that $I - \mathcal{P}' = I - \mathcal{W}(\tau) \in \mathcal{S}_+(\overline{D(\mathcal{L})})$. Furthermore, consider $w(\tau, w_0, \mathcal{H})$ as a bounded solution of Eq. (1.1) over the interval $[0, +\infty)$, then, the sequence $(\mathcal{P}_\tau^n w_0)_{n \in \mathbb{N}}$ with $\mathcal{P}_\tau^n w_0 = w(n\tau, w_0, \mathcal{H})$ is bounded on $\overline{D(\mathcal{L})}$. Consequently, from Theorem 3.2 we deduce that $P_{\mathcal{P}_\tau} \neq \varnothing$. Which guaranteed the τ-periodicity of solutions for Eq. (1.1). $\square$

By employing Remark 2.5, we get the following corollary.

Corollary 3.4. *Suppose that $(\mathbf{C_0})$, $(\mathbf{C_1})$ and $(\mathbf{C_2})$ hold. If $I - S_0(\tau) \in \mathcal{S}_+(\overline{D(\mathcal{L})})$ and $\|c\|_{Loc}$ satisfies the following estimate*

$$\|c\|_{Loc} < \frac{1}{2K_0 K[\tau + 1]} \ln\left(1 + \frac{e^{-\mu_0 \tau}}{2K_0 \delta (1 + \sqrt{n})}\right).$$

Then, the boundedness of solutions on the interval $[0, +\infty)$ implies the τ-periodicity of solutions for Eq. (1.1).

4. The periodicity in the special case where $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ where $\mathcal{L}_1 \in HY(\mathbb{E})$ and $\mathcal{L}_2 \in L(\mathbb{E})$

Now, consider the abstract evolution equation:

$$\begin{cases} \dfrac{d}{dt}w(t) &= (\mathcal{L} + \mathcal{C}(t))w(t) + \mathcal{H}(t) \qquad for \quad t \geq 0 \\ \quad w(0) &= w_0, \end{cases}$$

where $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ such that the operator $\mathcal{L}_1 : D(\mathcal{L}) \to \mathbb{E}$ is linear and $\mathcal{L}_2 \in L(\mathbb{E})$. We assume that:

($\mathbf{C_0'}$) The operator $\mathcal{L}_1 \in HY(\mathbb{E})$:

(i) There exists $\overline{\mu} \in \mathbb{R}$ with $\rho(\mathcal{L}_1) \supset (\overline{\mu}, +\infty)$, such that $\rho(\mathcal{L}_1)$ is the resolvent set of $\mathcal{L}_1$.
(ii) There exists a constant $\overline{K} \geq 1$ verifies

$$|\,(\eta I - \mathcal{L}_1)^{-m}| \quad \leq \frac{\overline{K}}{(\eta - \overline{\mu})^m} \qquad \text{for} \quad \eta > \overline{\mu} \ \ and\ m \in \mathbb{N}.$$

Theorem 4.1. [7] *If the condition* ($\mathbf{C_0'}$) *is satisfies. Then,* ($\mathbf{C_0}$) *holds with* $K = \overline{K}$ *and* $\widehat{\mu} = \overline{\mu} + \overline{K}|A_2|$.

Let $\overline{\mathcal{L}}_1$ be the part of $\mathcal{L}_1$ on $\overline{D(\mathcal{L})}$ and $(T_0(t))_{t\geq 0}$ be the C_0-semigroup generated by $\overline{\mathcal{L}}_1$ on $\overline{D(\mathcal{L})}$. Let $w(0) = w_0$ and consider the abstract equation:

$$\frac{d}{dt}w(t) = \mathcal{L}_1 w(t) + \mathcal{L}_2 w(t) \qquad \text{for} \quad t \geq 0. \tag{4.1}$$

Theorem 4.2. *Assume that the condition* ($\mathbf{C_0'}$) *holds. Then, for* $w_0 \in \overline{D(\mathcal{L}_1)}$, *Eq.* (4.1) *has a unique solution* $w(., w_0) \in \overline{D(\mathcal{L})}$ *satisfies:*

$$w(t, w_0) = T_0(t)w_0 + \lim_{\eta\to+\infty} \int_0^t T_0(t-\sigma)\eta(\eta I - \mathcal{L}_1)^{-1}\mathcal{L}_2 w(\sigma, w_0)\, d\sigma \quad for \quad t \geq 0.$$

Define the operator $\widetilde{\mathcal{R}}(t)$ from $\overline{D(\mathcal{L})}$ to $\mathbb{E}$ by:

$$\widetilde{\mathcal{R}}(t)w_0 = \lim_{\eta\to+\infty} \int_0^t T_0(t-\sigma)\eta(\eta I - \mathcal{L}_1)^{-1}\mathcal{L}_2 w(\sigma, w_0)\, d\sigma.$$

Lemma 4.3. *Assume that* ($\mathbf{C_0'}$) *holds. Let* $\|T_0(t)\| \leq \widetilde{K}e^{-\widetilde{\mu}t}$ *for* $t \geq 0$, $\widetilde{K} \geq 1$ *and* $\widetilde{\mu} > 0$. *Then, the operator* $\widetilde{\mathcal{R}}(t)$ *satisfies*

$$|\widetilde{\mathcal{R}}(t)| \leq \widetilde{K}e^{-\widetilde{\mu}t}(e^{\overline{K}\widetilde{K}|\mathcal{L}_2|t} - 1).$$

Proof. Let $t \geq 0$. Then,

$$\|\widetilde{\mathcal{R}}(t)z_0\| \leq \overline{K}\widetilde{K}|\mathcal{L}_2| \int_0^t e^{-\widetilde{\mu}(t-\sigma)}\|w(\sigma, w_0)\|\, d\sigma.$$

Since

$$\|w(\sigma, w_0)\| \leq \widetilde{K}e^{-\widetilde{\mu}\sigma}\|w_0\| + \overline{K}\widetilde{K}|\mathcal{L}_2| \int_0^\sigma e^{-\widetilde{\mu}(\sigma-s)}\|w(s, w_0)\|\, ds.$$

and

$$e^{\widetilde{\mu}\sigma}\|w(\sigma,w_0)\| \leq \widetilde{K}\|w_0\| + \overline{K}\widetilde{K}|\mathcal{L}_2|\int_0^\sigma e^{\widetilde{\mu}s}\|w(s,w_0)\|\,ds.$$

Applying Gronwall's Lemma, we obtain that

$$e^{\widetilde{\mu}\sigma}\|w(\sigma,w_0)\| \leq \widetilde{K}e^{\overline{K}\widetilde{K}|\mathcal{L}_2|\sigma}\|w_0\|,$$

hence,

$$\|w(\sigma,w_0)\| \leq \widetilde{K}e^{(\overline{K}\widetilde{K}|\mathcal{L}_2|-\widetilde{\mu})\sigma}\|w_0\|.$$

Then,

$$\begin{aligned}\|\widetilde{\mathcal{R}}(t)w_0\| &\leq \overline{K}\widetilde{K}^2|\mathcal{L}_2|e^{-\widetilde{\mu}t}\int_0^t e^{\overline{K}\widetilde{K}|\mathcal{L}_2|\sigma}\,d\sigma\|w_0\| \\ &\leq \widetilde{K}e^{-\widetilde{\mu}t}(e^{\overline{K}\widetilde{K}|\mathcal{L}_2|t}-1)\|w_0\|,\end{aligned}$$

which prove the result. □

Now, suppose that $\|T_0(t)\| \leq \widetilde{K}e^{-\widetilde{\mu}t}$ for $t \geq 0$, $\widetilde{K} \geq 1$ and $\widetilde{\mu} > 0$. We introduce the following new norm $|.|_{\widetilde{\mu}}$ on the space $\mathbb{E}$ defined by:

$$|v|_{\widetilde{\mu}} = \sup_{t\geq 0}\|e^{\widetilde{\mu}t}T_0(t)v\|,$$

clearly, $\|v\| \leq |v|_{\widetilde{\mu}} \leq \widetilde{K}\|v\|$, then the equivalence between the norms $\|.\|$ and $|.|_{\widetilde{\mu}}$ is assured. Let $t \geq 0$ and $v \in \mathbb{E}$, then one has

$$\begin{aligned}|T_0(t)v|_{\widetilde{\mu}} &= \sup_{s\geq 0}\|e^{\widetilde{\mu}s}T_0(s+t)v\| = e^{-\widetilde{\mu}t}\sup_{s\geq 0}\|e^{\widetilde{\mu}(s+t)}T_0(s+t)v\| \\ &= e^{-\widetilde{\mu}t}\sup_{\sigma\geq t}\|e^{\widetilde{\mu}\sigma}T_0(\sigma)v\| \leq e^{-\widetilde{\mu}t}\sup_{\sigma\geq 0}\|e^{\widetilde{\mu}\sigma}T_0(\sigma)v\| \leq e^{-\widetilde{\mu}t}|v|_{\widetilde{\mu}}.\end{aligned}$$

which means that the exponential stability of the semigroup $(T_0(t))_{t\geq 0}$ implies the invertibility of the operator $I - T_0(\tau)$.

Proposition 4.4. *Suppose that* $(\mathbf{C_0'})$ *holds. Let* $\|T_0(t)\| \leq \widetilde{K}e^{-\widetilde{\mu}t}$ *for* $t \geq 0$, $\widetilde{K} \geq 1$ *and* $\widetilde{\mu} > 0$. *If the operator* $\mathcal{L}_2$ *fulfills*

$$|\mathcal{L}_2| < \frac{\ln\left(1 + \dfrac{1}{2\widetilde{K}\delta_1 e^{-\widetilde{\mu}\tau}}\right)}{\overline{K}\widetilde{K}\tau}, \tag{4.2}$$

where $\delta_1 > 0$ *and satisfies for every* $v \in \overline{D(\mathcal{L})}$:

$$|(I - T_0(\tau))^{-1}v|_{\widetilde{\mu}} \leq \delta_1|v|_{\widetilde{\mu}}.$$

Then,

$$I - S_0(\tau) \in \mathcal{S}_+(\overline{D(\mathcal{L})}).$$

Furthermore, $\dim\ker(I - S_0(\tau)) = 0$.

Proof. From the decomposition of the operator $\mathcal{L}$, we get that

$$I - S_0(\tau) = I - T_0(\tau) - \lim_{\eta\to+\infty} \int_0^\tau T_0(\tau-\sigma)\eta(\eta I - \mathcal{L}_1)^{-1}\mathcal{L}_2 S_0(\sigma)\, d\sigma. \tag{4.3}$$

The exponential stability of $(T_0(t))_{t\geq 0}$ guarantees that $I - T_0(\tau)$ is bijective, which implies that $I - T_0(\tau) \in \mathcal{S}_+(\overline{D(\mathcal{L})})$. On the other hand, since

$$|\mathcal{L}_2| < \frac{\ln\left(1 + \dfrac{1}{2\widetilde{K}\delta_1 e^{-\widetilde{\mu}\tau}}\right)}{\overline{K}\widetilde{K}\tau},$$

it follows that

$$\widetilde{K}e^{-\widetilde{\mu}\tau}(e^{\overline{K}\widetilde{K}|\mathcal{L}_2|\tau} - 1) < \frac{1}{2\delta_1},$$

using Lemma 4.3, we get that

$$|\widetilde{\mathcal{R}}(\tau)| < \frac{1}{2\delta_1}.$$

Moreover, the exponential stability of $(T_0(t))_{t\geq 0}$ confirms that $\dim\ker(I - T_0(\tau)) = 0$. Hence, applying Theorem 1.1 to formula (4.3), we deduce that $I - S_0(\tau) \in \mathcal{S}_+(\overline{D(\mathcal{L})})$ and $\dim\ker(I - S_0(\tau)) = 0$. □

Theorem 4.5. *Assume that* $(\mathbf{C_0'})$, $(\mathbf{C_1})$ *and* $(\mathbf{C_2})$ *hold. Let* $\|T_0(t)\| \leq \widetilde{K}e^{-\widetilde{\mu}t}$ *for* $t \geq 0$, $\widetilde{K} \geq 1$ *and* $\widetilde{\mu} > 0$. *If*

$$|\mathcal{L}_2| < \frac{\ln\left(1 + \dfrac{1 - e^{-\widetilde{\mu}\tau}}{2\widetilde{K}e^{-\widetilde{\mu}\tau}}\right)}{\overline{K}\widetilde{K}\tau}, \tag{4.4}$$

and if the function c *satisfies the following inequality*

$$\|c\|_{Loc} < c^*.$$

Then, the boundedness of solutions on the interval $[0, +\infty)$ *implies the* τ*-periodicity of solutions for Eq. (1.1).*

Proof. We begin the proof by computing the value of δ_1 given in Proposition 4.4. For $h \in \overline{D(\mathcal{L})}$, we have

$$\begin{aligned} |h|_{\widetilde{\mu}} &\leq |h - T_0(\tau)h|_{\widetilde{\mu}} + |T_0(\tau)h|_{\widetilde{\mu}} \\ &\leq |h - T_0(\tau)h|_{\widetilde{\mu}} + e^{-\widetilde{\mu}\tau}|h|_{\widetilde{\mu}}. \end{aligned}$$

Then,

$$(1 - e^{-\widetilde{\mu}\tau})|h|_{\widetilde{\mu}} \leq |h - T_0(\tau)h|_{\widetilde{\mu}} \quad \text{for} \quad h \in \overline{D(\mathcal{L})}$$

which means that

$$|h|_{\widetilde{\mu}} \leq \frac{1}{(1 - e^{-\widetilde{\mu}\tau})}|h - T_0(\tau)h|_{\widetilde{\mu}}.$$

and hence,

$$|(I - T_0(\tau))^{-1}h|_{\widetilde{\mu}} \leq \frac{1}{(1 - e^{-\widetilde{\mu}\tau})}|h|_{\widetilde{\mu}}.$$

Finally, the value of δ_1 can be selected such that

$$\delta_1 \leq \frac{1}{(1-e^{-\widetilde{\mu}\tau})}.$$

Then, for $\delta_1 \leq \dfrac{1}{(1-e^{-\widetilde{\mu}\tau})}$, the inequality (4.4) confirms that inequality (4.2) is satisfied and consequently, it follows from Proposition 4.4 that $I - S_0(\tau)$ is semi-Fredholm on $\overline{D(\mathcal{L})}$. Then, Theorem 3.3 implies that Eq. (1.1) has a τ-periodic solution. □

Combining Theorem 4.5 and Remark 2.5, we get that:

Lemma 4.6. *Assume that* $(\mathbf{C_0'})$, $(\mathbf{C_1})$ *and* $(\mathbf{C_2})$ *hold. Let* $\|T_0(t)\| \leq \widetilde{K}e^{-\widetilde{\mu}t}$ *for* $t \geq 0$, $\widetilde{K} \geq 1$ *and* $\widetilde{\mu} > 0$. *If the operator* $\mathcal{L}_2$ *satisfies*

$$|\mathcal{L}_2| < \frac{\ln\left(1 + \dfrac{1-e^{-\widetilde{\mu}\tau}}{2\widetilde{K}e^{-\widetilde{\mu}\tau}}\right)}{\overline{K}\widetilde{K}\tau}, \tag{4.5}$$

and if the function c satisfies the following inequality

$$\|c\|_{Loc} < \frac{1}{2K_0K[\tau+1]} \ln\left(1 + \frac{e^{-\mu_0\tau}}{2\delta K_0}\right). \tag{4.6}$$

Then, the boundedness of solutions on the interval $[0, +\infty)$ *implies the* τ*-periodicity of solutions for Eq. (1.1).*

5. Applications and numerical simulations

To apply the obtained theoretical results, we propose to study this model:

$$\begin{cases} \dfrac{\partial}{\partial t}w(t,\zeta) = \dfrac{\partial}{\partial \zeta}w(t,\zeta) - \alpha w(t,\zeta) + \vartheta(t)w(t,\zeta) + \displaystyle\int_0^{+\infty} \psi(\zeta,\xi)w(t,\xi)\,d\xi + G(t,\zeta)\ , \ t \geq 0, \quad \zeta \geq 0, \\ w(t,0) = \lim\limits_{\zeta\to+\infty} w(t,\zeta) = 0, \quad t \geq 0, \\ w(0,\zeta) = w_0(\zeta), \qquad\qquad \zeta \geq 0, \end{cases} \tag{5.1}$$

where $\alpha > 0$, the function $\vartheta(.)$ belongs to $L^1_{loc}(\mathbb{R}^+)$, τ-periodic and satisfies $|\vartheta(t)| \leq \nu$ for $t \geq 0$, $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function and the function $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\psi(\zeta,.) \in L^1(\mathbb{R}^+)$. $\mathbf{C}([0,+\infty])$ is the space of all continuous functions with $\lim\limits_{\zeta\to+\infty} w(\zeta)$ exists. Let $\mathbb{E} = \mathbf{C}([0,+\infty])$. Then, $\mathbb{E}$ is a Banach space provided with the norm

$$\|z\|_\infty = \sup_{0\leq\zeta<+\infty} |z(\zeta)|.$$

Define the operator $\widetilde{\mathcal{L}}$ from $D(\widetilde{\mathcal{L}}) \subset \mathbb{E}$ to $\mathbb{E}$ as follows:

$$\begin{cases} D(\widetilde{\mathcal{L}}) &= \{w \in \mathbf{C}^1([0,+\infty]) : w(0) = \lim\limits_{\zeta\to+\infty} w(\zeta) = 0\}, \\ \widetilde{\mathcal{L}}w &= w'. \end{cases}$$

We have the following lemmas on the operator $\widetilde{\mathcal{L}}$ and its part $\widetilde{\mathcal{L}}_0$ on $\overline{D(\widetilde{\mathcal{L}})}$.

Lemma 5.1. [4]

$$\rho(\widetilde{\mathcal{L}}) \supset (0,+\infty) \ \ \textit{and} \ |(\eta I-\widetilde{\mathcal{L}})^{-1}| \leq \frac{1}{\eta} \ \ \textit{for} \ \eta>0.$$

Furthermore,

$$\overline{D(\widetilde{\mathcal{L}})} = \{w \in \mathbf{C}\left([0,+\infty]\right) : w(0) = \lim_{\zeta\to+\infty} w(\zeta) = 0\} \neq \mathbb{E}.$$

In addition $\widetilde{\mathcal{L}}_0$ the part of $\widetilde{\mathcal{L}}$ in $\overline{D(\widetilde{\mathcal{L}})}$ is expressed by

$$\begin{cases} D(\widetilde{\mathcal{L}}_0) &= \{w \in \mathbf{C}^1\left([0,+\infty]\right) : w(0) = \lim\limits_{\zeta\to+\infty} w(\zeta) = 0 \ \text{ and } w'(0) = \lim\limits_{\zeta\to+\infty} w'(\zeta) = 0\}, \\ A_0 w &= w'. \end{cases}$$

Lemma 5.2. [4] *$\widetilde{\mathcal{L}}_0$ is the generator of a C_0-semigroup $(T_{\widetilde{L}_0}(t))_{t\geq 0}$ on $\overline{D(\widetilde{\mathcal{L}})}$. In addition,*

$$|T_{\widetilde{\mathcal{L}}_0}(t)| = 1 \quad \textit{for} \quad t \geq 0.$$

Consider the operator $\mathcal{L}_1$ defined from $D(\mathcal{L}) \subset \mathbb{E}$ to $\mathbb{E}$ by

$$\begin{cases} D(\mathcal{L}_1) = \left\{ w \in \mathbf{C}^1\left([0,+\infty]\right) : w(0) = \lim\limits_{\zeta\to+\infty} w(\zeta) = 0 \right\}, \\ \\ \mathcal{L}_1 w = w' - \alpha\ w. \end{cases}$$

From Lemma 5.1, one has that

Lemma 5.3.

$$\rho(\mathcal{L}_1) \supset (-\alpha,+\infty) \ \ \textit{and} \ |(\eta I-\mathcal{L}_1)^{-1}| \leq \frac{1}{\eta+\alpha} \ \ \textit{for} \ \eta>-\alpha.$$

Hence, from Lemma 5.3, the operator $\mathcal{L}_1 \in HY(\mathbb{E})$ and then the hypothesis $\overline{(\mathbf{C_0'})}$ is satisfied with $\overline{K} = 1$ and $\overline{\mu} = -\alpha$, furthermore, $\overline{\mathcal{L}}_1$ the part of $\mathcal{L}_1$ on $\overline{D(\mathcal{L})}$ generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ verifies

$$|T_0(t)| \leq e^{-\alpha t}, \ t \geq 0.$$

Now, let $\mathcal{L}_2$ the operator defined on $\mathbb{E}$ by

$$(\mathcal{L}_2 w)(\zeta) = \int_0^{+\infty} \psi(\zeta,\xi) w(\xi)\, d\xi \ \text{ for all } \ \zeta \in [0,+\infty).$$

Since $\psi(\zeta,.) \in L^1(\mathbb{R}^+)$, the operator $\mathcal{L}_2$ is well defined from $\mathbb{E}$ to $\mathbb{E}$. Moreover, if $\sup\limits_{\zeta\in[0,+\infty)} \int_0^{+\infty} \psi(\zeta,\xi)\, d\xi < \infty$, clearly $\mathcal{L}_2 \in L(\mathbb{E})$ and satisfies

$$|\mathcal{L}_2| \leq \sup_{\zeta\in[0,+\infty)} \int_0^{+\infty} \psi(\zeta,\xi)\, d\xi.$$

In the next, we take $\gamma = \sup\limits_{\zeta\in[0,+\infty)} \int_0^{+\infty} \psi(\zeta,\xi)\, d\xi.$

Lemma 5.4. [7] *The part $\mathcal{L}_0$ of the linear operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ generates a C_0-semigroup $(S_0(t))_{t\geq 0}$ verifying:*

$$|S_0(t)| \leq e^{(\gamma-\alpha)t}.$$

Let $\{\mathcal{C}(t), t \geq 0\}$ be the family of operators defined on $\overline{D(\mathcal{L})}$ by $\mathcal{C}(t) = \vartheta(t)I$, $t \geq 0$. Since $\vartheta(.) \in L^1_{loc}(\mathbb{R}^+)$ and $|\vartheta(.)| \leq \nu$, then $(\mathbf{C_1})$ is verified, furthermore, consider the following notations:

$$\begin{cases} w(t)(\zeta) &= w(t,\zeta) \ \text{ for } \ t \geq 0, \quad \zeta \geq 0, \\ \mathcal{H}(t)(\zeta) &= G(t,\zeta) \ \text{ for } \ t \geq 0, \quad \zeta \geq 0. \end{cases}$$

So, Eq. (5.1) takes the form:

$$\begin{cases} \dfrac{d}{dt}w(t) &= (\mathcal{L} + \mathcal{C}(t))w(t) + \mathcal{H}(t), \qquad t \geq 0, \\ w(0) &= w_0. \end{cases} \tag{5.2}$$

To examine the boundedness of solutions for Eq. (5.2), we introduce the variation of constant formula associate to Eq. (1.1):

$$w(t) = T_0(t)w(0) + \lim_{\eta\to+\infty}\int_0^t T_0(t-\sigma)\eta(\eta I - \mathcal{L}_1)^{-1}\left(\mathcal{L}_2 w(\sigma) + \mathcal{C}(\sigma)w(\sigma) + \mathcal{H}(\sigma)\right) d\sigma \quad for \quad t \geq 0.$$

Moreover, suppose that:

$(\mathbf{C_3})$ $\gamma + \nu < \alpha$.

If we put $\lambda = 1 + \dfrac{|\mathcal{H}|_\infty}{\alpha - \gamma - \nu}$ such that $|\mathcal{H}|_\infty = \sup\limits_{0\leq t\leq\tau} \|\mathcal{H}(t)\|_\infty$. Then, the following result is obtained.

Lemma 5.5. *Suppose that $(\mathbf{C_3})$ holds. If $\|w(0)\|_\infty < \lambda$, then, the integral solution of Eq. (5.2) satisfies $\|w(t, w_0)\|_\infty \leq \lambda$, $t \geq 0$.*

Proof. Let $t_1 = \inf\{t > 0 : \|w(t, w_0)\|_\infty > \lambda\}$. The continuity of w implies that

$$\|w(t_1, w_0)\|_\infty = \lambda$$

and there exists $\epsilon > 0$ with

$$\|w(t, w_0)\|_\infty > \lambda, \qquad t \in (t_1, t_1 + \epsilon).$$

Hence, for $t_1 \geq 0$

$$\begin{aligned}
\|w(t_1, w_0)\|_\infty &\leq \|T_0(t_1)\| \, \|w(0)\|_\infty + \int_0^{t_1} e^{-\alpha(t_1-\sigma)} \left(\|\mathcal{L}_2 w(\sigma)\|_\infty + \nu \|w(\sigma)\|_\infty + \|\mathcal{H}(\sigma)\|_\infty\right) d\sigma \\
&\leq \lambda e^{-\alpha t_1} + \lambda |\mathcal{L}_2| \int_0^{t_1} e^{-\alpha(t_1-\sigma)} \, d\sigma + (|\mathcal{H}|_\infty + \lambda\nu) \int_0^{t_1} e^{-\alpha(t_1-\sigma)} \, d\sigma \\
&\leq \lambda e^{-\alpha t_1} + \frac{(1 - e^{-\alpha t_1})}{\alpha} \left(|\mathcal{H}|_\infty + \lambda(|\mathcal{L}_2| + \nu)\right) \\
&\leq \lambda e^{-\alpha t_1} + \frac{(1 - e^{-\alpha t_1})}{\alpha} \left((\lambda - 1)(\alpha - \gamma - \nu) + \lambda(\gamma + \nu)\right) \\
&\leq \lambda e^{-\alpha t_1} + \left(\lambda - 1 + \frac{\gamma + \nu}{\alpha}\right) (1 - e^{-\alpha t_1}) \\
&\leq \lambda e^{-\alpha t_1} + \lambda(1 - e^{-\alpha t_1}) \\
&\leq \lambda,
\end{aligned}$$

this contradicts the above definition of t_1, then

$$\|w(t, w_0)\|_\infty \leq \lambda \qquad for \quad each \quad t \geq 0.$$

□

To examine the problem of periodicity of solutions for Eq. (5.2), suppose that:

($\mathbf{C_4}$) G is τ-periodic.

Clearly ($\mathbf{C_4}$) implies that ($\mathbf{C_2}$) is satisfied. Then, the following theorem can be stated.

Theorem 5.6. *Suppose that* ($\mathbf{C_3}$) *and* ($\mathbf{C_4}$) *hold. If*

$$\gamma < \frac{1}{\tau} \ln \left(1 + \frac{1 - e^{-\alpha\tau}}{2e^{-\alpha\tau}}\right), \tag{5.3}$$

and

$$\nu < \frac{1}{2[\tau + 1]} \ln \left(1 + \frac{e^{-(\alpha-\gamma)\tau}(1 - e^{-(\alpha-\gamma)\tau})}{2}\right). \tag{5.4}$$

Then, the solution of Eq. (5.2) is τ-periodic.

Proof. Applying Lemma 4.6, since $\overline{K} = \widetilde{K} = 1$ and $\widetilde{\mu} = \alpha$, it follows from (5.3) that

$$|\mathcal{L}_2| < \frac{1}{\tau} \ln \left(1 + \frac{1 - e^{-\widetilde{\mu}\tau}}{2e^{-\widetilde{\mu}\tau}}\right),$$

and hence condition (4.5) is satisfied. Moreover, as in the proof of Theorem 4.5, we estimate δ by

$$\delta \leq \frac{1}{1 - e^{-(\alpha-\gamma)\tau}}.$$

From hypothesis $(\mathbf{C_3})$, we get

$$\frac{1}{2[\tau+1]}\ln\left(1+\frac{e^{-(\alpha-\gamma)\tau}(1-e^{-(\alpha-\gamma)\tau})}{2}\right)<\frac{1}{2[\tau+1]}\ln\left(1+\frac{e^{-(\alpha-\gamma)\tau}}{2\delta}\right),$$

and then condition (5.4) implies that

$$\nu<\frac{1}{2[\tau+1]}\ln\left(1+\frac{e^{-(\alpha-\gamma)\tau}}{2\delta}\right).$$

Consequently, all condition of Lemma 4.6 are satisfied. Then, the solution of Eq. (5.2) is τ-periodic. □

The objective now is checking the validity of our theoretical results via some numerical simulations. In the example of application, we consider the following quantities:

$$\alpha=2 \text{ and } \psi(\zeta,\xi)=\frac{3}{2}e^{-\xi},$$

the initial condition is given by

$$w(0,\zeta)=\zeta e^{-\zeta},$$

the functions G and $\vartheta(.)$ are given respectively by

$$G(t,\zeta)=\sin(\frac{2\pi}{5}t+\zeta) \text{ and } \vartheta(t)=3\times10^{-3}\cos(\frac{2\pi}{5}t).$$

clearly:

- G and $\vartheta(.)$ are 5-periodic functions,
- $\gamma=\sup\limits_{\zeta\in[0,+\infty)}\int_0^{+\infty}\psi(\zeta,\xi)\,d\xi=1.5.$

Then, for $\gamma=1.5$ and $\nu=0.003$, all conditions of Theorem 5.6 are verified which guaranteed the 5-periodicity of solution for Eq.(5.1). This result is illustrated by Figure 1 in $3D$ and $2D$ in figures 2 and 3 for $\zeta=0.25$ and 0.5 respectively.

6. Conclusion

In conclusion, this study establishes sufficient conditions for the existence of τ-periodic solutions for a class of evolution equations by leveraging the semi-Fredholm operator theory and fixed-point techniques. The results extend existing approaches by relaxing compactness and exponential stability requirements. These theoretical contributions are supported by illustrative applications and numerical simulations, demonstrating their practicality in addressing periodicity in non-densely defined systems.

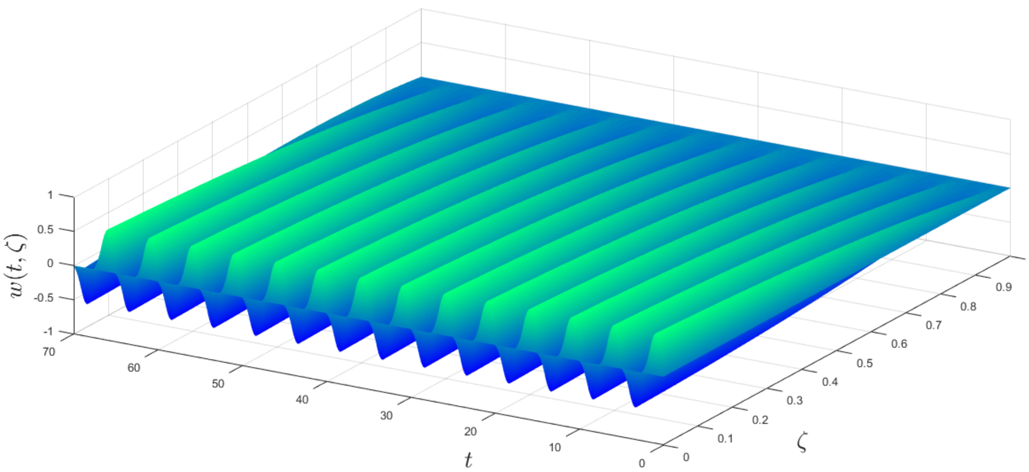

FIGURE 1. The graph of the 5-periodic solution of Eq. (5.1) in $3D$ with respect to the variables t and ζ.

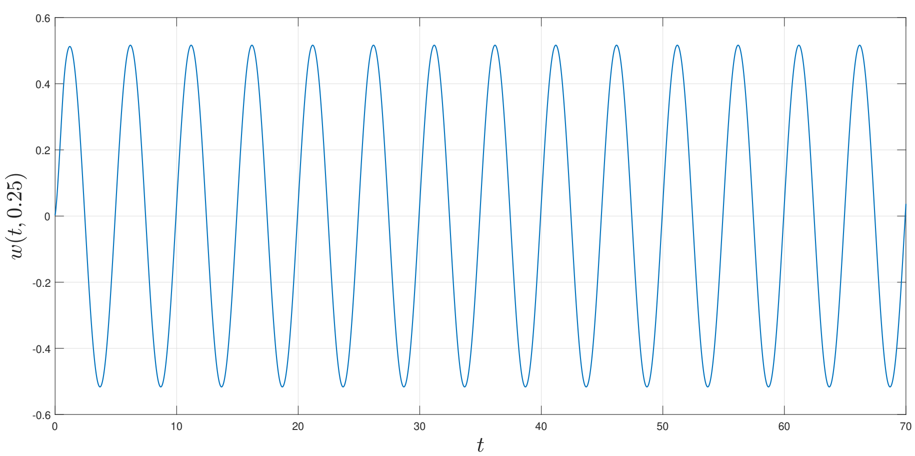

FIGURE 2. The graph of the 5-periodic solution of Eq. (5.1) in $2D$ for $\zeta = 0.25$.

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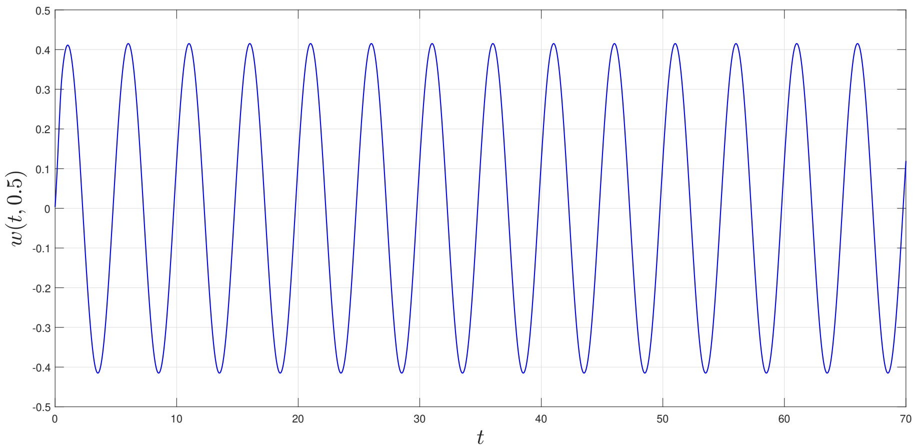

FIGURE 3. The graph of the 5-periodic solution of Eq. (5.1) in $2D$ for $\zeta = 0.5$.

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