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PAIRS OF RINGS WITH THE SAME IDEAL AND KRONECKER FUNCTION RINGS

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Abstract. In this paper, we study the classical Kronecker function rings in a pair of rings $R \subseteq T$ sharing the same ideal M, assumed to be maximal in T and contained in the Jacobson radical of R or that of T. We apply this to the $D+M$ construction and the b-operations on D and R.

1. Introduction

Let R be a subring of a ring T, and let $*$ be a star operation on the extension $R \subseteq T$. In their book [6], Knebusch and Kaiser introduced the notion of Kronecker function ring of the extension $R \subseteq T$ with respect to the operation $*$. In [7], L. Paudel and S. Tchamna studied the Kronecker function ring of a ring extension arising from a pullback diagram of type $\square$, where the extension is assumed to be Prüfer. Motivated by this study, we investigate the classical Kronecker function ring in a pair of rings $R \subseteq T$ sharing the same ideal. We also discuss the classical Kronecker function ring (Definition 2.3) in a pair of rings in pullback diagrams of type $\square$. In commutative algebra, pullbacks, and in particular pullbacks of type $\square$, are an important tool in the classical ideal theory. These types of diagrams were studied by S. Gabelli and E. Houston in [4].

In this article, all rings are assumed to be commutative with identity. For any ring R, we denote by $Jac(R)$ the Jacobson radical of R, $L = F_r(R)$ the field of fractions of R, $F(R)$ the set of all nonzero fractional ideals of R and finally $I(R)$ the set of nonzero ideals of R.

In Section 2, we give definitions, some notations, and preliminary results. In Proposition 2.7, we give an important result related to pullbacks of type $\square$. In Remark 2.9, we prove an analogue of Gilmer's result [5, Theorem 32.7] on Kronecker function rings relative to maps. The proof of Gilmer's theorem was limited to using some of the conditions of a star operation, but not all. In this work, we will only use an application that satisfies the conditions under which the Kronecker function ring is well-defined.

It is well known, for example, that if $R \subseteq T$ is an extension of domains sharing the same ideal M, which is assumed to be maximal in T and contained in the

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Jacobson radical of either R or T, and if $\Psi : T \to T/M$ is a surjective homomorphism, then the restriction $\alpha : R \to R/M$ of Ψ is also surjective, and the horizontal maps are inclusions. Consider the following commutative diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\alpha"'] & T \arrow[d, "\Psi"] \\ D = R/M \arrow[r] & T/M \end{tikzcd}$$

Since $\ker(\alpha) = \ker(\Psi) = M$ is a maximal ideal of T, the commutative diagram is called a pullback diagram of type $\square$.

Consider the map $* : I(D) \longrightarrow I(D)$, $A \longmapsto A^*$ defined for all $A,\ B,\ C \in I(D)$ by:

- $A \subseteq A^*$.
- $A \subseteq B \ \Rightarrow A^* \subseteq B^*$.
- $(A^*)^* = A^*$.
- $AB^* \subseteq (AB)^*$.
- If $A,\ B,\ C$ are finitely generated with $(AB)^* \subseteq (AC)^*$, then $B^* \subseteq C^*$.

We define a new application $*'$: $I(R) \longrightarrow I(R)$, $A \longmapsto A^{*'} := \Psi^{-1}(\Psi(A)^*)$ that satisfies the same conditions as above, and under the additional condition that $M \subseteq Jac(R)$ or $M \subseteq Jac(T)$.

Let $R \subseteq T$ be a ring extension, and denote $R^{*'}$ the classical Kronecker function ring of R with respect to the map $*'$, $T(X)$ the Nagata function ring of T and b_R the star operation on R. In Section 3, we prove that $R^{*'} \cap T(X) \subseteq T(X)$ is a pair of rings sharing the same ideal $M(R^{*'} \cap T(X)) = MT(X)$. In addition, if M is a maximal ideal of T, then $R^{*'} \cap T(X) = D^* \times_{(T/M)(X)} T(X)$ (where D^* denotes the classical Kronecker function ring of D with respect to the map $*$), in particular, if $T = K + M$ is a valuation domain, K is a field subring of T, M is the maximal ideal of T, D is subring of K with $K = F_r(D)$ and $R := D + M$, then we get the following nice result: $R^{b_R} = (D+M)^{b_R} = D^{b_D} \times_{K(X)} T(X)$, i.e., $(D \times_K T)^b = D^b \times_{K^b} T^b$.

2. Preliminaries

We will recall some definitions and results concerning star operations and Kronecker function rings.

Definition 2.1. A star operation on R is the application $* : F(R) \to F(R) : A \to A^*$ such that

c_1) $(x)^* = (x)$ and $(xA)^* = xA^*$.
c_2) $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.
c_3) $(A^*)^* = A^*$ for each x in $L - \{0\}$ and all A, B in $F(R)$.

Definition 2.2. ([5, Sections 32, p.394])

A star operation $*$ is e.a.b., if for any finitely generated $I,\ J,\ H \in F(R)$, $(IJ)^* \subseteq (IH)^*$ implies $J^* \subseteq H^*$.

It is known that

Definition 2.3. a) Given an e.a.b. star operation the Kronecker function ring of R with respect to $*$ is defined by: $Kr(R,*)$ or $R^* = \{\frac{f}{g} \mid f,g \in R[X], c_R(f)^* \subseteq c_R(g)^* \}$.
b) $Kr(R,*)$ is a Bézout domain with quotient field $L(X)$ such that $Kr(R,*) \cap L = R$.

Proof. see [5, Theorem 32.7, p.398]. □

We now recall the definition of the Nagata ring.

Definition 2.4. ([5, Proposition 33.1, p.410]) If R is a ring and X is an indeterminate over R, then the ring:
$R(X) := \{\frac{f}{g} \mid f,g \in R[X] \text{ and } c_R(g) = R \}$ is called the Nagata ring of R.

Proposition 2.5. *If b_R denotes the star operation of R defined on the fractional ideals I of R by $I \to I^{b_R} := \cap\{IV \mid V \text{is a valuation overring of } R\}$, then R is a Prüfer domain if and only if $R(X) = R^{b_R}$.*

Proof. see ([5, Sections 32, p.414]). □

For more details about star operations, we refer the reader to [5, Sections 32 and 34].

We define pullback diagrams and fiber products, which are the main tools in this paper.
Let $R \subseteq T$ be a ring extension, and let M be a common ideal of R and T. When M is a maximal ideal of T, the commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\alpha"'] & T \arrow[d, "\Psi"] \\ R/M \arrow[r] & T/M \end{tikzcd}$$

is called a pullback diagram of type □.
We recall the following definition:

Definition 2.6. If $f : A \to C$, $g : B \to C$ are ring homomorphisms, the subring $T := A \times_C B := \{(a,b) \in A \times B \mid f(a) = g(b)\}$ of the direct product $A \times B$ is called the pullback (or the fiber product) of A and B over C.

We begin this section with the following result, which will be relevant in the sequel.

Proposition 2.7. *Let $R \subseteq S$ and $L \subseteq T$ be two ring extensions. If*

$$\begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "\alpha"'] & S \arrow[d, "\Psi"] \\ L \arrow[r, "i_2"] & T \end{tikzcd}$$

is a Pullback diagrams of type $\square$, then $R \cong L \times_T S$.

Proof. Set $W = \{s \in S/\Psi(s) \in i_2(L)\ \}$.

a) It is clear that W is a subring of S.

b) Let f be the map $L \times_T S \longrightarrow W$, $(l, s) \longmapsto s$.
If $(l, s) \in L \times_T S$, then $(l, s) \in L \times S$ and $i_2(l) = \Psi(s)$. Hence $s \in S$ and $\Psi(s) = i_2(l) \in i_2(L)$, then $s \in W$.
If $(l, s) = (l', s') \in L \times_T S$, then $s = s'$ in S and $\Psi(s) = i_2(l)$, with $\Psi(s') = i_2(l')$ in $i_2(L)$. Therefore, $s = s'$ in W.
This implies that f is well-defined.

c) It is trivial that f is a homomorphism.
f is injective. Indeed, let $(l, s), (l', s') \in L \times_T S$ such that $f((l, s)) = f((l', s'))$. We have $s = s'$, so $\Psi(s) = \Psi(s')$. Therefore, $i_2(l) = i_2(l')$, so $l = l'$. Thus, f is injective.
f is surjective. Indeed, let $s \in W$; then $\Psi(s) \in i_2(L)$. So $\Psi(s) = i_2(l)$ with $l \in L$. Therefore, $(l, s) \in L \times_T S$. Then, for $s \in W$, there exists $y = (l, s) \in L \times_T S$ such that $f(y) = s$, thus f is surjective.
This implies that f is an isomorphism and $L \times_T S \cong W$.

d) We show that $R = W$.
If $r \in R$, then $r \in S$ and $\Psi(s) = \alpha(s) \in L = i_2(L)$. Thus $R \subseteq W$.
If $w \in W$, then $w \in S$ and $\Psi(w) \in i_2(L) = L$. Hence $w \in S$ and $\overline{w} = \Psi(w) \in L = R/Ker\Psi$, so $w \in S$ and $\overline{w} = \overline{r}$ with $r \in R$. Therefore, $w \in S$ and $w - r \in Ker\Psi \subseteq R$, hence $w \in R$, so $W \subseteq R$. Thus, $R = W$.
The previous argument shows that $R \cong L \times_T S$.

$\square$

First of all, let us fix the notations that we will use in this section. Let $R \subseteq T$ be an ring extension of integral domains. Let M be a common ideal of R and T with M be a prime ideal of R, then the following is a commutative diagram.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & T \arrow[d, "\Psi"] \\ D = R/M \arrow[r] & T/M \end{tikzcd}$$

Let $*$ be the map $*$: $I(D) \longrightarrow I(D)$, defined by $A \longmapsto A^*$, such that for all $A,\ B,\ C \in I(D)$:
$A \subseteq A^*$, $A \subseteq B \Rightarrow A^* \subseteq B^*$, $(A^*)^* = A^*$ and $AB^* \subseteq (AB)^*$. Additionally, if A, B, C are finitely generated and $(AB)^* \subseteq (AC)^*$, then $B^* \subseteq C^*$. We will consider the map $*' : I(R) \longrightarrow I(R)$, defined by $A \longmapsto A^{*'} := \Psi^{-1}(\Psi(A)^*)$.

Proposition 2.8. (1) *$*'$ is a well-defined map.*

(2) *For all $A,\ B,\ C \in I(R)$, we have: $A \subseteq A^{*'}$, $A \subseteq B \Rightarrow A^{*'} \subseteq B^{*'}$, $(A^{*'})^{*'} = A^{*'}$, and $AB^{*'} \subseteq (AB)^{*'}$.*

(3) *If $A,\ B,\ C \in I(R)$ are finitely generated and if $(AB)^{*'} \subseteq (AC)^{*'}$, then $B^{*'} \subseteq C^{*'}$.*

Proof. (1) If $A \in I(R)$, since Ψ is surjective from $R \longrightarrow R/M = D$, then $\Psi(A)$ is an ideal of D. Therefore, $\Psi(A) \in I(D)$, which implies that $\Psi(A)^* \in I(D)$, and thus $A^{*'} = \Psi^{-1}(\Psi(A)^*) \in I(R)$. Hence, $*'$ is well-defined.

(2) Let $A, B, C \in I(R)$.
- We have $A \subseteq \Psi^{-1}(\Psi(A)) \subseteq \Psi^{-1}(\Psi(A)^*) = A^{*'}$.
- If $A \subseteq B$, then $\Psi(A) \subseteq \Psi(B)$. Therefore, $\Psi(A)^* \subseteq \Psi(B)^*$, so $\Psi^{-1}(\Psi(A)^*) \subseteq \Psi^{-1}(\Psi(B)^*)$, and thus $A^{*'} \subseteq B^{*'}$.
- We pose $C = A^{*'}$. Then $(A^{*'})^{*'} = C^{*'} = \Psi^{-1}(\Psi(C)^*)$, so $(A^{*'})^{*'} = \Psi^{-1}((\Psi(\Psi^{-1}(\Psi(A)^*)))^*) = \Psi^{-1}((\Psi(A)^*)^*)$. Since $(\Psi(A)^*)^* = \Psi(A)^*$, it follows that $(A^{*'})^{*'} = \Psi^{-1}(\Psi(A)^*) = A^{*'}$.
- We have $AB^{*'} \subseteq \Psi^{-1}(\Psi(A))\Psi^{-1}(\Psi(B)^*) \subseteq \Psi^{-1}(\Psi(A)\Psi(B)^*) \subseteq \Psi^{-1}((\Psi(A)\Psi(B))^*) \subseteq \Psi^{-1}(\Psi(AB)^*) = (AB)^{*'}$.

(3) Suppose that A, B, and C are finitely generated in $I(R)$ with $(AB)^{*'} \subseteq (AC)^{*'}$. Since A is a finitely generated ideal of R, then $\Psi(A)$ is a finitely generated ideal of $\Psi(R) = D$, because Ψ is surjective. Likewise, $\Psi(B)$ and $\Psi(C)$ are finitely generated ideals of D.
If $(AB)^{*'} \subseteq (AC)^{*'}$, then $\Psi^{-1}(\Psi(AB)^*) \subseteq \Psi^{-1}(\Psi(AC)^*)$, so $\Psi(\Psi^{-1}(\Psi(AB)^*)) \subseteq \Psi(\Psi^{-1}(\Psi(AC)^*))$. Since Ψ is surjective, we have $\Psi(AB)^* \subseteq \Psi(AC)^*$. Therefore, $(\Psi(A)\Psi(B))^* \subseteq (\Psi(A)\Psi(C))^*$, and thus $\Psi(B)^* \subseteq \Psi(C)^*$, which implies that $\Psi^{-1}(\Psi(B)^*) \subseteq \Psi^{-1}(\Psi(C)^*)$. Hence, $B^{*'} \subseteq C^{*'}$.

□

Remark 2.9. a) Let R be an integral domain with quotient field L. Let $*_1 : I(R) \longrightarrow I(R)$, $A \longmapsto A^{*_1}$, be a map that satisfies the following conditions:
For all $A,\ B,\ C \in I(R)$:
- $A \subseteq A^{*_1}$, if $A \subseteq B \Rightarrow A^{*_1} \subseteq B^{*_1}$, $(A^{*_1})^{*_1} = A^{*_1}$, and $AB^{*_1} \subseteq (AB)^{*_1}$.
- If $A,\ B$, and $C \in I(R)$ are finitely generated and if $(AB)^{*_1} \subseteq (AC)^{*_1}$, then $B^{*_1} \subseteq C^{*_1}$.

According to the proof [5, Theorem 32.7, p. 398], the set denoted by $R^{*_1} := \{\frac{f}{g} \in L(X) \mid f, g \in R[X] - (0), c_R(f)^{*_1} \subseteq c_R(g)^{*_1} \ \} \cup \{0_L\}$, where $c_R(f)$ denotes the content of the polynomial $f(X)$ (i.e., $c_R(f)$ is the ideal of R generated by the coefficients of f), which is well-defined and is a Bézout domain with quotient field $L(X)$ and containing $R[X, \dfrac{1}{X}]$, which we will call the Kronecker function ring of R relative to the map $*_1 : I(R) \longrightarrow I(R)$.

b) Note that the map $*' : I(R) \longrightarrow I(R)$, $A \longmapsto A^{*'} := \Psi^{-1}(\Psi(A)^*)$, defined in the Proposition 2.8, cannot be extended into a classical star operation on R. Otherwise, taking $m \in M - (0)$, it assumes that $M \neq (0)$, and $A = M \in I(R) - (0) \subseteq F(R)$. Then $(mA)^{*'} = mA^{*'}$, so $\Psi^{-1}(\Psi(mA)^*) = m.\Psi^{-1}(\Psi(A)^*)$, so $\Psi^{-1}((0)^*) = m.\Psi^{-1}((0)^*)$, and with the assumption $(0)^* = (0)$, we have $M = mM$, then there exists $m' \in M$ such that $m = mm'$ and $1 = m' \in M$, which leads to a contradiction.

3. Results

In this section, we study the behavior of Kronecker function rings in pullback diagrams. Let $R \subseteq T$ be a ring extension of integral domains. Let M be a common ideal of R and T, where M is a prime ideal of R. Thus, the following diagram is commutative.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & T \arrow[d, "\Psi"] \\ D = R/M \arrow[r] & T/M \end{tikzcd}$$

Let $*$ be the map $I(D) \longrightarrow I(D)$, $A \longmapsto A^*$ such that, for all $A,\ B,\ C \in I(D)$:

$A \subseteq A^*$, $A \subseteq B \Rightarrow A^* \subseteq B^*$, $(A^*)^* = A^*$, and $AB^* \subseteq (AB)^*$. Additionally, if $A,\ B,\ C$ are finitely generated and $(AB)^* \subseteq (AC)^*$, then $B^* \subseteq C^*$.

Let $*'$ be the map $I(R) \longrightarrow I(R)$, $A \longmapsto A^{*'} := \Psi^{-1}(\Psi(A)^*)$.

For $f = \sum_{k=0}^{n} a_k X^k \in R[X]$, we write $\overline{f} = \sum_{k=0}^{n} \overline{a_k} X^k = \sum_{k=0}^{n} \Psi(a_k) X^k \in D[X]$.

Proposition 3.1. *$\overline{\alpha}$:$R^{*'} \cap T(X) \to D^*$, $\frac{f}{g} \to \frac{\overline{f}}{\overline{g}}$ is well-defined.*

Proof. Let $\frac{f}{g} \in R^{*'} \cap T(X)$, where $f, g \in R[X]$, $c_T(g) = T$, and $c_R(f)^{*'} \subseteq c_R(g)^{*'}$. Then $\Psi^{-1}(\Psi(c_R(f))^*) \subseteq \Psi^{-1}(\Psi(c_R(g))^*)$. Since Ψ is surjective, it follows that $\Psi(c_R(f))^* \subseteq \Psi(c_R(g))^*$.
Write $f = \sum_{i=0}^{n} a_i X^i \in R[X]$. Then, we have $c_R(f) = \sum_{i=0}^{n} Ra_i$ and $\Psi(c_R(f)) = \sum_{i=0}^{n} \Psi(R)\Psi(a_i) = \sum_{i=0}^{n} D\overline{a_i} = c_D(\overline{f})$, so $c_D(\overline{f})^* \subseteq c_D(\overline{g})^*$. Since $c_T(g) = T$, then $\overline{g} \neq \overline{0}$ (because if $\overline{g} = \overline{0}$, then $g \in M[X]$, and if $g = \sum_{j=0}^{m} b_j X^j$, then for all $j, b_j \in M$ and $T = c_T(g) = \sum_{j=0}^{m} Tb_j \subseteq M$, since $Tb_j \subseteq TM = M$, this leads to a contradiction). Therefore, $\frac{\overline{f}}{\overline{g}} \in D^*$.

If $\frac{f}{g} = \frac{f'}{g'}$ in $R^{*'} \cap T(X)$, where f, f' in $R[X]$ and $g, g' \in R[X] - (0)$, then $c_T(g) = c_T(g') = T$ and $fg' = gf'$ in $R[X]$, so $\overline{fg'} = \overline{gf'}$, and therefore $\overline{f}\,\overline{g'} = \overline{g}\,\overline{f'}$ in $D[X]$. Since $c_T(g) = c_T(g') = T$, we have $\overline{g} \neq \overline{0}$ and $\overline{g'} \neq \overline{0}$, so $\overline{g}$, $\overline{g'} \in D[X] - (0)$, and hence $\frac{\overline{f}}{\overline{g}} = \frac{\overline{f'}}{\overline{g'}}$ in D^*. It follows that $\overline{\alpha}$ is well-defined. $\square$

We have the following result: the extension $R^{*'} \cap T(X) \subseteq T(X)$ is a pair of rings sharing the same ideal $M(R^{*'} \cap T(X)) = MT(X)$.

Proposition 3.2. *$M(R^{*'} \cap T(X)) = MT(X)$.*

Proof. Since $R^{*'} \cap T(X) \subseteq T(X)$ and M is an ideal in $R \subseteq R^{*'} \cap T(X) \subseteq T(X)$, then $M(R^{*'} \cap T(X)) \subseteq MT(X)$.

For the containment $MT(X) \subseteq M(R^{*'} \cap T(X))$, it suffices to show that each element of the form $a = \sum_{i=1}^{r} u_i \frac{f_i}{g_i}$ with $u_i \in M - (0)$ and $\dfrac{f_i}{g_i} \in T(X)$ ($c_T(g_i) = T$) is also an element of $M(R^{*'} \cap T(X))$.

Write $f_i = \sum_{k=0}^{n} a_{i,k}X^k \in T[X]$ and $g_i = \sum_{l=0}^{m} b_{i,l}X^l \in T[X]$, and like $\dfrac{f_i}{g_i} = \dfrac{u_i f_i}{u_i g_i}$ in $F_r(T[X])$, and $u_i f_i \in MT[X] \subseteq R[X]$ (likewise $u_i g_i \in MT[X] \subseteq R[X]$). So, for all $l \in \{0, ..., m\}$: $u_i b_{i,l} \in R[X]$ and $c_R(u_i b_{i,l}) = Ru_i b_{i,l} \subseteq c_R(u_i g_i) = Ru_i b_{i,0} + ... + Ru_i b_{i,m}$, we have $c_R(u_i b_{i,l})^{*'} \subseteq c_R(u_i g_i)^{*'}$, then $\dfrac{u_i b_{i,l}}{u_i g_i} \in R^{*'}$. This implies that $\dfrac{u_i b_{i,l}}{u_i g_i} = \dfrac{b_{i,l}}{g_i} \in R^{*'} \cap T(X)$. Since $c_T(g_i) = T$, then $1 = t_{i,0}b_{i,0} + ... + t_{i,m}b_{i,m}$ with $t_{i,l} \in T$. So $u_i\frac{u_i f_i}{u_i g_i} = (u_i t_{i,0}b_{i,0} + ... + u_i t_{i,m}b_{i,m})\frac{u_i f_i}{u_i g_i} = u_i t_{i,0}b_{i,0}\frac{u_i f_i}{u_i g_i} + ... + u_i t_{i,m}b_{i,m}\frac{u_i f_i}{u_i g_i} = u_i t_{i,0}f_i\frac{u_i b_{i,0}}{u_i g_i} + ... + u_i t_{i,m}f_i\frac{u_i b_{i,m}}{u_i g_i}$. As M is an ideal of T and $u_i \in M$, $t_{i,l} \in T$, $a_{i,k} \in T$, then $u_i t_{i,l} f_i \in M[X]$. Then for all i, $u_i\frac{f_i}{g_i} = u_i\frac{u_i f_i}{u_i g_i} \in M[X](R^{*'} \cap T(X))$.

Notice that if $h = \sum_{k=0}^{d} c_k X^k \in M[X]$ and $\frac{A}{B} \in R^{*'} \cap T(X)$, then $h(\dfrac{A}{B}) = \sum_{k=0}^{d} c_k(\frac{X^k A}{B})$ and $c_R(A)^{*'} \subseteq c_R(B)^{*'}$. Since $c_R(X^k A)^{*'} = c_R(A)^{*'}$, then $c_R(X^k A)^{*'} \subseteq c_R(B)^{*'}$. Therefore, $\frac{X^k A}{B} \in R^{*'} \cap T(X)$. Hence, $h(\dfrac{A}{B}) = \sum_{k=0}^{d} c_k(\frac{X^k A}{B}) \in M.(R^{*'} \cap T(X))$. Thus, for all i, $u_i\frac{f_i}{g_i} \in M[X](R^{*'} \cap T(X)) = M(R^{*'} \cap T(X))$, we have $MT(X) \subseteq M(R^{*'} \cap T(X))$. Therefore, we have $M(R^{*'} \cap T(X)) = MT(X)$. □

Proposition 3.3. *Suppose that M is a prime in T and that $M \subseteq Jac(R)$ or $M \subseteq Jac(T)$. Then:*

1. *The map $\overline{\Psi}$:$T(X) \to (T/M)(X)$, $\frac{f}{g} \to \frac{\overline{f}}{\overline{g}}$ is a surjective ring homomorphism whose kernel is the ideal $MT(X)$.*
2. *The map $\overline{\alpha}$:$R^{*'} \cap T(X) \to D^*$, $\frac{f}{g} \to \frac{\overline{f}}{\overline{g}}$ is a surjective ring homomorphism whose kernel is the ideal $M(R^{*'} \cap T(X))$.*
3. *The diagram*

$$\begin{tikzcd} R^{*'} \cap T(X) \arrow[r, "i_1"] \arrow[d, "\overline{\alpha}"'] & T(X) \arrow[d, "\overline{\Psi}"] \\ D^* \arrow[r, "i_2"] & (T/M)(X) \end{tikzcd}$$

is commutative.

Proof. (1) Let $\frac{f}{g} \in T(X)$ with f, $g \in T[X]$ and $c_T(g) = T$. Write $f = \sum_{i=0}^{n} a_i X^i, g = \sum_{j=0}^{m} b_j X^j$. Since $c_T(g) = \sum_{j=0}^{m} Tb_j = T$, then $T/M = \Psi(T) = \Psi(c_T(g)) = \Psi(\sum_{j=0}^{m} Tb_j) = \sum_{j=0}^{m} \Psi(T)\Psi(b_j) = \sum_{j=0}^{m}(T/M)\overline{b_j} = c_{T/M}(\overline{g})$, so $\frac{\overline{f}}{\overline{g}} \in (T/M)(X)$.

Let $\frac{f}{g}$, $\frac{f_1}{g_1} \in T(X)$, where f, f_1, g and g_1 are in $T[X]$ and $c_T(g) = c_T(g_1) = T$. If $\frac{f}{g} = \frac{f_1}{g_1}$, and since $T(X) \subseteq F_r(T)(X)$, then $fg_1 = f_1 g$ and $\overline{fg_1} = \overline{f_1 g}$, so $\overline{f}\overline{g_1} = \overline{f_1}\overline{g}$ in $(T/M)[X] \subseteq (T/M)(X) \subseteq F_r(T/M)(X)$. As

$c_{T/M}(\overline{g}) = c_{T/M}(\overline{g_1}) = T/M$, then $\frac{\overline{f}}{\overline{g}} = \frac{\overline{f_1}}{\overline{g_1}}$, so $\overline{\Psi}(\frac{f}{g}) = \overline{\Psi}(\frac{f_1}{g_1})$ in $(T/M)(X)$. This shows that $\overline{\Psi}$ is well-defined. It is easy to see that $\overline{\Psi}$ is a ring homomorphism.

Let $\frac{u}{v} \in (T/M)(X)$ where $u \in (T/M)[X]$, $v \in (T/M)[X] - (0)$, and $c_{T/M}(v) = T/M$.
Write $v = \sum_{j=0}^{m} \overline{b_j}X^j$ with $b_j \in T$. Since $c_{T/M}(v) = T/M$, then $\sum_{j=0}^{m}(T/M)\overline{b_j} = T/M$, so $\overline{1} \in \sum_{j=0}^{m}(T/M)\overline{b_j}$, and $\overline{1} = \sum_{j=0}^{m} \overline{t_j}\overline{b_j} = \sum_{j=0}^{m} \overline{t_j b_j}$ with $t_j \in T$. Then $1 - \sum_{j=0}^{m} t_j b_j \in M \subseteq Jac(R)$ (respectively $M \subseteq Jac(T)$), and in both cases, we have $\sum_{j=0}^{m} t_j b_j$ is a unit in T. Hence, there exists $t \in T$ such that $\sum_{j=0}^{m} tt_j b_j = 1$, so $c_T(g = \sum_{j=0}^{m} b_j X^j) = T$. Thus, $\frac{f}{g} \in T(X)$ and $\overline{\Psi}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} = \frac{u}{v}$. Finally, $\overline{\Psi}$ is surjective.

Let $\frac{f}{g} \in Ker(\overline{\Psi}) \subseteq T(X)$ where $f, g \in T[X]$, $c_T(g) = T$ and $\overline{\Psi}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} = \overline{0}_{(T/M)(X)}$. Then $\overline{f} = \overline{0}_{(T/M)[X]}$. If $f = \sum_{i=0}^{n} a_i X^i$ where $a_i \in T$, then $\sum_{i=0}^{n} \overline{a_i} X^i = \overline{0}_{(T/M)[X]}$, so $\overline{a_i} = \overline{0}$. Therefore, $a_i \in M$, and hence $\frac{f}{g} = \frac{\sum_{i=0}^{n} a_i X^i}{g} = a_0\frac{1}{g} + a_1\frac{X}{g} + ... + a_n\frac{X^n}{g} \in MT(X)$. For all i, $\frac{X^i}{g} \in T(X)$, we conclude that $Ker(\overline{\Psi}) \subseteq MT(X)$.
Conversely, if $h \in MT(X)$, then $h = \sum_{i=1}^{l} m_i \frac{f_i}{g_i}$, where $m_i \in M$, $f_i, g_i \in T[X]$, and $c_T(g_i) = T$, so $\overline{\Psi}(h) = \overline{\Psi}(\sum_{i=1}^{l} m_i \frac{f_i}{g_i}) = \sum_{i=1}^{l} \Psi(m_i)\Psi(\frac{f_i}{g_i}) = \overline{0}_{(T/M)(X)}$. Therefore, $h \in Ker(\overline{\Psi})$ and $MT(X) \subseteq Ker(\overline{\Psi})$. This shows that $Ker(\overline{\Psi}) = MT(X)$.

(2) Let $\frac{u}{v} \in D^* - (0)$, then $u, v \in D[X] - (0)$ and $(c_D(u))^* \subseteq (c_D(v))^*$.
Write $u = \sum_{i=0}^{n} \overline{a_i} X^i, v = \sum_{j=0}^{m} \overline{b_j} X^j$, $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{j=0}^{m} b_j X^j$, where $a_i, b_j \in R$. Now, we show that $\frac{f}{g} \in R^{*'} \cap T(X)$ and $\overline{\alpha}(\frac{f}{g}) = \frac{u}{v}$. Since $c_R(f) = \sum_{i=0}^{n} Ra_i$, then $\Psi(c_R(f)) = \Psi(\sum_{i=0}^{n} Ra_i) = \sum_{i=0}^{n} \Psi(R)\Psi(a_i) = \sum_{i=0}^{n} D\overline{a_i} = c_D(u)$ and likewise $\Psi(c_R(g)) = c_D(v)$. As $(c_D(u))^* \subseteq (c_D(v))^*$, then $\Psi(c_R(f))^* \subseteq \Psi(c_R(g))^*$. So $\Psi^{-1}(\Psi(c_R(f))^*) \subseteq \Psi^{-1}(\Psi(c_R(g))^*)$. Hence $(c_R(f))^{*'} \subseteq (c_R(g))^{*'}$, so $\frac{f}{g} \in R^{*'}$. Since $v \in D[X] - (0)$, then $v \in (T/M)[X] - (0)$ and $c_{T/M}(v) = T/M$, so $\overline{1} = \sum_{j=0}^{m} \overline{t_j b_j}$ with $t_j \in T$, then $1 - \sum_{j=0}^{m} t_j b_j \in M \subseteq Jac(R)$ (respectively $M \subseteq Jac(T)$), so $\sum_{j=0}^{m} t_j b_j$ is a unit in R (respectively $\sum_{j=0}^{m} t_j b_j$ is a unit in T), then there exists $t \in T$ such that $1 = \sum_{j=0}^{m} tt_j b_j \in c_T(g)$, so $c_T(g) = T$. Hence $\frac{f}{g} \in R^{*'} \cap T(X)$ and $\overline{\alpha}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} = \frac{u}{v}$. Finally $\overline{\alpha}$ is surjective.

Let $\frac{f}{g} \in Ker(\overline{\alpha}) \subseteq R^{*'} \cap T(X)$. We have $(f, g) \in (R[X]) \times (R[X] - (0))$, $(c_R(f))^{*'} \subseteq (c_R(g))^{*'}$, $c_T(g) = T$ and $\overline{\alpha}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} = \overline{0}_{D^*}$. Since $\frac{\overline{f}}{\overline{g}} \in D^* \subseteq F_r(D)(X)$, then $\overline{f} = \overline{0}$.

If $f = \sum_{i=0}^{n} a_i X^i$, then $\overline{f} = \sum_{i=0}^{n} \overline{a_i} X^i = \overline{0}$. Thus, $\overline{a_i} = \overline{0}$, so $a_i \in M$. Since for all i, $\frac{X^i}{g} \in T(X)$, we have $\frac{f}{g} = a_0\frac{1}{g} + a_1\frac{X}{g} + ... + a_n\frac{X^n}{g} \in MT(X)$, it follows that $Ker(\overline{\alpha}) \subseteq MT(X)$. Then, by Proposition 3.2, we have $Ker(\overline{\alpha}) \subseteq M(R^{*'} \cap T(X))$.

Conversely, we show that $M(R^{*'} \cap T(X)) \subseteq Ker(\overline{\alpha})$. Let $\frac{f}{g} = \sum_{i=1}^{r} u_i \frac{f_i}{g_i} \in M(R^{*'} \cap T(X))$, $u_i \in M$, $\frac{f_i}{g_i} \in R^{*'} \cap T(X)$. We have $(f_i, g_i) \in (R[X]) \times (R[X] - (0))$, $(c_R(f_i))^* \subseteq (c_R(g_i))^*$ and $c_T(g_i) = T$. Then for all i, we have $u_i f_i \in M[X] \subseteq R^{*'} \cap T(X)$ and $\overline{\alpha}(u_i \frac{f_i}{g_i}) = \overline{\frac{u_i f_i}{g_i}} = \overline{0}$, it follows that $\overline{\alpha}(\frac{f}{g}) = \overline{\alpha}(\sum_{i=1}^{r} u_i \frac{f_i}{g_i}) = \sum_{i=1}^{r} \overline{\alpha}(u_i \frac{f_i}{g_i}) = \overline{0}$. Hence, $\frac{f}{g} \in Ker(\overline{\alpha})$. Thus, $Ker(\overline{\alpha}) = M(R^{*'} \cap T(X))$. Since $\overline{\alpha}$ is surjective, then $D^* \cong R^{*'} \cap T(X)/M(R^{*'} \cap T(X))$.

(3) Let $\frac{f}{g} \in R^{*'} \cap T(X) \subseteq T(X)$, we have $\overline{\Psi}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} \in (T/M)(X)$ and $\overline{\alpha}(\frac{f}{g}) = \frac{\overline{f}}{\overline{g}} \in D^* \subseteq (T/M)(X)$, then for all $\frac{f}{g} \in R^{*'} \cap T(X)$, $\overline{\Psi}(\frac{f}{g}) = \overline{\alpha}(\frac{f}{g})$. As for all $\frac{f}{g} \in R^{*'} \cap T(X)$, we have $i_1(\frac{f}{g}) = \frac{f}{g}$, then $\overline{\Psi}(i_1(\frac{f}{g})) = \overline{\alpha}(\frac{f}{g})$. Since for all $\frac{f}{g} \in R^{*'} \cap T(X)$; $\overline{\alpha}(\frac{f}{g}) \in D^*$, then for all $\frac{f}{g} \in R^{*'} \cap T(X)$: $i_2(\overline{\alpha}(\frac{f}{g})) = \overline{\alpha}(\frac{f}{g})$. Thus, for all $\frac{f}{g} \in R^{*'} \cap T(X)$, we have $\overline{\Psi} \circ i_1(\frac{f}{g}) = i_2 \circ \overline{\alpha}(\frac{f}{g})$. Therefore, the diagram commutes

□

Our next goal is to apply the previous results to establish an isomorphism $R^{*'} \cap T(X) \cong D^* \times_{(T/M)(X)} T(X)$. We are now ready for the main theorem of this section.

Theorem 3.4. *Suppose that M is a maximal ideal in T and $M \subseteq Jac(R)$ or $M \subseteq Jac(T)$. Then:*

(1) *The diagram*

$$\begin{tikzcd}
R^{*'} \cap T(X) \arrow[r, "i_1"] \arrow[d, "\overline{\alpha}"'] & T(X) \arrow[d, "\overline{\Psi}"] \\
D^* \arrow[r, "i_2"] & (T/M)(X)
\end{tikzcd}$$

is a pullback diagram of type □.

(2) $R^{*'} \cap T(X) \cong D^* \times_{(T/M)(X)} T(X)$.

Proof. (1) It suffices to see that $Ker(\overline{\Psi}) = MT(X)$ is a maximal ideal of $T(X)$. By Proposition 3.3, $T(X)/MT(X) \cong (T/M)(X)$, which is a field. Hence, $MT(X)$ is a maximal ideal of $T(X)$.

(2) Immediately from 1).

□

Next we exhibit an applications to the construction $D + M$ and the operations b on D and R.

Corollary 3.5. *Suppose that $T = K + M$ is a local domain with maximal ideal M, where K is a field subring of T. Let D be a subring of K, and let $R = D + M$. Then:*

(1) *The diagram*

$$\begin{tikzcd} R^{*'} \cap T(X) \arrow[r, "i_1"] \arrow[d, "\overline{\alpha}"'] & T(X) \arrow[d, "\overline{\Psi}"] \\ D^* \arrow[r, "i_2"] & (T/M)(X) = K(X) \end{tikzcd}$$

is a pullback diagram of type $\square$, thus

$$R^{*'} \cap T(X) \cong D^* \times_{K(X)} T(X)$$

(2) *If $R^{*'} \subseteq T(X)$, then the diagram*

$$\begin{tikzcd} R^{*'} \arrow[r, "i_1"] \arrow[d, "\overline{\alpha}"'] & T(X) \arrow[d, "\overline{\Psi}"] \\ D^* \arrow[r, "i_2"] & (T/M)(X) = K(X) \end{tikzcd}$$

is a pullback diagram of type $\square$ and in particular

$$R^{*'} \cong D^* \times_{K(X)} T(X)$$

.

Proof. (1) Since $M = Jac(T)$, it follows immediately from Theorem 3.4.
(2) Immediately from 1).

$\square$

Corollary 3.6. *Suppose that $T = K + M$ is a local domain with maximal ideal M, where K is a field subring of T. Let D be a subring of K that we suppose integrally closed domain and let $R = D + M$. Let $A \in I(R)$. Then:*

(1) *If $M \nsubseteq A$, then $A \subseteq M$.*
(2) *If $M \subseteq A$, then $A = (A \cap K) + M$.*
(3) *Suppose that $*'$:$I(R) \to I(R)$, $A \to A^{*'} = \Psi^{-1}((A \cap K)^{b_D})$ where b_D the b-operation on D. Then:*
   (a) *$*'$ is a well-defined map.*
   (b) *$A^{*'} = \Psi^{-1}(\Psi(A)^{b_D})$.*
   (c) *$A^{*'} = (A \cap K)^{b_D} + M$.*
   (d) *$R^{*'} \cap T(X) \cong D^{b_D} \times_{K(X)} T(X)$.*

Proof. (1) Let $y \in M - A$ and $x \in A$. If $x \notin M$, then x is a unit in T, so $x^{-1} \in T$ and $x^{-1}y \in M \subseteq R$. Thus $y = (x^{-1}y)x \in R.A \subseteq A$, which is a contradiction. Therefore, $x \in M$, and consequently, $A \subseteq M$.
(2) Let $z \in A \subseteq T = K + M$. Then $z = \alpha + m$, where $\alpha \in K$ and $m \in M$, then $\alpha = z - m \in A \cap K$ and $z = \alpha + m \in (A \cap K) + M$. Thus $A \subseteq (A \cap K) + M$. Conversely, since $A \cap K \subseteq A$ and $M \subseteq A$, it follows that $(A \cap K) + M \subseteq A$. Hence, $A = (A \cap K) + M$.

(3) (a) $A \cap K = A \cap D \in I(D)$, then $(A \cap K)^{b_D} = (A \cap D)^{b_D} \subseteq D^{b_D} = D$ and $(A \cap K)^{b_D} \in I(D)$, and so $\Psi^{-1}((A \cap K)^{b_D}) \in I(R)$, thus $*'$ is well-defined.

(b) • If $M \nsubseteq A$, by 1) we have $A \subseteq M$. So, $\Psi(A) = (0)$ and $\Psi(A)^{b_D} = \cap_\alpha(\Psi(A)D_\alpha) = (0)$, where $\{D_\alpha\}_\alpha$ is a family of all valuation overrings of D. Then $A^{*'} = \Psi^{-1}((A \cap K)^{b_D}) = \Psi^{-1}((0)) = M = \Psi^{-1}(\Psi(A)^{b_D})$.
• If $M \subseteq A$, then by 2), we have $A = (A \cap K) + M$. Thus, $\Psi(A) = \Psi((A \cap K) + M) = A \cap K$, and therefore $\Psi(A)^{b_D} = (A \cap K)^{b_D}$. Consequently, $A^{*'} = \Psi^{-1}((A \cap K)^{b_D}) = \Psi^{-1}(\Psi(A)^{b_D})$.

(c) Let $z = k + m \in T$, where $k \in K$ and $m \in M$. Then $z \in A^{*'} \iff z \in \Psi^{-1}((A \cap K)^{b_D}) \iff \Psi(z) = \Psi(k + m) = k \in (A \cap K)^{b_D} \iff z = k + m \in (A \cap K)^{b_D} + M$. Therefore, $A^{*'} = (A \cap K)^{b_D} + M$.

(d) By 1) from the Corollary 3.5, we have $R^{*'} \cap T(X) \cong D^{b_D} \times_{K(X)} T(X)$. □

Now, we present the following proposition which provides us an important example of the classical $D + M$, where $T = K + M$ is a valuation domain. We prove that there exists an isomorphism between the classical Kronecker function rings R^{b_R} and the fiber product $D^{b_D} \times_{K(X)} T(X)$.

Proposition 3.7. *With the notations of the Corollary 3.6, if T is additionally a valuation domain and $K = F_r(D)$, then for all $A \in I(R)$, we have:*

(1) *If $M \subseteq A$, then $A^{b_R} = A^{*'}$.*
(2) *$R^{*'} \cap T(X) = R^{b_R} \cap T(X)$.*
(3) *$R^{b_R} \cong D^{b_D} \times_{K(X)} T(X)$.*
(4) *In particular, if D is a Prüfer, then $R(X) \cong D(X) \times_{K(X)} T(X)$.*

Proof. (1) Let $(D_\alpha)_\alpha$ be the family of all valuation overrings of D , which we assume to be integrally closed (and thus, R is integrally closed). Since $M \subseteq A$, by 2) Corollary 3.6, we have $A = (A \cap K) + M$. According to b)3) of the same Corollary, $A^{*'} = (A \cap K)^{b_D} + M = \cap_\alpha((A \cap K).D_\alpha) + M = \cap_\alpha(((A \cap K) + M).(D_\alpha + M)) = \cap_\alpha(A.(D_\alpha + M)) = A^{b_R}$.

(2) Let $\frac{f}{g} \in R^{*'} \cap T(X)$, where $f, g \in R[X] - (0)$ and $c_R(f)^{*'} \subseteq c_R(g)^{*'}$, and $c_T(g) = T$.
• If $c_R(g) \subseteq M$, then the coefficients of g are in M, and $c_T(g) = T \subseteq M$, which leads to a contradiction. Thus, $M \subset c_R(g)$.
• If $M \subseteq c_R(f)$ and since $M \subseteq c_R(g)$, then $c_R(f)^{b_R} = c_R(f)^{*'} \subseteq c_R(g)^{*'} = c_R(g)^{b_R}$. Thus, $\frac{f}{g} \in R^{b_R} \cap T(X)$.
• If $M \nsubseteq c_R(f)$, then $c_R(f) \subseteq M \subseteq c_R(g)$, so $c_R(f)^{b_R} \subseteq c_R(g)^{b_R}$, and $\frac{f}{g} \in R^{b_R} \cap T(X)$.
It follows that $R^{*'} \cap T(X) \subseteq R^{b_R} \cap T(X)$.
Let $\frac{f}{g} \in R^{b_R} \cap T(X)$, where $f, g \in R[X] - (0)$ and $c_R(f)^{b_R} \subseteq c_R(g)^{b_R}$, and $c_T(g) = T$.
• Since $c_T(g) = T$, then $c_R(g) \nsubseteq M$ and so $M \subset c_R(g)$.
• If $M \subset c_R(f)$, then $c_R(f)^{*'} = c_R(f)^{b_R} \subseteq c_R(g)^{b_R} = c_R(g)^{*'}$ and $\frac{f}{g} \in R^{*'} \cap T(X)$.

• If $M \nsubseteq c_R(f)$, then $c_R(f) \subset M$, so $c_R(f) \subset M \subseteq c_R(g)$, and $c_R(f)^{*'} \subseteq c_R(g)^{*'}$. Hence $\frac{f}{g} \in R^{*'} \cap T(X)$. Thus $R^{b_R} \cap T(X) \subseteq R^{*'} \cap T(X)$. This shows that $R^{*'} \cap T(X) = R^{b_R} \cap T(X)$.

(3) Since T is of valuation and $R \subseteq T$, then $R^{b_R} \subseteq T^{b_T} = T(X)$. By 2), we have $R^{b_R} = R^{b_R} \cap T(X) = R^{*'} \cap T(X)$, by Corollary 3.6, $R^{*'} \cap T(X) \cong D^{b_D} \times_{K(X)} T(X)$. Thus $R^{b_R} \cong D^{b_D} \times_{K(X)} T(X)$.

(4) Since D and R are Prüfer domains, we have $R^{b_R} = R(X)$ and $D^{b_D} = D(X)$. Therefore, we conclude with (3).

□

We now present an example of how Proposition 3.5.

Example 3.8. Let $\mathbb{Q}$ be the field of rational numbers, and let X by an indeterminate over $\mathbb{Q}$. Define $T = \mathbb{Q}[X]_{(X)} = \mathbb{Q} + X.\mathbb{Q}[X]_{(X)}$. By [5, Exercise 11, p. 271], we have T is a valuation domain of $\mathbb{Q}(X)$. Let $M = X.\mathbb{Q}[X]_{(X)}$ be the maximal ideal of T, and let $D = \mathbb{Z}_{(2\mathbb{Z})} \cap \mathbb{Z}_{(3\mathbb{Z})}$ the subring of $\mathbb{Q}$ representing fractions whose denominators are not divisible by both 2 and 3. We have $F_r(D) = \mathbb{Q}$ and the valuation overrings of D are $\mathbb{Z}_{(2\mathbb{Z})}$, $\mathbb{Z}_{(3\mathbb{Z})}$ and $\mathbb{Q}$ (since $\mathbb{Z}_{(2\mathbb{Z})}$, and $\mathbb{Z}_{(3\mathbb{Z})}$ are overings of $\mathbb{Z}_{(2\mathbb{Z})} \cap \mathbb{Z}_{(3\mathbb{Z})}$, and no other overring can contain all the elements of $\mathbb{Z}_{(2\mathbb{Z})} \cap \mathbb{Z}_{(3\mathbb{Z})}$. Moreover, $\mathbb{Z}_{(2\mathbb{Z})}$, $\mathbb{Z}_{(3\mathbb{Z})}$ and $\mathbb{Q}$ are indeed valuation domains). Define $R = D + M$, R is not a valuation domain but is a Bézout domain, and therefore it is a Prüfer. Let $\Psi : T \to T/M = \mathbb{Q}$ be the canonical surjection and let $*'$ be the map from $I(R)$ to $I(R)$ defined in the following way: for any $A \in I(R)$; $A^{*'} = \Psi^{-1}(\Psi(A)^{b_D})$. By Corollary 3.6, for all $A \in I(R)$, we have $A^{*'} = (A \cap \mathbb{Q})^{b_D} + M = (A \cap \mathbb{Q}).\mathbb{Z}_{(2\mathbb{Z})} \cap (A \cap \mathbb{Q}).\mathbb{Z}_{(3\mathbb{Z})} + M$. The valuation overrings of R are $\mathbb{Z}_{(2\mathbb{Z})} + M$, $\mathbb{Z}_{(3\mathbb{Z})} + M$, and $T = \mathbb{Q} + M = \mathbb{Q}[X]_{(X)}$ (see [5, Exercise 11, p. 203]), and $F_r(R) = F_r(T) = \mathbb{Q}(X)$ (see [5, Exercise 13, p. 203]). Furthermore, we have $(M^2)^{*'} = M$ and $(M^2)^{b_R} = M^2.(\mathbb{Z}_{(2\mathbb{Z})}) \cap M^2.(\mathbb{Z}_{(3\mathbb{Z})}) \subseteq M^2 \neq M = (M^2)^{*'}$. However, according to the proposition 3.7, we have for all $A \in I(R)$ with $M \subseteq A$, $A^{*'} = A^{b_R}$ and $R^{b_R} \cong D^{b_D} \times_{\mathbb{Q}(Y)} T(Y)$.

4. Conclusion

In this paper, we studied the classical Kronecker function rings in the context of the $D + M$ construction, with a particular focus on the pair of rings $R \subseteq T$ sharing the same ideal M, assumed to be maximal in T and contained within the Jacobson radical of R or T. We applied this framework to the b-operations on the rings D and R, and derived several important results
We prove that $R^{*'} \cap T(X) \subseteq T(X)$ forms a pair of rings that share the same ideal, and that $M(R^{*'} \cap T(X)) = MT(X)$. In addition, if M is a maximal ideal of T, we prove that:
$R^{*'} \cap T(X) = D^{*} \times_{(T/M)(X)} T(X)$. In particular, if $T = K + M$ is a valuation domain, where K is a field subring of T, M is the maximal ideal of T, D is a subring of K with $K = \mathrm{Fr}(D)$, and $R := D + M$, then we obtain the following elegant result:

$$(D \times_K T)^b = D^b \times_{K^b} T^b.$$

These results contribute to a deeper understanding of the interaction between the structures of the rings involved, particularly in the case of valuation domains and their subrings. By establishing the properties of the classical Kronecker function rings in these settings, we also highlight the intricate relationships between R, T, and their respective ideal structures, offering insights into their behavior with respect to the b-operations.

Ultimately, these findings suggest new directions for future research, particularly in the extension of these methods to other types of rings and the further exploration of their properties within more general frameworks.

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