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SEMIDERIVATIONS ON RINGS WITH CERTAIN ALGEBRAIC IDENTITIES

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ABSTRACT. This paper presents a new type of semiderivation in ring, called a (f, g)-semiderivation, where f and g are are self-mappings of a ring. We explore various properties of (f, g)-semiderivations and demonstrate that if a semiprime ring R satisfies specific differential identities involving a (f, g)-semiderivation D, then D must be a zero map on R. Additionally, we examine the commutativity of prime rings that admit nonzero (f, g)-semiderivations which satisfy certain differential identities.

1. Introduction

The theory of derivation is one of the most fascinating research areas in rings. It has a long history and has been studied by many mathematicians over the years. The concept of derivation was first introduced by the German mathematician Hermann Grassmann. He defined a derivation of a rings R as a map $D : R \to R$ which satisfies the following two conditions: (i) $D(x+y) = D(x)+D(y)$, and (ii) $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$.

For a long time, derivations were only studied in commutative rings. However, in the 1950s, several mathematicians began to study derivations in non-commutative rings. One of the most important results in this area was proved by the American mathematician Eliakim Carmon Posner in 1957. Posner [8] demonstrated that a prime ring R with a nonzero derivation D must necessarily be commutative. This result was a major breakthrough in the theory of derivations, and it has had a significant impact on the study of non-commutative rings. Since Posner's result, there has been a lot of research on derivations in non-commutative rings. Many mathematicians have studied the properties of derivations in various types of non-commutative rings, such as semiprime rings, prime rings, and near rings. Furthermore, there has been considerable interest in studying the commutativity of rings, particularly prime and semiprime rings that satisfy certain algebraic identities involving derivations.

A mapping D from a ring R into R is called a generalized derivation if there exists a derivation $d : R \to R$ such that (i) $D(x + y) = D(x) + D(y)$, and (ii) $D(xy) = D(x)y + xd(y)$ for all $x, y \in R$. In this case, D is referred to as

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the generalized derivation associated with d. For any mapping f from a ring R into itself, an f-derivation is a mapping $D : R \to R$ satisfying (i) $D(x+y) = D(x)+D(y)$, and (ii) $D(xy) = D(x)f(y)+f(x)D(y)$ for all $x, y \in R$. The concepts of derivation and f-derivation have also been generalized in various directions.

Recently, Leerawat and Toka [9] proved two main results that provide new insights into the relationship between f-derivations and commutativity in prime rings. The first result is that if a prime ring R satisfies specific differential identities involving an f-derivation d on a nonzero ideal of R, then d is a zero map on R. The second result demonstrates that if a prime ring R has a nonzero f-derivation d such that $d[x, y] = [d(x), f(y)] + [d(y), f(x)]$ for all x, y in a nonzero ideal of R, then R must be commutative.

The concept of a semiderivation was introduced by J. Bergen in [1]. For any mapping f from a ring R into itself, a semiderivation is a mapping $D : R \to R$ that satisfies the following conditions for all x, y in R (i) $D(x+y) = D(x)+D(y)$ (ii) $D(xy) = D(x)f(y)+xD(y) = D(x)y+f(x)D(y)$ and (iii) $D(f(x)) = f(D(x))$. In this case, D is referred to as the semiderivation associated with f. When f is the identity map on R, D becomes a derivation. However, a semiderivation is not necessarily a derivation. For example, consider $D(x) = x - f(x)$, where f is any ring endomorphism of R. In this case, D is a semiderivation of R with associated map f, but it is not a derivation. Bresar [5] proved that the only semiderivations of prime rings are derivations. Additionally, Chang [6] showed that if D is a nonzero semiderivation of a prime ring R, then the associated mapping must be a ring endomorphism of R.

Boua et al. [3] introduced the concept of a generalized semiderivation of a near ring N as follows: A mapping $D : N \to N$ is called a generalized semiderivation of N if there exists a semiderivation $d : N \to N$ associated with a map $f : N \to N$ such that for all $x, y \in N$ (i) $D(x + y) = D(x) + D(y)$, (ii) $D(xy) = D(x)y + f(x)d(y) = d(x)f(y) + xD(y)$, and (iii) $D(f(x)) = f(D(x))$. Notably, every semiderivation is also a generalized semiderivations. For further details on semiderivations, generalized semiderivations and related commutativity properties, see Boua et al. [4], Chang [6], Bharathi et al. [2], and Nabiel [7].

This work aims to introduce a new type of semiderivation, called a (f, g)-semiderivation, in a ring R, where f and g are mappings from R into itself. We show that if a semiprime ring R satisfies certain differential identities involving a (f, g)-semiderivation D, then D acts as the zero map on R. Additionally, we prove that prime near rings satisfying some algebraic identities involving nonzero (f, g)-semiderivations are commutative rings.

2. (f, g)-SEMIDERIVATION

In this section, we introduce a special type of semiderivation; known as a (f, g)-semiderivation on a ring, and prove some fundamental properties.

Definition 2.1. Let f and g be mappings on a ring R. A mapping $d : R \to R$ is called a (f, g)-semiderivation of R if for all $x, y \in R$

(i) $d(x + y) = d(x) + d(y)$,

(ii) $d(xy) = d(x)f(y) + g(x)d(y) = d(x)g(y) + f(x)d(y)$, and
(iii) $d(f(x)) = f(d(x))$ and $d(g(x)) = g(d(x))$.

Example 2.2. Let $R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$ be a ring of 2×2 matrices over $\mathbb{R}$. Define $f, g, d : R \to R$ by

$$f\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, g\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \text{and } d\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix},$$

respectively. It is easy to verify that d is a (f, g)-semiderivation of R.

In this paper, let R denote a ring. Recall that a ring R is prime if for any $a, b \in R, aRb = \{0\}$ implies that either $a = 0$ or $b = 0$. Additionally, R is semiprime if for any $a \in R, aRa = \{0\}$ implies $a = 0$. A prime ring is obviously semiprime, but the converse is not true always.

From now on, let $f, g : R \to R$ be mappings. The following lemmas are essential for establishing the proof of our results.

Lemma 2.3. *Let R be a prime ring and let $f : R \to R$ be a surjective mapping. Let d be a (f, g)-semiderivation of R, and $a \in R$. If $ad(x) = 0$ or $d(x)a = 0$ for all $x \in R$, then $a = 0$ or $d(x) = 0$ for all $x \in R$.*

Proof. Assume $ad(x) = 0$ for all $x \in R$. Replacing x by yx in the previous equation yields $af(y)d(x) = 0$ for all $x, y \in R$. Since f is surjective, $aRd(x) = \{0\}$ for all $x \in R$. By the primness of the ring R, we obtain $a = 0$ or $d(x) = 0$ for all $x \in R$.

Similar arguments may be used to get the required result, if $d(x)a = 0$ for all $x \in R$. □

Lemma 2.4. *Let R be a prime ring and let $g : R \to R$ be a surjective mapping. Let d be a (f, g)-semiderivation of R, and $a \in R$. If $ad(x) = 0$ or $d(x)a = 0$ for all $x \in R$, then $a = 0$ or $d(x) = 0$ for all $x \in R$.*

The proof of this lemma is quite the same as in the proof of Lemma 2.3.

Lemma 2.5. *Let R be a prime ring and let $f : R \to R$ be a surjective mapping. Let d be a (f, g)-semiderivation of R. Then f is a homomorphism of R if and only if g is homomorphism of R.*

Proof. Suppose that f is a homomorphism of R. Let $x, y, z \in R$. Then

$$d((x + y)z) = d(x)f(z) + d(y)f(z) + g(x + y)d(z),$$

and

$$d(xz + yz) = d(x)f(z) + g(x)d(z) + d(y)f(z) + g(y)d(z).$$

It follows that

$$(g(x + y) - g(x) - g(y))d(z) = 0$$

for all $x, y \in R$. Since R is a prime ring, and d is a nonzero (f, g)-semiderivation of R, $g(x + y) = g(x) + g(y)$ for all $x, y \in R$.

Now, we consider

$$d((xy)z) = d(x)f(y)f(z) + g(x)d(y)f(z) + g(xy)d(z), \text{and}$$
$$d(x(yz)) = d(x)f(y)f(z) + g(x)d(y)f(z) + g(x)g(y)d(z).$$

Then

$$(g(xy) - g(x)g(y))d(z) = 0$$

for all $x, y, z \in R$. But R is prime and d is nonzero, $g(xy) = g(x)g(y)$ for all $x, y \in R$. Hence, g is homomorphism of R.

Conversely, suppose that g is a homomorphism of R. By using the same argument as above yields f is a homomorphism of R. This completes the proof. □

3. Main results

In this section, for any elements x and y in a ring R, we define the commutator $[x, y]$ as $xy - yx$, and the skew-commutator $x \circ y$ as $xy + yx$. Throughout, we will frequently use fundamental identities for commutators and skew-commutators:

$$[xy, z] = x[y, z] + [x, z]y,$$
$$[x, yz] = y[x, z] + [x, y]z,$$
$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z], \text{and}$$
$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z,$$

for all $x, y, z \in R$.

Theorem 3.1. *Let R be a semiprime ring. Let $f : R \to R$ be an epimorphism and $g : R \to R$ be a homomorphism such that $[f(x), g(y)] = 0$ for all $x, y \in R$. Suppose that d is a (f, g)-semiderivation of R satisfying any one of the following conditions:*
(i) $d(x)f(y) - d(y)g(x) = 0$ for all $x, y \in R$,
(ii) $d(x)f(y) + d(y)g(x) = 0$ for all $x, y \in R$,
(iii) $g(x)d(y) - f(y)d(x) = 0$ for all $x, y \in R$,
(iv) $g(x)d(y) + f(y)d(x) = 0$ for all $x, y \in R$.
Then $d(x) = 0$ for all $x \in R$.

Proof. (i) Assume

$$d(x)f(y) - d(y)g(x) = 0$$

for all $x, y \in R$. Replacing x by rx where $r \in R$ in the previous equation, and applying the previous equation again, we obtain

$$d(r)g(x)f(y) + f(r)d(x)f(y) = d(y)g(r)g(x) = d(r)f(y)g(x)$$

for all $r, x, y \in R$. This implies that $f(r)d(x)f(y) = d(r)[f(y), g(x)]$ for all $r, x, y \in R$. Since $[f(x), g(y)] = 0$ for all $x, y \in R, f(r)d(x)f(y) = 0$ for all $x, y, r \in R$. Hence, $d(x)f(y)f(r)d(x)f(y) = 0$ for all $x, y, r \in R$. Since f is surjective, $d(x)f(y)Rd(x)f(y) = \{0\}$ for all $x, y \in R$. Since R is semiprime, $d(x)f(y) = 0$ for all $x, y \in R$. Then $d(x)f(y)d(x) = 0$ for all $x, y \in R$. Since f is a surjective mapping, $d(x)Rd(x) = \{0\}$ for all $x \in R$. Since R is semiprime, $d(x) = 0$ for all $x \in R$.

(ii) Assume that $d(x)f(y) + d(y)g(x) = 0$ for all $x, y \in R$. By using the same technique in the proof of (i), we can conclude that $d(x) = 0$ for all $x \in R$.

(iii) Suppose that

$$g(x)d(y) - f(y)d(x) = 0$$

for all $x, y \in R$. Replacing x by xr where $r \in R$ in the previous equation, and applying the previous equation again, we obtain

$$g(x)g(r)d(y) = f(y)d(x)f(r) + f(y)g(x)d(r)$$

for all $r, x, y \in R$. It follows that $f(y)d(x)f(r) = [f(y), g(x)]d(r)$ for all $r, x, y \in R$. Since $[f(x), g(y)] = 0$, we have $f(y)d(x)f(r) = 0$. Once again using the same technique in the proof of (i), we can conclude that $d(x) = 0$ for all $x \in R$.

(iv) Assume $g(x)d(y)+f(y)d(x) = 0$ for all $x, y \in R$. By using the same method in the proof of (iii), we can conclude that $d(x) = 0$ for all $x \in R$. □

The proof of the following theorem proceeds in a similar manner to that of Theorem 3.1.

Theorem 3.2. *Let R be a semiprime ring. Let $f : R \to R$ be a homomorphism, and $g : R \to R$ be an epimorphism such that $[f(x), g(y)] = 0$ for all $x, y \in R$. Suppose that d is a (f, g)-semiderivation of R satisfying any one of the following conditions:*

(i) $d(x)f(y) - d(y)g(x) = 0$ for all $x, y \in R$,
(ii) $d(x)f(y) + d(y)g(x) = 0$ for all $x, y \in R$,
(iii) $g(x)d(y) - f(y)d(x) = 0$ for all $x, y \in R$,
(iv) $g(x)d(y) + f(y)d(x) = 0$ for all $x, y \in R$.

Then $d(x) = 0$ for all $x \in R$.

Corollary 3.3. *Let R be a semiprime ring. Let $f : R \to R$ be an epimorphism, and $g : R \to R$ be a homomorphism (or $f : R \to R$ be a homomorphism, and $g : R \to R$ be an epimorphism) such that $[f(x), g(y)] = 0$ for all $x, y \in R$. Suppose that d is a (f, g)-semiderivation of R satisfying any one of the following conditions:*

(i) $d(xy) = d(x) \circ f(y)$ for all $x, y \in R$,
(ii) $d(xy) = d(x) \circ g(y)$ for all $x, y \in R$,
(iii) $d(xy) = [d(x), f(y)]$ for all $x, y \in R$,
(iv) $d(xy) = [g(x), d(y)]$ for all $x, y \in R$,
(v) $d([x, y]) = d(x) \circ f(y) - d(y) \circ g(x)$ for all $x, y \in R$, and $char(R) \neq 2$,
(vi) $d([x, y]) = -d(x) \circ f(y) + d(y) \circ g(x)$ for all $x, y \in R$, and $char(R) \neq 2$.

Then $d(x) = 0$ for all $x \in R$.

Theorem 3.4. *Let R be a prime ring. Let $f : R \to R$ be an epimorphism, and $g : R \to R$ be an injective mapping such that $[f(x), g(y)] = 0$ for all $x, y \in R$. Suppose that d is a nonzero (f, g)-semiderivation of R satisfying any one of the following conditions:*

(i) $d(y) \circ g(x) = 0$ for all $x, y \in R$,
(ii) $[d(y), g(x)] = 0$ for all $x, y \in R$.

Then R is commutative.

Proof. (i) Assume
$$d(y) \circ g(x) = 0$$
for all $x, y \in R$. Replacing y by yz where $z \in R$ in the previous equation, we have
$$[g(y), g(x)]d(z) = 0$$
for all $x, y, z \in R$. Replacing z by rz where $r \in R$ in the previous equation, we obtain
$$[g(y), g(x)]f(r)d(z) = 0$$
for all $r, x, y, z \in R$. Since f is surjective, $[g(y), g(x)]Rd(z) = \{0\}$ for all $x, y, z \in R$. Since R is prime, and d is nonzero on R, we have $[g(y), g(x)] = 0$ for all $x, y \in R$. By Lemma 2.5, g is a homomorphism of R. Thus, $g([y, x]) = 0$ for all $x, y \in R$. Since g is injective, $[y, x] = 0$ for all $x, y \in R$. This implies that R is commutative.

(ii) Suppose that $[d(y), g(x)] = 0$ for all $x, y \in R$. By using the same technique in the proof of (i), we can conclude that R is commutative. □

The proof of the following lemma follows a similar approach to the proof of Theorem 3.4

Theorem 3.5. *Let R be a prime ring. Let $f : R \to R$ be an injective mapping, and $g : R \to R$ be an epimorphism such that $[f(x), g(y)] = 0$ for all $x, y \in R$. Suppose that d is a nonzero (f, g)-semiderivation of R satisfying any one of the following conditions:*

(i) $d(y) \circ f(x) = 0$ for all $x, y \in R$,

(ii) $[d(y), f(x)] = 0$ for all $x, y \in R$.

Then R is commutative.

Corollary 3.6. *Let R be a prime ring. Let $f : R \to R$ be an epimorphism, and $g : R \to R$ be an injective mapping such that $[f(x), g(y)] = 0$ for all $x, y \in R$. Suppose that d is a nonzero (f, g)-semiderivation of R satisfying any one of the following conditions:*

(i) $d(x \circ y) = d(x) \circ f(y)$ for all $x, y \in R$,

(ii) $d([x, y]) = [d(x), f(y)]$ for all $x, y \in R$,

(iii) $d(x \circ y) = d(x) \circ f(y) - d(y) \circ g(x)$ for all $x, y \in R$, and $char(R) \neq 2$,

(iv) $d[x, y] = [d(x), f(y)] + [d(y), g(x)]$ for all $x, y \in R$, and $char(R) \neq 2$.

Then R is commutative.

Corollary 3.7. *Let R be a prime ring. Let $f : R \to R$ be an injective mapping, and $g : R \to R$ be an epimorphism such that $[f(x), g(y)] = 0$ for all $x, y \in R$. Suppose that d is a nonzero (f, g)-semiderivation of R satisfying any one of the following conditions:*

(i) $d(x \circ y) = g(x) \circ d(y)$ for all $x, y \in R$,

(ii) $d([x, y]) = [g(x), d(y)]$ for all $x, y \in R$.

Then R is commutative.

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References

1. Bergen, J. (1983). Derivations in prime rings. Canadian Mathematical Bulletin,26, 267–270. https://doi.org/10.4153/CMB-1983-042-2
2. Bharathi, M. V. L., & Jayalakshmi K. (2017). Semiderivations in prime rings. International Journal of Pure and Applied Mathematics, 113(6), Special Issue. 1 – 8.
3. Boua, A., Raji, A., Ali, A. , & Ali, F. (2015). On generalized semiderivations of prime near rings. International Journal of Mathematics and Mathematical Sciences. Article ID 867923, 7 pages. https://doi.org/10.1155/2015/867923
4. Boua, A., & Oukhtite, L. (2013). Semiderivations satisfying certain algebraic identities on prime near-rings. Asian-European Journal of Mathematics, 6(3), Article ID 1350043, 8 pages. https://doi.org/10.1142/S1793557113500435
5. Bresar, M. (1990). Semiderivation of prime rings. Proc. Amer. Math. Soc. 108(4), 859-860. https://doi.org/10.1090/S0002-9939-1990-1007488-X
6. Chang, J. C. (1984). On semiderivations in prime rings. Chinese J. Math., 12(4), 255-262. https://www.jstor.org/stable/43836229
7. Nabiel, H. (2013). Semiderivations and commutativity in semiprime rings. Gen. Math. Notes, 19(2), 71-82.
8. Posner, E. (1957). Derivations in prime rings. Proc. Amer. Math. Soc. 8, 1093-1100. https://doi.org/10.2307/2032686
9. Leerawat, U., & Toka, P. (2022). Some differential identities with f-derivations on prime rings. JP Journal of Algebra, Number Theory and Applications, 57, 39 – 52. http://dx.doi.org/10.17654/0972555522029

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