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HILBERT SPACE FRAMES, RIESZ BASES AND TENSOR SUMS

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Abstract. In this article we study Riesz bases and frames using tensor sum of linear operators on Hilbert spaces. Moreover, we develope a theory for Riesz bases in tensor product of Hilbert spaces and discuss their properties.

1. INTRODUCTION

Duffin and Schaeffer [7] introduced Hilbert space frames in 1952 to address some deep questions in non - harmonic Fourier series. The idea was to weaken Parseval's identity.

Definition 1.1. If $\{f_i\}_{i\in I}$ is an orthonormal sequence in a Hilbert space H, then for all $f \in H$ we have

$$\sum_{i\in I} | < f, f_i > |^2 = ||f||.$$

It is not necessary to have an orthonormal sequence for the equality in Parsevel's identity. For example, if $\{e_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are orthonormal bases for a Hilbert space H, then $\left\{\frac{1}{\sqrt{2}}e_i, \frac{1}{\sqrt{2}}g_i\right\}$ satisfies Parsevel's identity. Duffin and Schaeffer weakened Parsevel's identity to produce what they called a Frame. Any vector in a Hilbert space can be written as a linear combination of the frame elements where the expansion coefficients can be computed via a frame operator.[see Sec 2]

As discussed in [8], frames work as an alternative to orthonormal bases in Hilbert space which has many advantages. Frame theory plays an important role not just in signal processing but also in dozen of applied areas like filter bank theory, sigma - delta quantization, wireless communication, sampling theory, internet coding and more [3, 4].

Tensor products have important applications, for example, they are useful in the approximation of multi - variate functions of combinations of univariate ones. Tensor products of frames have been studied by some authors recently [1, 2].

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In this article, we study cases in which new frames and Riesz bases can be obtained by using tensor products and tensor sums and discuss their properties.

2. DEFINITIONS AND BASIC RESULTS

We recall the definitions and basic properties of frames and bases. For more information we refer to the survey article by Casazza [5] and the book by Christensen [6].

Let H be a separable Hilbert space and let I be an indexing set. A family $\{f_i\}_{i\in I}$ is a frame for H if there exists constants $0 < A \leq B < \infty$ such that for all $f \in H$, $A||f||^2 \leq \sum_{i\in I} | < f, f_i > |^2 \leq B||f||^2$.

The constants A and B are called a lower and upper frame bound for the frame. Those sequences which satisfy only the upper inequality are called B - Bessel sequences.

A frame is A - tight if $A = B$. If $A = B = 1$ it is called a Parsevel frame. Every orthonormal basis is a tight frame with $A = B = 1$.

If $\{f_i\}_{i\in I}$ is a B - Bessel sequence we define its analysis operator $T : H \to l^2(I)$ by

$$T(f) = \{< f, f_i >\}_{i\in I}$$

Clearly $\{f_i\}_{i\in I}$ is a frame if and only if T is invertible on H.

The adjoint of T is the synthesis operator given by

$$T^*(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i f_i$$

If the analysis operator is bounded, we define the frame operator by $S = T^*T$. Clearly S is positive, self - adjoint operator which is invertible on H if and only if $\{f_i\}_{i\in I}$ is a frame for H.

A collection of vectors in a Hilbert space is a Riesz basis if it is the image of an orthonormal basis under an invertible linear transformation.

We now recall the definitions and basic results of tensor products. For more information we refer to [9].

Let H and K be non - zero complex Hibert spaces. The tensor product of $x \in H$ and $y \in K$ is a conjugate bilinear functional $x \otimes y : H \otimes K \to C$ defined by

$$(x \otimes y)(u, v) =< x, u >< y, v > \text{ for every } (u, v) \in H \otimes K$$

The tensor product space $H \otimes K$ is the completion of the inner product space consisting of all sums of single tensors, which is a Hilbert space with respect to the inner product

$$< \sum_i (x_i \otimes y_i), \sum_j (w_j \otimes z_j) >= \sum_i \sum_j < x_i, w_j >< y_i, z_j >$$

for every $\sum_i (x_i \otimes y_i)$ and $\sum_j (w_j \otimes z_j)$ in $H \otimes K$. The norm on $H \otimes K$ is the one generated by the above inner product.

Let $B(H)$ be the Banach algebra of all bounded linear operators on H. For $A \in B(H)$ and $B \in B(K)$, the tensor product $A \otimes B : H \otimes K \to H \otimes K$ is defined by

$$(A \otimes B) \sum_i (x_i \otimes y_i) = \sum_i (Ax_i \otimes By_i)$$

for every $\sum_i (x_i \otimes y_i) \in H \otimes K$ which belongs to $B(H \otimes K)$.

We list some basic properties of tensor product of operators.

Proposition 2.1. *For every $\alpha, \beta \in \mathbb{C}$, $A, A_1, A_2 \in B(H)$ and $B, B_1, B_2 \in B(K)$ we have*
a) $\alpha\beta(A \otimes B) = \alpha A \otimes \beta B$.
b) $(A_1 + A_2) \otimes (B_1 + B_2) = (A_1 \otimes B_1) + (A_2 \otimes B_1) + (A_1 \otimes B_2) + (A_2 \otimes B_2)$
c) $A_1 A_2 \otimes B_1 B_2 = (A_1 \otimes B_1)(A_2 \otimes B_2)$
d) $(A \otimes B)^* = A^* \otimes B^*$
e) $||A \otimes B|| = ||A|| ||B||$
If A and B are invertible then so is $A \otimes B$ and
f) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

3. TENSOR PRODUCT OF FRAMES

In this section, we show that we can form new frames from the given frames on different Hilbert spaces.

We start with a known result [2].

Theorem 3.1. *Let $\{f_i\}_{i \in I}$ and $\{g_j\}_{j \in J}$ be frames for H and K respectively with frame bounds $A \leq C$ and $B \leq D$ respectively. Then $\{f_i \otimes g_j\}_{i \in I, j \in J}$ is a frame for $H \otimes K$ with frame bounds $AB \leq CD$. Moreover, if S_1 and S_2 are the frame operators for $\{f_i\}_{i \in I}$, $\{g_j\}_{j \in J}$ respectively then $S_1 \otimes S_2$ is the frame operator for $\{f_i \otimes g_j\}_{i \in I, j \in J}$.*

Proof. Since,

$$A||f||^2 \leq \sum_{i \in I} | < f, f_i > |^2 \leq B||f||^2 \text{ for } f \in H \text{ and}$$
$$C||g||^2 \leq \sum_{j \in J} | < g, g_j > |^2 \leq D||g||^2 \text{ for } g \in K.$$

We have

$$AC||f||^2||g||^2 \leq \sum_{i \in I} | < f, f_i > |^2 \sum_{j \in J} | < g, g_j > |^2 \leq BD||f||^2||g||^2$$

showing that

$$AC||f \otimes g||^2 \leq \sum_{i \in I, j \in J} | < f \otimes g, f_i \otimes g_j > |^2 \leq BD||f \otimes g||^2$$

Moreover, $AI_1 \leq S_1 \leq CI_1$ and $BI_2 \leq S_2 \leq DI_2$ imply that

$$AB(I_1 \otimes I_2) \leq S_1 \otimes S_2 \leq CD(I_2 \otimes I_1)$$

where I_1 and I_2 are identity operators on H and K respectively. □

In the converse direction we have,

Theorem 3.2. *If $\{f_i \otimes g_j\}_{i \in I, j \in J}$ is a frame for $H \otimes K$ then $\{f_i\}_{i \in I}$ and $\{g_j\}_{j \in J}$ are frames for H and K respectively.*

Proof. Since,

$$A||f \otimes g||^2 \leq \sum_{i\in I,j\in J} | < f \otimes g, f_i \otimes g_j > |^2 \leq B||f \otimes g||^2$$

We have

$$A||f||^2||g||^2 \leq \sum_{i\in I} | < f, f_i > |^2 \sum_{j\in J} | < g, g_j > |^2 \leq B||f||^2||g||^2$$

Take $g = g_j$ for some fixed j to get $\{f_i\}_{i\in I}$ as a frame for H. □

Proposition 3.3. [10] *Let $\{f_i\}_{i\in I}$ be a frame for H with frame operator S, frame bounds $A \leq B$ and let $L : H \to H$ be a bounded operator. Then $\{Lf_i\}_{i\in I}$ is a frame for H if and only if L is surjective. Moreover, in this case the frame operator for $\{Lf_i\}_{i\in I}$ is LSL^* and the new frame bounds are $A||L^\dagger||^{-2}$ and $B||L||^2$ where $L^\dagger$ is the pseudo - inverse of L.*

This result is now extended to the tensor product case.

Theorem 3.4. *Let $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ be frames for H and K, with frame operators S_1 and S_2, frame bounds $A \leq C$ and $B \leq D$ respectively and let $L_1 : H \to H$ and $L_2 : K \to K$ be bounded operators. Then $\{L_1f_i\}_{i\in I}$ and $\{L_2g_j\}_{j\in J}$ are frames for H and K respectively if and only if $\{(L_1 \otimes L_2)(f_i \otimes g_j)\}_{i\in I,j\in J}$ is a frame for $H \otimes K$ with frame bounds $||L_1^\dagger||^{-2}||L_2^\dagger||^{-2}AB$, $||L_1||^2||L_2||^2CD$ and frame operator $L_1S_1L_1^* \otimes L_2S_2L_2^*$ where $L_i^\dagger$ is the pseudo - inverse of L_i for $i = 1, 2$.*

Proof. By Proposition 3.3, $\{L_1f_i\}_{i\in I}$ and $\{L_2g_j\}_{j\in J}$ are frames for H and K respectively if and only if $L_1$and L_2 are surjective on H and K respectively, if and only if $L_1 \otimes L_2$ is surjective on $H \otimes K$ if and only if $\{(L_1 \otimes L_2)(f_i \otimes g_j)\}_{i\in I,j\in J}$ is a frame for $H \otimes K$. Moreover, if $L_1 \otimes L_2$ is surjective, then

$$\sum_{i\in I,j\in J} | < (f \otimes g), (L_1 \otimes L_2)(f_i \otimes g_j) > |^2$$

lies between $||L_1^\dagger||^{-2}||L_2^\dagger||^{-2}AB||f \otimes g||^2$ and $||L_1||^2||L_2||^2CD||f \otimes g||^2$ as desired. Further, the frame operator of $\{(L_1 \otimes L_2)(f_i \otimes g_j)\}_{i\in I,j\in J}$ is given by

$$\begin{aligned}
&\sum_{i\in I,j\in J} < (f \otimes g), (L_1 \otimes L_2)(f_i \otimes g_j) > (L_1 \otimes L_2)(f_i \otimes g_j) \\
&\qquad = L_1[\sum_{i\in I} < L_1^*f, f_i > f_i] \otimes L_2[\sum_{j\in J} < L_2^*g, g_j > g_j] \\
&\qquad = (L_1S_1L_1^* \otimes L_2S_2L_2^*)(f \otimes g) \\
&\qquad = (L_1 \otimes L_2)(S_1 \otimes S_2)(L_1 \otimes L_2)^*(f \otimes g).
\end{aligned}$$

□

Remark 3.5. Taking $L_1 = I_1$, the identity on H and $L_2 = I_2$, the identity on K, we get Theorem 3.1. Again, using associative property of tensor product and by induction, the theorem can be extended to n frames for n Hilbert spaces.

We now give simple necessary and sufficient condition on the Bessel sequences $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ and the operators L_1 and L_2 on the Hilbert spaces H and K so that $\{(L_1 \otimes L_2)(f_i \otimes g_j)\}_{i\in I,j\in J}$ is a Riesz basis for $H \otimes K$.

Theorem 3.6. *Let $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ be Bessel sequences in H and K, with analysis operators T_1 , T_2 and frame operators S_1, S_2 respectively. Also let L_1 :*

$H \to H$ and $L_2 : K \to K$. Then the followings are equivalent
(1) $\{(L_1 \otimes L_2)(f_i \otimes g_j)\}_{i\in I, j\in J}$ is a Riesz basis for $H \otimes K$
(2) $T_1L_1^* \otimes T_2L_2^*$ is invertible on $H \otimes K$.

Proof. (1) $\Leftrightarrow$ (2)
$(L_1 \otimes L_2)(f_i \otimes g_j)$ is a Riesz basis for $H \otimes K$ if and only if its analysis operator T on $H \otimes K$ is invertible where

$$\begin{aligned} T(f \otimes g) &= \{< (f \otimes g), (L_1 \otimes L_2)(f_i \otimes g_j) >\}_{i\in I, j\in J} \\ &= \{< f, L_1 f_i >< g, L_2 g_j \}_{i\in I, j\in J} \\ &= (T_1L_1^* \otimes T_2L_2^*)(f \otimes g) \end{aligned}$$

Since $T_1L_1^* \otimes T_2L_2^* = (T_1 \otimes T_2)(L_1 \otimes L_2)^*$ and T_1 , T_2 are invertible, it follows that $L_1 \otimes L_2$ is invertible on $H \otimes K$. □

Theorem 3.7. *Let $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ be Riesz bases for H and K, with analysis operators T_1 and T_2 and Riesz basis bounds $A \le C$ and $B \le D$ respectively. Let $L_1 : H \to H$ and $L_2 : K \to K$ be bounded operators. Then $\{L_1 f_i\}_{i\in I}$ and $\{L_2 g_j\}_{j\in J}$ are Riesz bases for H and K respectively if and only if $\{(L_1 \otimes L_2)(f_i \otimes g_j)\}_{i\in I, j\in J}$ is a Riesz basis for $H \otimes K$ with analysis operators $T_1L_1^* \otimes T_2L_2^*$ and Riesz basis bounds $||L_1^{-1}||^{-2}||L_2^{-1}||^{-2}AB$, $||L_1||^2||L_2||^2CD$.*

Proof. By [10, Proposition 2.8], $\{L_1 f_i\}_{i\in I}$ and $\{L_2 g_j\}_{j\in J}$ are Riesz bases if and only if L_1 and L_2 are invertible on $H \otimes K$ if and only if $\{(L_1 \otimes L_2)(f_i \otimes g_j)\}_{i\in I, j\in J}$ is a Riesz basis for $H \otimes K$. □

Remark 3.8. $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ are Riesz bases for H and K respectively if and only if $\{f_i \otimes g_j\}_{i\in I, j\in J}$ is a Riesz basis for $H \otimes K$.

4. TENSOR SUMS AND FRAMES

Let A and B be arbitrary operators on H and K respectively. The tensor sum of A and B is the transformation $A \oplus B : H \otimes K \to H \otimes K$, defined by

$$A \oplus B = (A \otimes I) + (I \otimes B)$$

which is an operator in $B(H \otimes K)$.

Tensor sum is not commutative and we have

Proposition 4.1. [9] *For every $\alpha, \beta \in \mathbb{C}$, $A, A_1, A_2 \in B(H)$ and $B, B_1, B_2 \in B(K)$*
a) $(\alpha + \beta)(A \oplus B) = \alpha A \oplus \beta B + \beta A \oplus \alpha B$.
b) $(A_1 + A_2) \oplus (B_1 + B_2) = (A_1 \oplus B_1) + (A_2 \oplus B_1)$
c) $(A_1 \oplus B_1)(A_2 \oplus B_2) = A_1B_2 + A_2B_1 + A_1(A_2 \oplus B_1B_2$
d) $(A \oplus B)^* = A^* \oplus B^*$
e) $||A \oplus B|| \le ||A|| + ||B||$

Let $L_1 : H \to H$ and $L_2 : K \to K$ be bounded operators and let $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ be frames for H and K respectively. If $\{L_1 f_i\}_{i\in I}$ and $\{L_2 g_j\}_{j\in J}$ are frames for H and K respectively then by Theorem 3.4, $\{(L_1 \otimes L_2)(f_i \otimes g_j)\}_{i\in I, j\in J}$ is a frame for $H \otimes K$. The following example shows that $\{(L_1 \oplus L_2)(f_i \otimes g_j)\}_{i\in I, j\in J}$ need not be a frame for $H \otimes K$.

Example 4.2. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for a Hilbert space H. Define a shift operator L on H by $L(e_1) = 0$, and $L(e_n) = e_{n-1}$ if $n > 1$. Then $\{L(e_n)\}_{n=1}^{\infty}$ is a frame for H. Take $K = H$, $L_1 = L_2 = L$ and $f_n = g_n = e_n$ for each $n \in N$ so that $\{L_1(f_n)\}_{n=1}^{\infty}$ and $\{L_2(g_n)\}_{n=1}^{\infty}$ are frames. Since the analysis operator of $\{(L_1 \oplus L_2)(f_n \otimes g_n)\}_{n=1}^{\infty}$ is not surjective, it is not a frame.

Theorem 4.3. *Let $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ be Bessel sequences in H and K, with analysis operators T_1 , T_2 and frame operators S_1, S_2 respectively. Also let $L_1 : H \to H$ and $L_2 : K \to K$. Then the followings are equivalent*
(1) $\{(L_1 \oplus L_2)(f_i \otimes g_j)\}_{i\in I, j\in J}$ is a Riesz basis for $H \otimes K$
(2) $T_1L_1^* \otimes T_2 + T_1 \otimes T_2L_2^*$ is invertible on $H \otimes K$.

Proof. (1) $\Leftrightarrow$ (2)
$\{(L_1 \oplus L_2)(f_i \otimes g_j)\}$ is a Riesz basis for $H \otimes K$ if and only if its analysis operator T is invertible on $H \otimes K$ where

$$\begin{aligned} T(f \otimes g) &= \{< (f \otimes g), (L_1 \oplus L_2)(f_i \otimes g_j) >\}_{i\in I, j\in J} \\ &= \{< f \otimes g, L_1f_i \otimes g_j > + < f \otimes g, f_i \otimes L_2g_j >\}_{i\in I, j\in J} \\ &= \{< L_1^*f, f_i >< g, g_j > + < f, f_i >< L_2^*g, g_j >\}_{i\in I, j\in J} \\ &= ((T_1L_1^* \otimes T_2 + (T_1 \otimes T_2L_2^*))(f \otimes g)) \end{aligned}$$

Since $T = T_1L_1^* \otimes T_2 + T_1 \otimes T_2L_2^* = (T_1 \otimes T_2)(L_1 \oplus L_2)^*$ it follows that T is invertible if and only if $L_1 \oplus L_2$ is invertible. □

Corollary 4.4. *Let $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ be Riesz bases for H and K, with analysis operators T_1 and T_2 and Riesz basis bounds $A \le C$ and $B \le D$ respectively. Also let $L : H \to H$ and $O : K \to K$. Then $\{(L \oplus O)(f_i \otimes g_j)\}_{i\in I, j\in J}$ is a Riesz basis for $H \otimes K$ if and only if $\{Lf_i\}_{i\in I}$ is a Riesz basis for H.*

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