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# ULAM-HYERS STABILITY ANALYSIS FOR HYBRID $\phi$-CAPUTO TIME-FRACTIONAL SYSTEMS OF ORDER $\delta \in (1;2]$

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Abstract. This study sought to explore the existence, uniqueness, and Ulam-Hyers stability results for a class of hybrid $\phi$-Caputo time-fractional systems of order $\delta$ in $(1, 2]$. The desired conclusions were achieved by applying well-known fixed-point theorems. We study the four different types of Ulam-Hyers stability for the proposed fractional problems. An example is provided at the end to demonstrate the effectiveness of the results.

## 1. Introduction

In recent decades, there has been a significant increase in interest in fractional calculus, largely due to its wide-ranging applications in various fields. This heightened interest can be attributed to its inherent capability to extend classical calculus to non-integer orders, making it essential for accurately modeling complex phenomena. In academic literature, fractional differential operators are described through a variety of methodologies, incorporating approaches such as Riesz, Caputo, Hilfer, Hadamard,and Riemann-Liouville, among others (see [12, 9, 1]). It's worth noting that the Caputo and Riemann-Liouville definitions are the most commonly used in practice. The Riemann-Liouville approach entails examining the limiting behavior of fractional derivatives within initial conditions, albeit with challenges in interpretation. Lately, researchers have been increasingly interested in generalized fractional calculus. Particularly noteworthy are the contributions of Almeida [3], who has proposed a new definition for the Caputo fractional derivative. This novel approach entails examining the derivative of a function with respect to another function, represented by $\phi$. By choosing a suitable function $\phi$, this method can greatly enhance model accuracy [19]. Alongside introducing this new definition, Almeida performed extensive research on the numerous properties of fractional calculus. In contemporary times, the application of fractional calculus has expanded into numerous domains of technological expertise, including biology, physics, chemistry, mechanics, engineering, bioengineering and electrochemistry (see [2, 5, 7, 13, 15]). The theory

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of fractional evolution equations is one area of study within fractional calculus. It is predicated on the description of equations that undergo changes throughout time. Readers are directed to references [17, 16] for further information on the theory of fractional evolution problems. In the realm of fractional differential equations, a crucial consideration regarding solution stability is the concept of data dependence. This notion, recognized as the Ulam-Hyers stability, has garnered considerable attention as a pivotal topic, as documented in references [11]. The fixed point theorem, along with the existence and stability of Ulam-Hyers solution to the system of fractional differential equations of order $\delta \in (0,1)$, was established by Suechoei and Sa Ngiamsunthorn [17].

$$\begin{cases} {}^{\mathbb{C}}D^{\delta,\phi}_{0^+}\Im(x) = A\Im(x) + \hbar(x,\Im(x)), \qquad x \in \xi = [0,X], \\ \Im(0) = \Im_f, \end{cases}$$

where${}^{\mathbb{C}}D^{\delta,\phi}_{0^+}$ denotes the $\phi$-Caputo fractional derivative with , A is the infinitesimal generator of a $C_0-$semi group $\{T(x)\}_{x\geq 0}$ on a Banach space E, and $\hbar : [0,\infty) \times E \to E$.

Based on the aforementioned works, we analyze the following system of $\phi-$Caputo fractional semilinear of order $1 < \delta \leq 2$ in a Banach space $E$.

$$\begin{cases} {}^{\mathbb{C}}D^{\delta,\phi}_{0^+}\left(\dfrac{\Im(x)}{\aleph(x,\Im(x))}\right) = A\left(\dfrac{\Im(x)}{\aleph(x,\Im(x))}\right) + \hbar(x,\Im(x)), \qquad x \in \xi = [0,X], \\ \dfrac{d}{dx}\left(\dfrac{\Im(0)}{\aleph(0,\Im(0))}\right) = \Im_d \quad \dfrac{\Im(0)}{\aleph(0,\Im(0))} = \Im_f, \end{cases} \tag{1.1}$$

where ${}^{\mathbb{C}}D^{\delta,\phi}_{0^+}$ denotes the $\phi$-Caputo fractional derivative with $\delta \in (1,2]$, $A$ represents the infinitesimal generator of a strongly continuous cosine family of uniformly bounded linear operators $C(s)_{s\geq 0}$ on a Banach space $E$, $X$ is a positive real number, $\aleph$ is a function in $\mathbb{C}(\xi \times \mathbb{R}, \mathbb{R} \setminus 0)$, and $\hbar$ is a function in $\mathbb{C}_c(\xi \times \mathbb{R}, \mathbb{R})$.

This paper is structured as follows: Section 2 introduces fundamental definitions and properties of $\phi$-Caputo fractional derivatives and functional analysis, which will be referenced throughout. Section 3 presents the mild solution to problem (1.1), with a focus on the existence of such solutions for the problem. Section 4 explores various stability results for the solutions to problem (1.1), and Section 5 provides an illustrative example. The paper concludes with Section 6.

## 2. Preliminaries

This part, we reflect on essential definitions, notations and results on $\phi$-fractional integrals and $\phi$-fractional derivatives, as well as early findings that will be utilized in later proofs. Given $X$ as a strictly positive real number, let $\xi = [0,X]$ represent a time interval of the real line $\mathbb{R}$. We denote by $L(E)$ the space of all linear bounded operator from E to itself and $\mathbb{C}(\xi,E)$ the Banach space of all continuous functions on this interval, equipped with the norm

$$\|\aleph\| = \sup\{|\aleph(x)|, x \in \xi\}.$$

We denote by $\mathbb{C}_c(\xi \times E, E)$ the class of all functions $\hbar : \xi \times E \to E$ such that for each $\Im \in E$, the map $x \to \hbar(x,\Im)$ is measurable and for every $x \in \xi$, the map

$\Im \to \hbar(x, \Im)$ is continuous . Let $A : D(A) \subset E \to E$ be an operator with $\rho(A)$ the resolvent set, and we use $D(A)$ to denote the domain of $A$.

**Definition 2.1.** [4] Let $\delta > 0$, $\Im \in \mathfrak{L}^1(\xi, \mathbb{R})$ and $\phi \in \mathbb{C}^n(\xi, \mathbb{R})$ where $\phi'(x) > 0$ for every $x \in \xi$. The fractional integral of order $\delta$ in the context of the $\phi$-Riemann-Liouville integral defined by

$$\mathfrak{I}_{0^+}^{\delta,\phi}\Im(x) = \frac{1}{\Gamma(\delta)} \int_0^x \phi'(s)(\phi(x) - \phi(s))^{\delta-1}\Im(s)\, ds.$$

**Definition 2.2.** [4] Let $\delta > 0$, $\Im \in \mathbb{C}^{n-1}(\xi, \mathbb{R})$ and $\phi \in \mathbb{C}^n(\xi, \mathbb{R})$ where $\phi'(x) > 0$ for every $t \in \xi$. The fractional derivative af order $\delta$ in the sens of the $\phi$-Caputo operator for the function $\Im$ with respect to function $\phi$ is defined by

$${}^{\mathbb{C}}D_{0^+}^{\delta,\phi}\Im(x) = \frac{1}{\Gamma(n-\delta)} \int_0^x \phi'(s)(\phi(x) - \phi(s))^{n-\delta-1}\Im_\phi^{[n]}(s)ds,$$

Where

$$\Im_\phi^{[n]}(s) = \left(\frac{1}{\phi'(s)}\frac{d}{ds}\right)^n \Im(s) \quad \textit{and} \quad n = [\delta] + 1.$$

And $[\delta]$ represents the integer part of $\delta$.

**Lemma 2.3.** [4] *Given $\delta$ as a strictly positive real number and $\Im \in \mathbb{C}^{n-1}(\xi, \mathbb{R})$. Then, we have*

(1) $\mathfrak{I}_{0^+}^{\delta,\phi}$ *is linear operator and bounded from $\mathbb{C}(\xi, \mathbb{R})$ to $\mathbb{C}(\xi, \mathbb{R})$.*
(2) ${}^{C}D_{0^+}^{\delta,\phi}\mathfrak{I}_{0^+}^{\delta,\phi}\Im(x) = \Im(x)$,
(3) $\mathfrak{I}_{0^+}^{\delta,\phi\mathbb{C}}D_{0^+}^{\delta,\phi}\Im(x) = \Im(x) - \sum_{k=0}^{n-1} \frac{\Im_\phi^{[k]}(0)\,(\phi(x)-\phi(0))^k}{k!}$,

Let us present the definition and some significant results concerning the sine and cosine families.

**Definition 2.4.** [18] A $C_0$-cosine operator function is given as a family of one-parameter operators $\{C(s),\ s \in \mathbb{R}\}$, having these properties

(1) $C(0) = \mathbb{I}$, $\mathbb{I}$ denotes the identity operator on the set $E$,
(2) $s \to C(s)y$ is continuous on $\mathbb{R}$ for every $y$ in $E$,
(3) $C(s + \tau) + C(s - \tau) = 2C(s)C(\tau)$ for any $\tau, s \in \mathbb{R}$.

If $(C(s))_{s\in\mathbb{R}}$ is an infinitesimal generating operator $A$ is given by

$$D(A) = \left\{y \in E \setminus C(s)y \in \mathbb{C}^2(\mathbb{R}, E)\right\}$$

and

$$Ay = \frac{d^2}{ds}C(s)y.$$

We define $(S(x))_{x\in\mathbb{R}}$ is the sine operator function associated with $(C(s))_{s\in\mathbb{R}}$, the strongly continuous cosine family as follows

$$S(x)y = \int_0^x C(\tau)yd\tau,\ y \in E,\ x \in \mathbb{R}.$$

**Proposition 2.5.** [18] *Let $(S(x))_{x\in\mathbb{R}}$ be the sine family associated with $(C(s))_{s\in\mathbb{R}}$, the strongly continuous cosine family. Then*

- *If $S(x)y$ in $D(A)$ for every $y \in E$. Then, we have* $\frac{d}{dx}C(x) = AS(x)y$,
- $\lim\limits_{h\to 0} \dfrac{S(h)y}{h} = y$ *for any* $y \in E$.

**Lemma 2.6.** [18]

*Consider $A$ as the infinitesimal generator of the family $(C(s))_{s\in\mathbb{R}}$. Then, for each $\lambda \in \rho(A)$ with $Re(\lambda) > 0$ and $\lambda^2 \in \rho(A)$, the following hold*

- $\lambda R(\lambda^2; A)y = \displaystyle\int_0^\infty exp(-\lambda s)C(s)y\, ds,$
- $R(\lambda^2; A)y = \displaystyle\int_0^\infty exp(-\lambda x)S(x)y\, dx,$

  *where $y \in E$ and $R(\lambda; A) = (\lambda I - A)^{-1}$.*

In this study, we consider $(C(s))_{s\in\mathbb{R}}$, the strongly continuous cosine family such that there exists a constant $\mathfrak{M} \geq 1$ for any $s \in \mathbb{R}$, we have $\|C(s)\|_{L^1(E)} \leq 1$. Moreover, $\|S(x_1) - S(x_2)\| \leq \mathfrak{M}|x_1 - x_2|$ for all $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$.

**Definition 2.7.** [10] The generalized Laplace transform of a real valued function $\Im : [0, \infty[ \to \mathbb{R}$ is defined by

$$\mathfrak{L}_\phi(\Im(x))(\lambda) = \int_0^\infty \phi'(s)\, exp(-\lambda(\phi(s) - \phi(0)))\Im(s)ds,$$

for all values $\lambda$.

**Definition 2.8.** [10] Let $u$ and $\Im$ be two functions that are piecewise continuous with exponential order on $\xi$. We define the generalized $\phi$-convolution of $\Im$ and $u$ as follows

$$(\Im *_\phi u)(x) = \int_0^x \Im(s)(\phi^{-1}(\phi(x) + \phi(0) - \phi(s))\phi'(s)\, ds.$$

**Lemma 2.9.** [10]*Let $\delta > 0$, $\Im$, and $u$ be two piecewise continuous functions on each interval $\xi$, and suppose that $\Im$ and $u$ are of $\phi(x)$-exponential order. Then, we have*

- $\mathfrak{L}_\phi\{(1)\}(\lambda) = \frac{1}{\lambda}$;
- $\mathfrak{L}_\phi(\mathfrak{I}^{\delta,\phi}_{0^+}\Im(x))(\lambda) = \frac{\mathfrak{I}^{\delta,\phi}_{0^+}\Im(x)}{\lambda^\delta}$;
- $\mathfrak{L}_\phi((\Im *_\phi u)(x))(\lambda) = \mathfrak{L}_\phi(\Im(x))(\lambda) \times \mathfrak{L}_\phi(u(x))(\lambda)$.

**Definition 2.10.** [14] Let $0 < \beta < 1$ and $z > 0$. We define the Wright-type function by

$$\Theta_\beta(z) = \sum_{i=0}^{+\infty} \frac{(-z)^i}{i!\Gamma(-\beta(i+1)+1)}$$
$$= \sum_{i=0}^{+\infty} \frac{(-z)^i\Gamma(\beta(i+1))sin(\pi\beta(k+1))}{i!}.$$

**Lemma 2.11.** [14] *Let $\Theta_\beta$. The Wright function, then we have the following properties:*

- $\Theta_\beta(z) > 0$ *for all* $z \geq 0$;

- $\int_0^{+\infty} z^n \Theta_\beta(z)dz = \dfrac{\Gamma(1+n)}{\Gamma(1+n\beta)}$, $n > -1$, $0 < \beta < 1$.

We define the multiplication operator in $E$ as follows

$$(\Im u)(t) = \Im(x)u(t), \quad \text{for every} \quad \Im, u \in E.$$

It is obvious that $E = \mathbb{C}(\xi; \mathbb{R})$ forms a Banach algebra with respect to the above norm and multiplication.

**Theorem 2.12.** [6]*Let $\boldsymbol{S}$ be a non-empty, convex, closed, and bounded subset of a Banach algebra $E$. Consider two operators $\Upsilon : E \to E$ and $\phi : \boldsymbol{S} \to E$ such that*

(1) *$\phi$ is completely continuous,*
(2) *$\Upsilon$ is Lipschitzian with a Lipschitz constant $\gamma$,*
(3) *$\Im = \Upsilon\Im_1\phi\Im_2 \Rightarrow \Im_1 \in \boldsymbol{S}$ for all $\Im_2 \in \boldsymbol{S}$,*
(4) *$\gamma K < 1$ where $K = \|\phi(\boldsymbol{S})\| = \sup\{|\phi(\Im)|, \Im \in \boldsymbol{S}\}$.*
*Then, the equation $\Upsilon\Im\phi\Im = \Im$ has a solution in $\boldsymbol{S}$.*

## 3. Construction of mild solution using semigroup

In this partition, we formulate a new version of the mild solution for the system (1.1) by utilizing a probability density.

**Definition 3.1.** [8] Let $z$ be a positive real number. The one-sided stable probability density is defined as follows

$$\omega_\beta(z) = \frac{1}{\pi}\sum_{i=0}^{+\infty} \frac{(-1)^{i-1}(-z)^{-\beta i-1}\Gamma(i\beta+1)sin(i\pi\beta)}{i!}.$$

**Lemma 3.2.** [8] *The Laplace transform of the function $\omega_\beta(z)$ is given as follows*

$$\int_0^{+\infty} exp(-\lambda z)\omega_\beta(z)dz = exp(-\lambda^\beta), \ \beta \in (0,1).$$

**Lemma 3.3.** *The problem (1.1) is equivalent to the equation given below*

$$\Im(x) = \aleph(x, \Im(x))\left[\Im_f + \frac{\phi(x)-\phi(0)}{\phi'(0)}\Im_d + \mathfrak{I}_{0^+}^{\delta,\phi}\left(A\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right) + \hbar(x,\Im(x))\right)\right] \tag{3.1}$$

*Proof.* Let $\Im$ be a solution of the system (1.1). By applying the $\phi$-fractional integral operator to both sides of (1.1) and using Lemma 2.3, we can get

$$\begin{aligned}\frac{\Im(x)}{\aleph(x,\Im(x))} &= \frac{\Im(0)}{\aleph(0,\Im(0))} + \frac{\phi(x)-\phi(0)}{\phi'(0)}\frac{d}{dx}\left(\frac{\Im(0)}{\aleph(0,\Im(0))}\right) \\ &\quad + \mathfrak{I}_{0^+}^{\delta,\phi}\left(A\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right) + \hbar(x,\Im(x))\right) \\ &= \Im_f + \frac{\phi(x)-\phi(0)}{\phi'(0)}\Im_d + \mathfrak{I}_{0^+}^{\delta,\phi}\left(A\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right) + \hbar(x,\Im(x))\right).\end{aligned}$$

Thus, (3.1) holds. Now, assume that $\Im$ satisfies the (3.1), Applying the $\phi$-Caputo fractional operator derivative at order $\delta$, we obtain the first equation of (1.1).

By substituting $x = 0$ into (3.1), we get $\frac{\Im(0)}{\aleph(0,\Im(0))} = \Im_f$
and $\frac{d}{dx}\left(\frac{\Im(0)}{\aleph(0,\Im(0))}\right) = \frac{\phi'(0)}{\phi'(0)}\Im_d = \Im_d$. $\square$

**Theorem 3.4.** *If* (3.1) *holds, then*

$$\begin{aligned}\Im(x) &= \aleph(x,\Im(x)) \\ &\times \left[K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\,\hbar(s,\Im(s))ds\right],\end{aligned} \tag{3.2}$$

*where*

- $K_\beta^\phi(x,0)y = \displaystyle\int_0^{+\infty} \Theta_\beta(s)C((\phi(x)-\phi(0))^\beta s)yds$, $\beta = \frac{\delta}{2}$, $y \in E$,
- $L_\beta^\phi(x,0)y = \displaystyle\int_0^x K_\beta^\phi(x,0)\frac{\phi'(s)}{\phi(0)}yds$,
- $M_\beta^\phi(x,s)y = \displaystyle\int_0^{+\infty} \beta\theta\Theta_\beta(\theta)S((\phi(x)-\phi(s))^\beta\theta)yd\theta$.

*Proof.* Let $\lambda > 0$ with $\lambda^\beta \in \rho(A)$. By applying the generalized Laplace transform to equation (3.2), we conclude that

$$\begin{aligned}\mathfrak{L}_\phi\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right)(\lambda) &= \mathfrak{L}_\phi(\Im_f)(\lambda) \\ &+ \mathfrak{L}_\phi(\frac{\phi(x)-\phi(0)}{\phi'(0)}\Im_d)(\lambda) \\ &+ \mathfrak{L}_\phi\left(\mathfrak{I}_{0^+}^{\delta,\phi}\left(A\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right) + \hbar(x,\Im(x))\right)\right)(\lambda).\end{aligned}$$

Using Lemma 2.9 and Definition 2.7, we obtain that $\mathfrak{L}_\phi(\Im_f)(\lambda) = \frac{\Im_f}{\lambda}$, $\mathfrak{L}_\phi(\frac{\phi(x)-\phi(0)}{\phi'(0)}\Im_d)(\lambda) = \frac{\Im_d}{\lambda^2\phi'(0)}$ and

$$\begin{aligned}\mathfrak{L}_\phi\left(\mathfrak{I}_{0^+}^{\delta,\phi}\left(A\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right) + \hbar(x,\Im(x))\right)\right)(\lambda) &= \frac{1}{\lambda^\delta}A\mathfrak{L}_\phi\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right) \\ &+ \frac{1}{\lambda^\delta}\mathfrak{L}_\phi(\hbar(x,\Im(x))(\lambda).\end{aligned}$$

Then,

$$\begin{aligned}
&\mathfrak{L}_\phi\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right)(\lambda)\\
&=\frac{\Im_f}{\lambda}+\frac{\Im_d}{\lambda^2\phi'(0)}+\frac{1}{\lambda^\delta}A\mathfrak{L}_\phi\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right)+\frac{1}{\lambda^\delta}\mathfrak{L}_\phi(\hbar(x,\Im(x))(\lambda)\\
&=\lambda^{\delta-1}(\lambda^\delta I-A)^{-1}\Im_f+\frac{\lambda^{\delta-2}}{\phi'(0)}(\lambda^\delta I-A)^{-1}\Im_d+(\lambda^\delta I-A)^{-1}\mathfrak{L}_\phi(\hbar(x,\Im(x))(\lambda)\\
&=\int_0^{+\infty}\lambda^{\beta-1}exp(-\lambda^\beta x)C(x)\Im_f dx+\frac{1}{\lambda}\int_0^{+\infty}\lambda^{\beta-1}exp(-\lambda^\beta x)C(x)\frac{\Im_d}{\phi'(0)}dx\\
&+\int_0^{+\infty}exp(-\lambda^\beta x)S(x)\chi_\phi(\lambda)dx,\\
&\text{where}\beta=\frac{\delta}{2}\text{and}\chi_\phi(\lambda)=\mathfrak{L}_\phi(\hbar(x,\Im(x))(\lambda).
\end{aligned}$$

Let$x=(\phi(\iota)-\phi(0))^\beta$, we obtain

$$\begin{aligned}
&\mathfrak{L}_\phi\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right)(\lambda)\\
&=\int_0^{+\infty}\beta(\lambda(\phi(\iota)-\phi(0)))^{\beta-1}\phi'(\iota)exp(-(\lambda(\phi(\iota)-\phi(0)))^\beta)C((\phi(\iota)-\phi(0))^\beta)\Im_f d\iota\\
&+\frac{1}{\lambda}\int_0^{+\infty}\Big[\beta(\lambda(\phi(\iota)-\phi(0)))^{\beta-1}\phi'(\iota)\\
&\quad\times exp(-(\lambda(\phi(\iota)-\phi(0)))^\beta)C((\phi(\iota)-\phi(0))^\beta)\frac{\Im_d}{\phi'(0)}\Big]d\iota\\
&+\int_0^{+\infty}\beta(\phi(\iota)-\phi(0))^{\beta-1}\phi'(\iota)exp(-(\lambda(\phi(\iota)-\phi(0)))^\beta)S((\phi(\iota)-\phi(0)))^\beta)\chi_\phi(\lambda)d\iota\\
&=\int_0^{+\infty}\frac{-1}{\lambda}\frac{d}{d\iota}(exp(-(\lambda(\phi(\iota)-\phi(0)))^\beta)C((\phi(\iota)-\phi(0))^\beta)\Im_f d\iota\\
&+\frac{1}{\lambda}\int_0^{+\infty}\frac{-1}{\lambda}\frac{d}{d\iota}(exp(-(\lambda(\phi(\iota)-\phi(0)))^\beta)C((\phi(\iota)-\phi(0))^\beta)\frac{\Im_d}{\phi'(0)}d\iota\\
&+\int_0^{+\infty}\beta(\phi(\iota)-\phi(0))^{\beta-1}\phi'(\iota)exp(-(\lambda(\phi(\iota)-\phi(0)))^\beta)S((\phi(\iota)-\phi(0)))^\beta)\chi_\phi(\lambda)d\iota\\
&=\int_0^{+\infty}\frac{-1}{\lambda}\frac{d}{d\iota}\int_0^{+\infty}exp(-\lambda(\phi(\iota)-\phi(0))z)\omega_\beta(z)dzC((\phi(\iota)-\phi(0))^\beta)\Im_f d\iota\\
&+\frac{1}{\lambda}\int_0^{+\infty}\frac{-1}{\lambda}\frac{d}{d\iota}\int_0^{+\infty}exp(-\lambda(\phi(\iota)-\phi(0))z)\omega_\beta(z)dzC((\phi(\iota)-\phi(0))^\beta)\frac{\Im_d}{\phi'(0)}d\iota\\
&+\int_0^{+\infty}\beta(\phi(\iota)-\phi(0))^{\beta-1}\phi'(\iota)\int_0^{+\infty}exp(-\lambda(\phi(\iota)-\phi(0))z)\omega_\beta(z)dz\\
&\quad\times S((\phi(\iota)-\phi(0)))^\beta)\chi_\phi(\lambda)d\iota
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty}\int_0^{+\infty} z\phi'(\iota)exp(-\lambda(\phi(\iota)-\phi(0))z)\omega_\beta(z)C((\phi(\iota)-\phi(0))^\beta)\Im_f dzd\iota \\
&+ \int_0^{+\infty}\int_0^{+\infty} \frac{z}{\lambda}\phi'(\iota)exp(-\lambda(\phi(\iota)-\phi(0))z)\omega_\beta(z)C((\phi(\iota)-\phi(0))^\beta)\frac{\Im_d}{\phi'(0)}dzd\iota \\
&+ \int_0^{+\infty}\beta(\phi(\iota)-\phi(0))^{\beta-1}\phi'(\iota)\int_0^{+\infty} exp(-\lambda(\phi(\iota)-\phi(0))z)\omega_\beta(z)dz \\
&\quad S((\phi(\iota)-\phi(0)))^\beta)\chi_\phi(\lambda)d\iota \\
&= \int_0^{+\infty}\int_0^{+\infty} exp(-\lambda(\phi(\tau)-\phi(0)))\omega_\beta(z)C\left(\left(\frac{\phi(\tau)-\phi(0)}{z}\right)^\beta\right)\phi'(\tau)\Im_f dzd\tau \\
&+ \int_0^{+\infty}\int_0^{+\infty} \frac{\phi'(\tau)}{\lambda}exp(-\lambda(\phi(\tau)-\phi(0)))\omega_\beta(z)C\left(\left(\frac{\phi(\tau)-\phi(0)}{z}\right)^\beta\right)\frac{\Im_d}{\phi'(0)}dzd\tau \\
&+ \int_0^{+\infty}\int_0^{+\infty} \beta\frac{(\phi(\tau)-\phi(0))^{\beta-1}}{z^\beta}exp(-\lambda(\phi(\tau)-\phi(0)))\omega_\beta(z) \\
&\quad S\left(\left(\frac{\phi(\tau)-\phi(0)}{z}\right)^\beta\right)\phi'(\tau)\chi_\phi(\lambda)d\tau dz \\
&= \mathfrak{L}_\phi\left(\int_0^{+\infty}\omega_\beta(z)C\left(\left(\frac{\phi(x)-\phi(0)}{z}\right)^\beta\right)(\lambda)\Im_f dz\right) \\
&+ \mathfrak{L}_\phi\left(\int_0^{+\infty}\frac{1}{\lambda}\omega_\beta(z)C\left(\left(\frac{\phi(x)-\phi(0)}{z}\right)^\beta\right)\frac{\Im_d}{\phi'(0)}dz\right)(\lambda) \\
&+ \mathfrak{L}_\phi\left(\int_0^{+\infty}\beta\frac{(\phi(x)-\phi(0))^{\beta-1}}{z^\beta}\omega_\beta(z)S\left(\left(\frac{\phi(x)-\phi(0)}{z}\right)^\beta\right)\chi_\phi(\lambda)dz\right)(\lambda) \\
&= \mathfrak{L}_\phi\left(\int_0^{+\infty}\frac{1}{\beta}\sigma^{-1-\frac{1}{\beta}}\omega_\beta(\sigma^{-\frac{1}{\beta}})C((\phi(x)-\phi(0))^\beta\sigma)\Im_f d\sigma\right)(\lambda) \\
&+ \mathfrak{L}_\phi\left(\int_0^{+\infty}\frac{1}{\lambda}\frac{1}{\beta}\sigma^{-1-\frac{1}{\beta}}\omega_\beta(\sigma^{-\frac{1}{\beta}})C((\phi(x)-\phi(0))^\beta\sigma)\frac{\Im_d}{\phi'(0)}d\sigma\right)(\lambda) \\
&+ \mathfrak{L}_\phi\left(\int_0^{+\infty}\beta\sigma\frac{(\phi(x)-\phi(0))^{\beta-1}\sigma^{-1-\frac{1}{\beta}}}{\beta}\omega_\beta(t^{-\frac{1}{\beta}})S((\phi(x)-\phi(0))^\beta\sigma)\chi_\phi(\lambda)d\sigma\right)(\lambda)
\end{aligned}
$$

where $\Theta_\beta(\sigma) = \frac{1}{\beta}\sigma^{-1-\frac{1}{\beta}}\omega_\beta(\sigma^{-\frac{1}{\beta}})$ is the probability density function defined on $\mathbb{R}^+$. We have

$$\begin{aligned}
&\mathfrak{L}_\phi\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right)(\lambda)\\
&= \mathfrak{L}_\phi\left(\int_0^{+\infty}\Theta_\beta(\sigma)C((\phi(x)-\phi(0))^\beta\sigma)\Im_f d\sigma\right)(\lambda)\\
&+ \mathfrak{L}_\phi\left(\int_0^{+\infty}\frac{1}{\lambda}\Theta_\beta(\sigma)C((\phi(x)-\phi(0))^\beta\sigma)\frac{\Im_d}{\phi'(0)}d\sigma\right)(\lambda)\\
&+ \mathfrak{L}_\phi\left(\int_0^{+\infty}\beta\sigma(\phi(x)-\phi(0))^{\beta-1}\Theta_\beta(\sigma)S((\phi(x)-\phi(0))^\beta\sigma)\chi_\phi(\lambda)d\sigma\right)(\lambda)
\end{aligned}$$

Since $\mathfrak{L}_\phi(1)(\lambda) = \frac{1}{\lambda}$, by using lemma (2.9) we obtain that

$$\begin{aligned}
&\mathfrak{L}_\phi\left(\int_0^{+\infty}\frac{1}{\lambda}\Theta_\beta(\sigma)C((\phi(x)-\phi(0))^\beta\sigma)\frac{\Im_d}{\phi'(0)}d\sigma\right)(\lambda)\\
&= \mathfrak{L}_\phi(1)(\lambda)\times\mathfrak{L}_\phi\left(\int_0^{+\infty}\Theta_\beta(\sigma)C((\phi(x)-\phi(0))^\beta\sigma)\frac{\Im_d}{\phi'(0)}d\sigma\right)(\lambda)\\
&= \mathfrak{L}_\phi\Big(\int_0^x\int_0^\infty\Theta_\beta(\sigma)C[(\phi(\phi^{-1}(\phi(x)+\phi(0)-\phi(\tau)))-\phi(0)))^\beta\sigma))\phi'(\tau)\frac{\Im_d d\tau d\sigma}{\phi'(0)}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{L}_\phi\left(\int_0^{+\infty}\beta\sigma(\phi(x)-\phi(0))^{\beta-1}\Theta_\beta(\sigma)S((\phi(x)-\phi(0))^\beta\sigma)\chi_\phi(\lambda)d\sigma\right)(\lambda)\\
=&\mathfrak{L}_\phi\left(\int_0^{+\infty}\beta\sigma(\phi(x)-\phi(0))^{\beta-1}\Theta_\beta(\sigma)S((\phi(x)-\phi(0))^\beta\sigma)d\sigma\right)(\lambda)\times\chi_\phi(\lambda)\\
=&\mathfrak{L}_\phi(\int_0^t g(\phi^{-1}(\phi(x)-\phi(\tau)+\phi(0)),\Im(\phi^{-1}(\phi(x)-\phi(\tau)+\phi(0))))(\phi(\tau)-\phi(0))^{\beta-1}\\
&\times\int_0^\infty\beta\sigma\Theta_\beta(\sigma)S((\phi(\tau)-\phi(0))^\beta\sigma)d\sigma\phi'(\tau)d\tau)(\lambda)\\
=&\mathfrak{L}_\phi\big(\int_0^t(\phi(x)-\phi(s))^{\beta-1}g(s,\Im(s))\phi'(s)\int_0^\infty\beta\sigma\Theta_\beta(\sigma)S((\phi(x)-\phi(s))^\beta\sigma)d\sigma ds\big)(\lambda).
\end{aligned}$$

Then,

$$\begin{aligned}
&\mathfrak{L}_\phi\left(\frac{\Im(x)}{\aleph(x,\Im(x))}\right)(\lambda)\\
=&\mathfrak{L}_\phi\left(\int_0^{+\infty}\Theta_\beta(\sigma)C((\phi(x)-\phi(0))^\beta\sigma)\Im_f d\sigma\right)(\lambda)\\
+&\mathfrak{L}_\phi\big(\int_0^t\int_0^\infty\Theta_\beta(\sigma)C[(\phi(\phi^{-1}(\phi(x)-\phi(\tau)+\phi(0)))-\phi(0)))^\beta\sigma))\frac{\phi'(\tau)\Im_d}{\phi'(0)}d\tau d\sigma\big)(\lambda)\\
+&\mathfrak{L}_\phi\big(\int_0^t(\phi(x)-\phi(s))^{\beta-1}g(s,\Im(s))\phi'(s)\int_0^\infty\beta\sigma\Theta_\beta(\sigma)S((\phi(x)-\phi(s))^\beta\sigma)d\sigma ds\big)(\lambda)\\
=&\mathfrak{L}_\phi\big(K_\beta^\phi(x,0)\Im_f+L_\beta^\phi(x,0)\Im_d+\int_0^x\phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\hbar(s,\Im(s))ds\big)(\lambda)
\end{aligned}$$

Thus, we obtain that through the use of the inverse generalized Laplace transform

$$\Im(x) = \aleph(x, \Im(x)) \\ \times \left[ K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1} M_\beta^\phi(x,s)\hbar(s,\Im(s))ds \right].$$

□

**Lemma 3.5.** *The properties of operators $K_\beta^\phi(x,0)$, $L_\beta^\phi(x,0)$ and $M_\beta^\phi(x,s)$ are as follows:*

(1) $\| K_\beta^\phi(x,0)y \| \leq \mathfrak{M} \| y \|$,

(2) $\| L_\beta^\phi(x,0)y \| \leq \frac{\mathfrak{M}(\phi(X)-\phi(0))}{\phi'(0)} \| y \|_E$,

(3) $\| M_\beta^\phi(x,s)y \| \leq \frac{\mathfrak{M}(\phi(X)-\phi(0))^\beta}{\Gamma(2\beta)} \| y \|_E$.

*Proof.* Let $y \in E$ and $x > s > 0$

1. For the first inequality, we have:

$$\begin{aligned} \| K_\beta^\phi(x,0)y \|_E &= \| \int_0^{+\infty} \Theta_\beta(s) C((\phi(x)-\phi(0))^\beta s) y ds \| \\ &\leq \int_0^{+\infty} \Theta_\beta(s) \| C((\phi(x)-\phi(0))^\beta s) \|_{L(E)} \| y \|_E \, ds \\ &\leq \mathfrak{M} \| y \|_E . \end{aligned}$$

2. It is evident that

$$\begin{aligned} \| L_\beta^\phi(x,0)y \|_E &= \| \int_0^{+\infty} K_\beta^\phi(x,0) \frac{\phi'(s)}{\phi(0)} y ds \|_E \\ &\leq \frac{\mathfrak{M} \| y \|_E}{\phi'(0)} \int_0^x \phi'(s) ds \\ &\leq \frac{\mathfrak{M}(\phi(X)-\phi(0))}{\phi'(0)} \| y \|_E . \end{aligned}$$

3. The third inequality can be formulated as follows:

$$\begin{aligned} \| M_\beta^\phi(x,s)y \|_E &= \| \int_0^{+\infty} \beta\theta\Theta_\beta(\theta) S((\phi(x)-\phi(s))^\beta \theta) y d\theta \|_E \\ &\leq \int_0^{+\infty} \beta\theta\Theta_\beta(\theta) \| \int_0^{\phi(x)-\phi(s))^\beta \theta} C(\tau) y d\tau \|_E \, d\theta \\ &\leq \frac{\mathfrak{M}(\phi(X)-\phi(0))^\beta}{\Gamma(2\beta)} \| y \|_E . \end{aligned}$$

□

**Lemma 3.6.** (1) *The operators $K_\beta^\phi$, $L_\beta^\phi$ and $M_\beta^\phi$ are strongly equicontinuous,*

(2) *If $S(x)$ and $C(x)$ are compact operators for each $x > 0$. Then, $K_\beta^\phi(x,0)$ and $L_\beta^\phi(x,0)$ are also operators compact for all $x > 0$.*

*Proof.* 1. We demonstrate the equicontinuity:

i. For every $\Im \in E$, $(C(x))_{x\in\mathbb{R}}$ is strongly continuous, i.e., for any $\varepsilon > 0$ and $x_1, x_2 \in \mathbb{R}$, it follows that

$$\| C(x_1)\Im - C(x_2)\Im \|_E \longrightarrow 0$$

whenever $x_1 - x_2 \longrightarrow 0$. Therefore, for any $0 \leq x_1 \leq x_2 \leq X$, we have:

$$\begin{aligned}
&\| K_\beta^\phi(x_1, 0)y - K_\beta^\phi(x_2, 0)y \|_E \leq \| \int_0^{+\infty} \Theta_\beta(s) C((\phi(x_1) - \phi(0))^\beta s) y ds \\
&- \int_0^{+\infty} \Theta_\beta(s) C((\phi(x_2) - \phi(0))^\beta s) y ds \|_E \\
&\leq \int_0^{+\infty} \Theta_\beta(s) \| C((\phi(x_1) - \phi(0))^\beta s)y - C((\phi(x_2) - \phi(0))^\beta s)y \|_E \, ds
\end{aligned}$$

Consequently, $\| K_\beta^\phi(x_1, 0)y - K_\beta^\phi(x_2, 0)y \|_E \longrightarrow 0$ whenever $x_1 - x_2 \longrightarrow 0$

ii. We have:

$$\begin{aligned}
\| L_\beta^\phi(x_1, 0)y - L_\beta^\phi(s, 0)y \|_E &\leq \| \int_0^{+\infty} K_\beta^\phi(x_1, 0) \frac{\phi'(s)}{\phi(0)} y ds \\
&\quad - \int_0^{x_2} K_\beta^\phi(s, 0) \frac{\phi'(s)}{\phi(0)} y ds \|_E \\
&\leq \| \int_{x_1}^{x_2} K_\beta^\phi(s, 0) \frac{\phi'(s)}{\phi(0)} y ds \|_E \\
&\leq \mathfrak{M} \| y \|_E \left| \frac{\phi(x_1) - \phi(x_2)}{\phi'(0)} \right| \longrightarrow 0, \\
&\text{whenever } x_1 - x_2 \longrightarrow 0.
\end{aligned}$$

iii. For every $\Im \in E$ and $0 \leq s \leq x_1 \leq x_2 \leq X$, we have:

$$\begin{aligned}
\| M_\beta^\phi(x_1, s) - M_\beta^\phi(x_2, s) \|_E &= \| \int_0^{+\infty} \beta\theta\Theta_\beta(\theta) S((\phi(x_1) - \phi(s))^\beta \theta) y d\theta \\
&\quad - \int_0^{+\infty} \beta\theta\Theta_\beta(\theta) S((\phi(x_2) - \phi(s))^\beta \theta) y d\theta \|_E \\
&\leq \frac{\mathfrak{M}}{\Gamma(\delta)} \| y \|_E \left| (\phi(x_1) - \phi(s))^\beta - (\phi(x_2) - \phi(s))^\beta \right|.
\end{aligned}$$

Consequently,$\| M_\beta^\phi(x_1, s) - M_\beta^\phi(x_2, s) \|_E \longrightarrow 0$ whenever $x_1 - x_2 \longrightarrow 0$

2. We demonstrate the compactness of the operator

i. Let $\Xi \geq 0$, set $B_\Xi = \{\Im \in E \,\|\, \Im \|_E \leq \Xi\}$, which is a bounded, closed, and convex subset of E. We demonstrate that for any positive constant $\Xi$ and $x > 0$, the sets

$$\Delta_1(x) = \{K_\beta^\phi(x, 0)\Im, \Im \in B_\Xi\}$$

and

$$\Delta_2(x) = \{L_\beta^\phi(x,0)\Im, \Im \in B_\Xi\},$$

are relatively compact in E. Let us consider $x > 0$ be fixed, for any $\omega > 0$, and $0 < \varepsilon \leq x$. We define the subsets

$$\Delta_1^{\omega,\varepsilon}(x) = \{C(\varepsilon^\beta \omega)\int_\omega^\infty \Theta_\beta(\theta)C((\phi(x)-\phi(0))^\beta\theta - \varepsilon^\beta\omega)\Im d\theta, \Im \in B_\Xi\},$$

and

$$\Delta_2^{\omega,\varepsilon}(x) = \{C(\varepsilon^\beta \omega)\int_0^{x-\varepsilon}\int_\omega^\infty \Theta_\beta(\theta)C((\phi(s)-\phi(0))^\beta\theta - \varepsilon^\beta\omega)\Im d\theta \frac{\phi'(s)}{\phi'(0)}ds, \Im \in B_\Xi\},$$

and we denote, for any $\Im \in B_\Xi$

$$K_\beta^{\phi,\omega,\varepsilon}(x,0)\Im = C(\varepsilon^\beta\omega)\int_\omega^\infty \Theta_\beta(\theta)C((\phi(x)-\phi(0))^\beta\theta - \varepsilon^\beta\omega)\Im d\theta,$$

and

$$L_\beta^{\phi,\omega,\varepsilon}(x,0) = C(\varepsilon^\beta\omega)\int_0^{x-\varepsilon}\int_\omega^\infty \Theta_\beta(\theta)C((\phi(s)-\phi(0))^\beta\theta - \varepsilon^\beta\omega)\Im d\theta \frac{\phi'(s)}{\phi'(0)}ds$$

By the uniform convergence of the Mainardi's Wright type function for $\theta \in (0,+\infty)$ and the uniform boundedness of the cosine family, we have:

$$\| K_\beta^{\phi,\omega,\varepsilon}(x,0)\Im \|_E \leq \mathfrak{M}^2 \| \Im \|_E$$

and

$$\| L_\beta^{\phi,\omega,\varepsilon}(x,0)\Im \|_E \leq \mathfrak{M}^2 \| \Im \|_E \ \frac{\phi(X)-\phi(0)}{\phi'(0)}$$

Hence, by the compactness of $C(x)$, we see that $\Delta_1^{\omega,\varepsilon}$ and $\Delta_2^{\omega,\varepsilon}$ are relatively compact in E,
for all $x > 0$. Furthermore,

$$\begin{aligned}
&\| K_\beta^{\phi,\omega,\varepsilon}(x,0)\Im - K_\beta^\phi(x,0)\Im \|_E \leq \| \int_0^\omega \Theta_\beta(\theta)C((\phi(x)-\phi(0))^\beta\theta)\Im d\theta \|_E \\
&+ \| K(\varepsilon^\beta\omega)\int_\omega^{+\infty} \Theta_\beta(\theta)C((\phi(x)-\phi(0))^\beta\theta - \varepsilon^\beta\omega)\Im d\theta \\
&- \int_\omega^{+\infty} \Theta_\beta(\theta)C((\phi(x)-\phi(0))^\beta\theta)\Im d\theta \|_E \\
&\leq \mathfrak{M} \| \Im \|_E \int_0^\omega \Theta_\beta(\theta)d\theta \\
&+ \mathfrak{M}\int_0^{+\infty} \Theta_\beta(\theta) \| ((\phi(x)-\phi(0))^\beta\theta - \varepsilon^\beta\omega) - C((\phi(x)-\phi(0))^\beta\theta)]\Im \|_E \, d\theta \\
&+ \mathfrak{M}\int_0^\omega \Theta_\beta(\theta)d\theta.
\end{aligned}$$

and

$$\begin{aligned}
&\| L_\beta^{\phi,\omega,\varepsilon}(x,0)\Im - L_\beta^{\phi}(x,0)\Im \|_E \\
&\leq \| \int_{x-\varepsilon}^{x}\int_0^{+\infty} \Theta_\beta(\theta)C((\phi(s)-\phi(0))^\beta\theta)\Im d\theta \frac{\phi'(s)}{\phi'(0)} ds \|_E \\
&+ \| \int_0^{x-\varepsilon}\int_\omega^{+\infty} \Theta_\beta(\theta)[C(\varepsilon^\beta\omega)C((\phi(s)-\phi(0))^\beta\theta \\
&-\varepsilon^\beta\omega) - C((\phi(s)-\phi(0))^\beta\theta)]\Im d\theta \frac{\phi'(s)}{\phi'(0)} ds \|_E \\
&+ \| \int_0^{x-\varepsilon}\int_0^{\omega} \Theta_\beta(\theta)C((\phi(s)-psi(0))^\beta\theta)\Im d\theta \frac{\phi'(s)}{\phi'(0)} ds \|_E
\end{aligned}$$

Using the strong continuity of the cosine family, we get $\| L_\beta^{\phi,\omega,\varepsilon}(x,0)\Im - L_\beta^{\phi}(x,0)\Im \|_E \longrightarrow 0 \ as \ \varepsilon \longrightarrow 0.$

Hence, $\Delta_1^{\omega,\varepsilon}(x)$ and $\Delta_2^{\omega,\varepsilon}(x)$ are relatively compact sets arbitrarily close to the sets $\Delta_1(x)$ and $\Delta_2(x)$ for every $x>0$. Thus, the sets $\Delta_1(x)$ and $\Delta_2(x)$ are relatively compact in E for every $x>0$. We now prove a relation between the operators $L_\beta^\phi$ and $K_\beta^\phi$. □

## 4. The existence of a mild solution

This section contains the findings that demonstrate the existence and uniqueness of the mild solution for problem (1.1). For the first result, we suppose the following

$(H_1)$ $S(x)$ and $C(x)$ are compact operators for all $x>0$,

$(H_2)$ There is a positive real constant $\gamma$ such that

$$|\aleph(x,\Im_1)-\aleph(x,\Im_2)| \leq \gamma|\Im_2-\Im|, \text{ for every } \Im_1,\Im_2\in E.$$

$(H_3)$ There is a function $\zeta$ in $L^1(\xi,\mathbb{R})$ with

$$\hbar(x,\Im)\leq\zeta(x).$$

**Theorem 4.1.** *Suppose that hypothesis $(H_1)$ through $(H_3)$ are verified. Furthermore, if*

$$\gamma\left(\mathfrak{M}|\Im_f| + \frac{(\phi(X)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| + \frac{2(\phi(X)-\phi(0))^\delta\mathfrak{M}}{\Gamma(\delta+1)} \|_E \zeta \|_E\right) < 1, \quad (4.1)$$

*then the fractional hybrid differential problem (1.1) has a solution on $\xi$.*

*Proof.* We define $\mathbf{S}$, the subset of $E=\mathbb{C}(\xi,R)$ as follows

$$\mathbf{S}=\left\{\Im\in E, \|\Im\|_E\leq M\right\},$$

where

$$M=\frac{\mathfrak{M}|\Im_f| + \frac{\aleph_s(\phi(X)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| + \frac{2(\phi(X)-\phi(0))^\delta\mathfrak{M}}{\Gamma(\delta+1)}\|\zeta\|}{1-\gamma\left(\mathfrak{M}|\Im_f| + \frac{(\phi(X)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| + \frac{2(\phi(X)-\phi(0))^\delta\mathfrak{M}}{\Gamma(\delta+1)}\|\zeta\|\right)}$$

and

$$\aleph_s = \sup_{x \in \xi}(\aleph(x,0)).$$

It is evident that $\mathbf{S}$ is a bounded, convex, and closed subset of $E$, the Banach space.

Consider two operators $\Psi : \mathbf{S} \to E$ and $\Upsilon : E \to E$ defined as follows

$$\Upsilon\Im(x) = \aleph(x, \Im(x)),$$

and

$$\Psi\Im(x) = K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\hbar(s,\Im(s))ds.$$

Since the system (1.1) is equivalent to the nonlinear fractional hybrid integral equation (3.2), the operator equation can be obtained by transforming equation (3.2)

$$\Upsilon\Im(x)\Psi\Im(x) = \Im(x) \quad \text{for all} \quad x \in \xi.$$

We will now show that operators $\Upsilon$ and $\Psi$ meet all the conditions specified in Theorem 2.12. The proof will be presented in several steps

**Step 1:** $\Upsilon : E \to E$ is Lipschitz. Let $\Im_1, \Im_2 \in E$, then from $(H_2)$ we can get

$$\begin{aligned}|\Upsilon\Im_1(x) - \Upsilon\Im_2(x)| &= |\aleph(x,\Im_1(x)) - \aleph(x,\Im_2(x))| \\ &\le \gamma|\Im_1(x) - \Im_2(x)|, \qquad \text{for every } x \in \xi.\end{aligned}$$

Taking the supremum with respect to $x$, we obtain

$$\| \Upsilon\Im_1 - \Upsilon\Im_2 \| \le \gamma \| \Im_1 - \Im_2 \|, \qquad \text{for all} \quad \Im_1, \Im_2 \in E.$$

**Step 2:** $\Psi : \mathbf{S} \to E$ is completely continuous. To establish this, it suffices to show that $\Psi$ is continuous and that $\Psi(\mathbf{S})$ is uniformly bounded and equicontinuous. We will first demonstrate that $\Psi$ is continuous. Let $\Im_n$ be a sequence in $S$ such that $\lim\limits_{n\to+\infty} \Im_n = \Im$, $\Im \in \mathbf{S}$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\mapsto+\infty} \Psi\Im_n(x) &= \lim_{n\mapsto+\infty}\Big[K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d \\ &\quad + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\hbar(s,\Im_n(s))ds\Big] \\ &= K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d \\ &\quad + \lim_{n\mapsto+\infty}\int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\aleph(s,\Im_n(s))ds \\ &= \Psi\Im(x).\end{aligned}$$

Then $\Psi$ is a continuous on $\mathbf{S}$.
Next, we prove that $\Psi(\mathbf{S})$ is a uniformly bounded set in $E$. Let $\Im \in \mathbf{S}$,

then from hypothesis $(H_2)$ and Proposition 3.5 we can get

$$
\begin{aligned}
&\|\Psi\Im(x)\|_E \\
&= \left\| K^{\phi}_{\beta}(x,0)v_0 + L^{\phi}_{\beta}(x,0)\Im_d + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1} M^{\phi}_{\beta}(x,s)\hbar(s,\Im(s))ds \right\|_E \\
&\leq \| K^{\phi}_{\beta}(x,0)\Im_f \|_E + \| L^{\phi}_{\beta}(x,0)\Im_d \|_E \\
&+ \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1} \| M^{\phi}_{\beta}(x,s)\hbar(s,\Im(s))\| ds \\
&\leq \mathfrak{M}|\Im_f| + \frac{(\phi(x)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| \\
&+ \frac{(\phi(X)-\phi(0))^{\beta}\mathfrak{M}}{\Gamma(\delta)} \| \zeta \|_E \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}ds \\
&\leq \mathfrak{M}|\Im_f| + \frac{(\phi(X)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| + \frac{2(\phi(X)-\phi(0))^{\delta}M}{\Gamma(\delta+1)} \| \zeta \|_E,
\end{aligned}
$$

for all $x \in \xi$. Thus

$$
\| \Psi\Im(x) \|_E \leq \mathfrak{M}|\Im_f| + \frac{(\phi(X)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| + \frac{2(\phi(X)-\phi(0))^{\delta}\mathfrak{M}}{\Gamma(\delta+1)} \| \zeta \|_E
$$

for all $\Im \in \mathbf{S}$. So, $\Psi$ is uniformly bounded on $\mathbf{S}$.

Now, we prove that $\Psi(\mathbf{S})$ is equicontinuous on $\xi$. Let $\Im \in \Psi(\mathbf{S})$ and $x_2,\, x_1 \in \xi$ with $x_1 < x_2$. Then, we have

$$
\begin{aligned}
&\| \Psi\Im(x_2) - \Psi\Im(x_1) \|_E \\
&\qquad = \Big\| K^{\phi}_{\beta}(x_2,0)\Im_f + L^{\phi}_{\beta}(x_2,0)\Im_d - K^{\phi}_{\beta}(x_1,0)\Im_f - L^{\phi}_{\beta}(x_1,0)\Im_d \\
&\qquad + \int_0^{x_2} \phi'(s)(\phi(x_2)-\phi(s))^{\beta-1} M^{\phi}_{\beta}(x_2,s)\hbar(s,\Im(s))ds \\
&\qquad - \int_0^{x_1} \phi'(s)(\phi(x_1)-\phi(s))^{\beta-1} M^{\phi}_{\beta}(x_1,s)\hbar(s,\Im(s))ds \Big\| \\
&\qquad \leq \| K^{\phi}_{\beta}(x_2,0)\Im_f - K^{\phi}_{\beta}(x_1,0)\Im_f \|_E \\
&\qquad + \| L^{\phi}_{\beta}(x_2,0)\Im_d - L^{\phi}_{\beta}(x_1,0)\Im_d \|_E \\
&\qquad + \Big\| \int_0^{x_1} \phi'(s)(\phi(x_2)-\phi(s))^{\beta-1} M^{\phi}_{\beta}(x_2,s)\hbar(s,\Im(s))ds \\
&\qquad - \int_0^{x_1} \phi'(s)(\phi(x_1)-\phi(s))^{\beta-1} M^{\phi}_{\beta}(x_1,s)\hbar(s,\Im(s))ds \\
&\qquad + \int_{x_1}^{x_2} \phi'(s)(\phi(x_2)-\phi(s))^{\beta-1} M^{\phi}_{\beta}(x_2,s)\hbar(s,\Im(s))ds \Big\|_E
\end{aligned}
$$

Using Proposition 3.5 we have

$$\begin{aligned}
&\| \Psi\Im(x_2) - \Psi\Im(x_1) \|_E \\
&\qquad \leq \| K^{\phi}_{\beta}(x_2,0)\Im_f - K^{\phi}_{\beta}(x_1,0)\Im_f \|_E + \| L^{\phi}_{\beta}(x_2,0)\Im_d - L^{\phi}_{\beta}(x_1,0)\Im_d \|_E \\
&\qquad + \frac{(\phi(X)-\phi(0))^{\beta}\mathfrak{M}}{\Gamma(\delta)} \| \zeta \|_E \\
&\qquad \times \left\| \int_0^{x_1} \phi'(s)(\phi(x_2)-\phi(s))^{\beta-1} - \phi'(s)(\phi(x_1)-\phi(s))^{\beta-1} ds \right. \\
&\qquad \left. + \int_{x_1}^{x_1} \phi'(s)(\phi(x_2)-\phi(s))^{\beta-1} ds \right\|_E \\
&\qquad \leq \| K^{\phi}_{\beta}(x_2,0)\Im_f - K^{\phi}_{\beta}(x_1,0)\Im_f \|_E + \| L^{\phi}_{\beta}(x_2,0)\Im_d - L^{\phi}_{\beta}(x_1,0)\Im_d \|_E \\
&\qquad + \frac{2(\phi(X)-\phi(0))^{\beta}\mathfrak{M}}{\Gamma(\delta+1)} \| \zeta \| \| (\phi(x_2)-\phi(0))^{\beta} - (\phi(x_1)-\phi(0))^{\beta} \|_E .
\end{aligned}$$

Since $\phi$ is a continuous. Then, we conclude that

$$\lim_{x_1 \to x_2} |\Psi\Im(x_1) - \Psi\Im(x_2)| = 0.$$

which shows that $\Psi(\mathbf{S})$ is equicontinuous.
Now, the set $\Psi(\mathbf{S})$ is equicontinuous and uniformly bounded. By using Arzelà–Ascoli theorem, $\Psi(\mathbf{S})$ is compact, which shows that $\Psi$ is completely continuous.
**Step 3:** Now it remains to demonstrate that the third hypothesis of Theorem 2.12 is verified. Let $\Im_1 \in E$, $\Im_2 \in S$ and $x \in \xi$ such that $\Im_1 = \Psi\Im_1\Upsilon\Im_2$

$$\begin{aligned}
&\| \Im_1(x) \|_E = \| \Psi\Im_1(x)\Upsilon\Im_2(x) \|_E \\
&= \| \aleph(x,\Im_1(x)) \|_E \\
&\times \left\|_E K^{\phi}_{\beta}(x,0)\Im_f + L^{\phi}_{\beta}(x,0)\Im_d + \int_0^t \phi'(s)(\phi(x)-\phi(s))^{\beta-1} M^{\phi}_{\beta}(x,s)\,\aleph(s,\Im_2(s))\,ds \right\|_E \\
&\leq \Big[ |\aleph(x,\Im_1(x)) - \aleph(x,0)| + |\aleph(x,0)| \Big] \\
&\times \left( \mathfrak{M}|\Im_f| + \frac{(\phi(X)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| + \frac{2(\phi(X)-\phi(0))^{\delta}\mathfrak{M}}{\Gamma(\delta+1)} \| \zeta \|_E \right) \\
&\leq \frac{\mathfrak{M}|\Im_f| + \frac{\aleph_s(\phi(x)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| + \frac{2(\phi(X)-\phi(0))^{\delta}\mathfrak{M}}{\Gamma(\delta+1)} \| \zeta \|_E}{1-\gamma\left( \mathfrak{M}|\Im_f| + \frac{(\phi(X)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| + \frac{2(\phi(X)-\phi(0))^{\delta}\mathfrak{M}}{\Gamma(\delta+1)} \| \zeta \|_E \right)} = M
\end{aligned}$$

Then, $\Im \in \mathbf{S}$.
**Step 4:** Let $K = \| \Psi(\mathbf{S}) \|_E = Sup\{\Psi(\Im)|, \Im \in \mathbf{S}\}$. Since

$$\| \Psi\Im(x) \|_E \leq \mathfrak{M}|\Im_f| + \frac{(\phi(x)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d| + \frac{2(\phi(x)-\phi(0))^{\delta}\mathfrak{M}}{\Gamma(\delta+1)} \| \zeta \|_E,$$

then, we get $\gamma K \leq 1$. Finally, all conditions of Theorem 2.12 are verified for the operators $\Upsilon$ and $\Psi$. Thus, the system (1.1) has a solution on $\xi$.

$\square$

## 5. Ulam–Hyers stability

This section examines the stability in the sense of the Ulam-Hyers of the mild solution to the problem (1.1).

**Definition 5.1.** The problem (1.1) is stable in the sense of Ulam-Hyers if there is $\kappa > 0$ such that for every $\varepsilon > 0$ and for all solution $\upsilon \in \mathbb{C}^1(\xi, E)$ of the following inequality

$$\left| {}^{\mathbb{C}}D_{0^+}^{\delta,\phi}\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) - A\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) - \hbar(x,\upsilon(x))\right| \leq \varepsilon, \tag{5.1}$$

there exists a mild solution $u \in \mathbb{C}^1(\xi, E)$ satisfying the problem (1.1) with

$$|\upsilon(x) - u(x)| \leq \kappa\varepsilon \quad \text{for all} \quad x \in \xi.$$

**Definition 5.2.** The problem (1.1) is stable in the sense of generalized Ulam-Hyers if there is $\omega \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that $\omega(0) = 0$, and for any solution $\upsilon \in \mathbb{C}^1(\xi, E)$ of the follwoing inequality

$$|{}^{\mathbb{C}}D_{0^+}^{\delta,\phi}\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) - A\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) - \hbar(x,\upsilon(x))| \leq \varepsilon, \tag{5.2}$$

there is a mild solution $u \in \mathbb{C}(\xi, E)$ satisfying the problem (1.1) with

$$|\upsilon(x) - u(x)| \leq \omega(\varepsilon) \quad \text{for all} \quad x \in \xi.$$

*Remark* 5.3. A function $\upsilon \in \mathbb{C}(\xi, E)$ satisfies (5.1) if and only if there is $\Lambda \in \mathbb{C}(\xi, E)$, depending on the solution $\upsilon$ with

(1) $|\Lambda(x)| \leq \varepsilon$ for all $x \in \xi$,
(2) ${}^{\mathbb{C}}D_{0^+}^{\delta,\phi}\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) = A\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) + \hbar(x,\upsilon(x)) + \Lambda(x)$.

**Definition 5.4.** The problem (1.1) is stable in the sense of Ulam-Hyers-Rassias with respect to the function $\omega_r \in \mathbb{C}(\xi, \mathbb{R}^+)$ there is a real number $\kappa > 0$ such that for any $\varepsilon > 0$ and for all solution $\upsilon \in \mathbb{C}^1(\xi, E)$ of the following inequality

$$| {}^{\mathbb{C}}D_{0^+}^{\delta,\phi}\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) - A\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) - \hbar(x,\upsilon(x))| \leq \varepsilon\omega_r(x), \tag{5.3}$$

there is a mild solution $u \in \mathbb{C}(\xi, E)$ satisfying the problem (1.1) such that

$$|\upsilon(x) - u(x)| \leq \kappa\varepsilon\omega_r(x) \quad \text{for every} \quad x \in \xi.$$

**Definition 5.5.** The problem (1.1) is stable in the sense of generalized Ulam-Hyers-Rassias with respect to the function $\omega_r \in \mathbb{C}(\xi, \mathbb{R}^+)$ there is a real number $\kappa_{\omega_r} > 0$ such that for any $\varepsilon > 0$ and for all solution $\upsilon \in \mathbb{C}^1(\xi, E)$ of the following inequality

$$| {}^{\mathbb{C}}D_{0^+}^{\delta,\phi}\left(\frac{\upsilon(t)}{\aleph(x,\upsilon(x))}\right) - A\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) - \hbar(x,\upsilon(x))| \leq \varepsilon\omega_r(x), \tag{5.4}$$

there is a mild solution $u \in \mathbb{C}(\xi, E)$ satisfying the problem (1.1) such that

$$|\upsilon(x) - u(x)| \leq \kappa_{\omega_r}\varepsilon\omega_r(x), \quad \text{for every} \quad x \in \xi$$

satisfies (5.1) if and only if there is $\Lambda \in \mathbb{C}(\xi, E)$, depending on the solution $y$ with

*Remark* 5.6. A function $\upsilon \in \mathbb{C}(\xi, E)$ satisfies (5.3) if and only if there is $\Lambda \in \mathbb{C}(\xi, E)$, depending on solution $\upsilon$ with

(1) $|\Lambda(x)| \leq \varepsilon \omega_r(x)$ for all $x \in \xi$,
(2) ${}^{\mathbb{C}}D_{0^+}^{\delta,\phi}\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) = A\left(\frac{\upsilon(x)}{\aleph(x,\upsilon(x))}\right) + \hbar(x, \upsilon(x)) + \Lambda(x)$.

**Lemma 5.7.** *Suppose that the hypotheses $(H_1)$ and $(H_2)$ are satisfied. Let $\varepsilon > 0$ and $\upsilon \in \mathbb{C}^1(\xi, E)$ be a solution of (5.1). Then,*

$$\begin{aligned}\Big\| \upsilon(x) - &\aleph(x, \upsilon(x)) \\ &\times \left[K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\hbar(s,\upsilon(s))\,ds\right]\Big\| \\ &\leq \frac{2\aleph_s \mathfrak{M}(\phi(X)-\phi(0))^\delta \varepsilon}{\Gamma(\delta+1)},\end{aligned}$$

*where*

$$\aleph_s = \sup_{x\in\xi} \aleph(x, \upsilon(x)).$$

*Proof.* We use Remark 5.3, Lemma 3.5 and Theorems 4.1, 3.4, we have

$$\begin{aligned}&\Big| \upsilon(x) - \aleph(x,\upsilon(x)) \times \Big[K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d \\ &\quad + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\hbar(s,\upsilon(s))ds\Big]\Big| \\ &\quad \leq \Big| \aleph(x,\upsilon(x)) \Big[K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d \\ &+ \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)(\hbar(s,\upsilon(s)) + \Lambda(s))ds\Big] - \aleph(x,\upsilon(x)) \\ &\times \left[K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\hbar(s,\upsilon(s))ds\right]\Big| \\ &\leq |\aleph(x,\upsilon(x))| \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\Lambda(s)ds \\ &\leq \frac{2\aleph_s\mathfrak{M}(\phi(X)-\phi(0))^\delta\varepsilon}{\Gamma(\delta+1)}.\end{aligned}$$

□

**Theorem 5.8.** *According to the hypotheses of Theorem 2.12, the problem (1.1) is stable in the sense of Ulam-Hyers in $\mathbb{C}(\xi, E)$ if the inequality (5.1) is satisfied and*

- *There is $\gamma > 0$ such that*
$$|\hbar(x,u) - \hbar(x,\upsilon)| \leq K|u - \upsilon|, \ \textit{for every } u, \upsilon \in E$$
- $\Gamma(\delta+1)(1 - \gamma(\mathfrak{M} + \frac{\mathfrak{M}(\phi(X)-\phi(0))}{\phi'(0)})) - 2\gamma\mathfrak{M}(\phi(X)-\phi(0))^\delta(\gamma + \aleph_s K) > 0$

*Proof.* Let $\varepsilon > 0$ and let $\upsilon \in \mathbb{C}(\xi, E)$ be a function satisfying (5.1) and $u$ is a mild solution of (1.1). Then,

$$
\begin{aligned}
&| \upsilon(x) - u(x)| \leq \bigg|\Big(\upsilon(x) - \aleph(x,\upsilon(x))\\
&\times\Big[K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\hbar(s,\upsilon(s))\,ds\Big]\Big)\bigg|\\
&+\bigg|\bigg(\aleph(x,\upsilon(x))\\
&\times\left[K_\beta^\phi(t,0)\Im_f + L_\beta^\phi(x,0)\Im_d + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\hbar(s,\upsilon(s))\,ds\right]\\
&-\aleph(x,u(x))\\
&\times\left[K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d + \int_0^x \phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(x,s)\hbar(s,u(s))\,ds\right]\bigg)\bigg|\\
&\leq \frac{2\aleph_s\mathfrak{M}(\phi(X)-\phi(0))^\delta\varepsilon}{\Gamma(\delta+1)} + |\aleph(x,\upsilon(x)) - \aleph(x,u(x))|\left|K_\beta^\phi(x,0)\Im_f + L_\beta^\phi(x,0)\Im_d\right|\\
&+\int_0^x\bigg[\phi'(s)(\phi(x)-\phi(s))^{\beta-1}\\
&\times\left(\aleph(x,u(x))M_\beta^\phi(x,s)\hbar(s,u(s)) - \aleph(x,\upsilon(x))M_\beta^\phi(x,s)\hbar(s,\upsilon(s))\right)\bigg]ds\\
&\leq \frac{2\aleph_s\mathfrak{M}(\phi(X)-\phi(0))^\delta\varepsilon}{\Gamma(\delta+1)} + |u(x)-\upsilon(x)|\\
&\times\left(\gamma(\mathfrak{M} + \frac{\mathfrak{M}(\phi(X)-\phi(0))}{\phi'(0)}) + \frac{2\gamma\mathfrak{M}(\phi(x)-\phi(0))^\delta}{\Gamma(\delta+1)} + \frac{2\aleph_s K\mathfrak{M}(\phi(X)-\phi(0))^\delta}{\Gamma(\delta+1)}\right)\\
&\leq \frac{2\aleph_s\mathfrak{M}(\phi(X)-\phi(0))^\delta\varepsilon}{\Gamma(\delta+1)(1-\gamma(\mathfrak{M}+\frac{\mathfrak{M}(\phi(X)-\phi(0))}{\phi'(0)})) - 2\gamma\mathfrak{M}(\phi(X)-\phi(0))^\delta(\gamma+\aleph_s K)}\\
&\leq \kappa\varepsilon.
\end{aligned}
$$

Then, the problem (1.1) is stable in the sense of Ulam−Hyers. □

*Remark* 5.9. If we set $\omega(\varepsilon) = 2\aleph_s\mathfrak{M}(\phi(X)-\phi(0))^\delta\varepsilon$, then $\omega(0) = 0$, and therefore the problem (1.1) becomes stable in the sense of generalized Ulam-Hyers

**Theorem 5.10.** *Suppose that the assumptions of Theorem 5.8 and there is $\kappa_\omega > 0$ such that*

$$I_{0^+}^{\beta,\phi}\omega_r(x) \leq \kappa_\omega\omega_r(x), \textit{for every } x \in \xi$$

*are verified. Then, the problem (1.1) is stable in the sense of Ulam-Hyers-Rassias with respect to the function $\omega_r$.*

*Proof.* Let $\varepsilon > 0$ and let $\upsilon \in \mathbb{C}(\xi, E)$ satisfying the inequality (5.1) and $u$ is a mild solution of (1.1). Then, using the same technique as Theorem 4.1's proof, we can rapidly show that

$$
\begin{aligned}
&|\upsilon(x)-u(x)|\\
&\le |\upsilon(x)-\aleph(x,u(x))|\\
&\times\left|\left[K_\beta^\phi(t,0)\Im_f+L_\beta^\phi(t,0)\Im_d+\int_0^t\phi'(s)(\phi(x)-\phi(s))^{\beta-1}M_\beta^\phi(t,s)g(s,u(s))ds\right]\right|\\
&\le \frac{2\aleph_s\mathfrak{M}(\phi(X)-\phi(0))^\delta\kappa_\omega\omega_r(x)\varepsilon}{\Gamma(\delta+1)}+|u(x)-\upsilon(x)|\\
&\times\left(\gamma(\mathfrak{M}+\frac{\mathfrak{M}(\phi(X)-\phi(0))}{\phi'(0)})+\frac{2\gamma\mathfrak{M}(\phi(X)-\phi(0))^\delta}{\Gamma(\delta+1)}+\frac{2\aleph_sK\mathfrak{M}(\phi(X)-\phi(0))^\delta}{\Gamma(\delta+1)}\right)\\
&\le \frac{2\aleph_s\mathfrak{M}(\phi(X)-\phi(0))^\delta\kappa_\omega\omega_r(x)\varepsilon}{\Gamma(\delta+1)(1-\gamma(\mathfrak{M}+\frac{\mathfrak{M}(\phi(X)-\phi(0))}{\phi'(0)}))-2\gamma\mathfrak{M}(\phi(X)-\phi(0))^\delta(\gamma+\aleph_sK)}\\
&\le \kappa\varepsilon\omega_r(x),
\end{aligned}
$$

where $\kappa=\frac{2\aleph_s\mathfrak{M}(\phi(X)-\phi(0))^\delta\kappa_\omega}{\Gamma(\delta+1)(1-\gamma(\mathfrak{M}+\frac{\mathfrak{M}(\phi(X)-\phi(0))}{\phi'(0)}))-2\gamma\mathfrak{M}(\phi(X)-\phi(0))^\delta(\gamma+\aleph_sK)}$,
which completes the proof of the theorem. □

Moreover, if $\varepsilon=1$ then, the equation (1.1) is stable in the sense of generalized Ulam-Hyers

## 6. Example

In this section, we provide an example to demonstrate our main result. Consider the hybrid fractional differential problem given below

$$
\begin{cases}
{}^{\mathbb{C}}D_{0^+}^{\frac{3}{2},t^2+\frac{1}{2}x}\left(\frac{\Im(x,y)}{\aleph(t,\Im(x,y))}\right)=A\left(\frac{\Im(x,y)}{\aleph(x,\Im(x,y))}\right)+\hbar(x,\Im(x,y)), \qquad x\in\xi=[0,1],\\
\Im(0,y)=\Im(1,y)=0,\\
\frac{\Im(0)}{\aleph(0,\Im(0))}=\Im_f(y)=0;\ \frac{d}{dx}\left(\frac{\Im(0)}{\aleph(0,\Im(0))}\right)=\Im_d(y)=0,
\end{cases}
\tag{6.1}
$$

where $\delta=\frac{3}{2}$, $E=L^2(\xi,\mathbb{R})$, $X=1$, $\phi(x)=x^2+\frac{1}{2}t$, $\hbar(x,\Im(x,y))=\dfrac{\sqrt{x}}{12}\sin(\Im(x,y))$ and

$$\aleph(x,\Im(x,y))=\frac{e^{-x}}{9+e^x}\left(\frac{\cos(x)|\Im(x,y)|}{1+|\Im(x,y)|}\right),$$

where $A=\frac{\partial^2}{\partial y}$ is the Laplace operator and $A:D(A)\subset E\to E$ be an operator given by

$$D(A)=H^2(\xi)\cap H_0^1(\xi).$$

In this situation, the operator $A$ generates a uniformly bounded strongly continuous cosine family $\{C(s)\}_{s\ge0}$. It is known that $\parallel C(s)\parallel_{L^1(E)}\le 1$ and the operator $S(x)$ is compact for avery $x\ge0$ [20]. We can take $\mathfrak{M}=1$, it is evident that $(H_1)$ is satisfied. To verify the assumption $(H_2)$, let $x\in\xi$ and consider $\Im_1(x,.)$ and

$\Im_2(x,.)$ in $\mathbb{C}(\xi,\mathbb{R})$. Then, we have

$$\begin{aligned}
|\aleph(x,\Im_2(x,.))-\aleph(x,\Im_1(x,.))| &= \left|\frac{e^{-x}}{9+e^x}\Big(\frac{\cos(x)|\Im_2(x,.)|}{1+|\Im_2(x,.)|}\Big)-\frac{e^{-x}}{9+e^x}\Big(\frac{\cos(x)|\Im_1(x,.)|}{1+|\Im_1(x,.)|}\Big)\right| \\
&\leq |\frac{e^{-x}}{9+e^x}|\left|\left(\frac{|\Im_2(x,.)|}{1+|\Im_2(x,.)|}\right)-\left(\frac{|\Im_1(x,.)|}{1+|\Im_1(x,.)|}\right)\right| \\
&\leq \frac{1}{10}\left|\frac{\Im_2(x,.)-\Im_1(x,.)}{(1+|\Im_2(x,.)|)\,(1+|\Im_1(x,.)|)}\right| \\
&\leq \frac{1}{10}|\Im_2(x,.)-\Im_1(x,.)|.
\end{aligned}$$

Therfore, the assumption $(H_2)$ is verified with $\gamma=\frac{1}{10}$. It remains to prove the hypothesis $(H_3)$. Let $t\in\xi$ and consider $\Im(t,.)\in E$. Then, we have

$$\begin{aligned}
|\hbar(x,\Im(x,.))| &= \left|\frac{\sqrt{x}}{12}\sin(\Im(x,.))\right| \\
&\leq \frac{\sqrt{x}}{12}\,|\sin(\Im(x,.))| \\
&\leq \frac{\sqrt{x}}{12}.
\end{aligned}$$

So, $(H_3)$ holds with $\zeta(x)=\frac{x^2}{12}$. Moreover, we have

$$\gamma\left(\mathfrak{M}|\Im_f|+\frac{(\phi(X)-\phi(0))\mathfrak{M}}{\phi'(0)}|\Im_d|+\frac{2(\phi(X)-\phi(0))^{\delta}\mathfrak{M}}{\Gamma(\delta+1)}\parallel\zeta\parallel\right) = \frac{2}{120\Gamma(\frac{5}{3})} \sim 0,017<1$$

Finally, from Theorem 2.12, the system (1.1) has a solution on $[0,1]$. Further, to derive the conditions for Ulam−Hyers stability, we have according to the simple calculations we verify that the conditions of the Theorems 5.8 and 5.10 are verifying then our solution of problem (1.1) is Ulam-Hyers-Rassias stability then is Ulam-Hyers stability.

## 7. Conclsion

This paper presents a moderate solution for a class of fractional hybrid systems with orders spanning $]1,\ 2]$, and a $\phi$-Caputo fractional derivative. The Laplace transform applied to the $\phi$-function is what our method uses. Our understanding of fractional calculus has been enhanced by highlighting the connection between the mild solution of the evolution system and its associated probability density function. Additionally, through the use of fixed-point theorem, we have derived conditions for the existence of a mild solution within a suitable spatial framework. To show the practical applicability is based on the Dhage fixed point theorem.However, following studying the four different types of Ulam-Hyers stability of solutions to a particular problem, we presented an example.

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