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SET SHARING AND UNIQUENESS OF MEROMORPHIC FUNCTIONS

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Abstract. The paper deals with the uniqueness problem of meromorphic functions sharing sets and the result of the paper improve, generalise and extend a recent result of Bhoosnurmath and Davanal [1].

1. Introduction and preliminaries

2. Introduction Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the whole complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [6].

It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \quad (r \longrightarrow \infty, r \notin E).$$

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, $r \notin E$.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) = a\}$, where each point is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z : f(z) = a\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values.

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In the late 1920s, R. Nevanlinna studied the conditions with which a meromorphic function can be determined by the theory just established by himself, and obtain his remarkable 5-point theorem which gives an upper bound on the number of distinct values, two different meromorphic functions can share ignoring multiplicities. His other two celebrated uniqueness theorems for meromorphic functions, are usually called Nevanlinnas four-value theorem and Nevanlinnas three-value theorem. This three theorems are the gateway to the investigation of uniqueness theory of meromorphic functions and in particular the shared values of meromorphic functions, which has grown up to an extensive subfield of the value distribution theory. Thus, the study of the relationship between two meromorphic functions via the pre-image sets of several distinct values in the range has a long history. Inspired by the Nevanlinna's four and five value theorems, in 1970s F. Gross and C.C. Yang started to study the similar but more general questions of two functions that share sets of distinct elements instead of values.

In [5] Gross asked the following question which is known as Gross's question in the literature.:

Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

Now a days the researchers are immensely involved to deal with the above question in case of meromorphic functions under weaker hypothesis. More explicitly they are interested to determine a set S with n elements and to make n as small as possible such that any two meromorphic functions f and g that share the value ∞ and the set S must be equal. {cf.[1]- [2], [4], [7], [11], [16]-[18]}.

In the direction to the question of Gross, in 1995 Yi [15] proved for meromorphic functions the following result.

Theorem A. [15] *Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If f and g are two non-constant entire functions satisfying $E_f(S) = E_g(S)$ then $f \equiv g$.*

Fang and Hua [3] extended *Theorem A* to meromorphic functions imposing on the ramification indexes of f and g. Fang and Hua [3] proved the following theorem.

Theorem B. [3] *Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If two meromorphic functions f and g are such that $\Theta(\infty; f) > \frac{11}{12}$, $\Theta(\infty; g) > \frac{11}{12}$ and $E_f(S) = E_g(S)$ then $f \equiv g$.*

In the direction to the question of Gross, in 1995 Yi [16] proved for meromorphic functions the following result.

Theorem C. [16] *Let $S = \{z : z^n + az^{n-m} + b = 0\}$ where n and m are two positive integers such that $m \geq 2$, $n \geq 2m + 7$ with n and m having no common factor, a and b be two nonzero constants such that $z^n + az^{n-m} + b = 0$ has no multiple root. If f and g are two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

In the same paper Yi [16] also asked the following question:

What can be said if $m = 1$ in *Theorem A*?

In connection to his question Yi [16] proved the following theorem.

Theorem D. [16] *Let* $S=\{z: z^n+az^{n-1}+b=0\}$ *where* $n(\geq 9)$ *be an integer and* a *and* b *be two nonzero constants such that* $z^n+az^{n-1}+b=0$ *has no multiple root. If* f *and* g *be two non-constant meromorphic functions such that* $E_f(S)=E_g(S)$ *and* $E_f(\{\infty\})=E_g(\{\infty\})$ *then either* $f\equiv g$ *or* $f\equiv \frac{-ah(h^{n-1}-1)}{h^n-1}$ *and* $g\equiv \frac{-a(h^{n-1}-1)}{h^n-1}$, *where* h *is a non-constant meromorphic function.*

To provide an answer to the question of Yi and to find under which condition $f\equiv g$ Lahiri [7] proved the following result.

Theorem E. [7] *Let* S *be defined as in* Theorem D *and* $n(\geq 8)$ *be an integer. If* f *and* g *be two non-constant meromorphic functions having no simple poles such that* $E_f(S)=E_g(S)$ *and* $E_f(\{\infty\})=E_g(\{\infty\})$ *then* $f\equiv g$.

Fang and Lahiri [4] further improved *Theorem E* by replacing the range set by a smaller one and proved the following theorem.

Theorem F. [4] *Let* S *be defined as in* Theorem D *and* $n(\geq 7)$ *be an integer. If* f *and* g *be two non-constant meromorphic functions having no simple poles such that* $E_f(S)=E_g(S)$ *and* $E_f(\{\infty\})=E_g(\{\infty\})$ *then* $f\equiv g$.

We now give the following definitions known as weighted sharing of sets and values which renders a useful tool for the purpose of relaxation of the nature of sharing the sets.

Definition 2.1. [9, 10] *Let* k *be a nonnegative integer or infinity. For* $a\in\mathbb{C}\cup\{\infty\}$ *we denote by* $E_k(a;f)$ *the set of all* a*-points of* f, *where an* a*-point of multiplicity* m *is counted* m *times if* $m\leq k$ *and* $k+1$ *times if* $m>k$. *If* $E_k(a;f)=E_k(a;g)$, *we say that* f,g *share the value* a *with weight* k.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0\leq p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 2.2. [9] *Let* S *be a set of distinct elements of* $\mathbb{C}\cup\{\infty\}$ *and* k *be a nonnegative integer or* ∞. *We denote by* $E_f(S,k)$ *the set* $E_f(S)=\bigcup_{a\in S}\{z: f(z)-a=0\}$. *Clearly* $E_f(S)=E_f(S,\infty)$ *and* $\overline{E}_f(S)=E_f(S,0)$.

Next the following definition is required.

Definition 2.3. [8] *For* $a\in\mathbb{C}\cup\{\infty\}$*we denote by* $N(r,a;f\mid=1)$ *the counting function of simple* a*-points of* f. *For a positive integer* m *we denote by* $N(r,a;f\mid\leq m)(N(r,a;f\mid\geq m))$ *the counting function of those* a*-points of* f *whose multiplicities are not greater(less) than* m *where each* a*-point is counted according to its multiplicity.*

$\overline{N}(r,a;f\mid\leq m)(\overline{N}(r,a;f\mid\geq m))$ *are defined similarly, where in counting the* a*-points of* f *we ignore the multiplicities.*

Also $N(r,a;f\mid<m), N(r,a;f\mid>m), \overline{N}(r,a;f\mid<m)$ *and* $\overline{N}(r,a;f\mid>m)$ *are defined analogously.*

Improving *Theorem F*, Lahiri [11] proved the following theorem.

Theorem G. [11] *Let S be defined as in* Theorem D *and $n(\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)+\Theta(\infty;g)>1$, $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ then $f\equiv g$.*

In 2006 to deal with a question of Gross, Yi and Lin [17] have proved the following results.

Theorem H. [17] *Let S be defined as in* Theorem D *and $n(\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)>\frac{1}{2}$. $E_f(S,\infty)=E_g(S,\infty)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ then $f\equiv g$.*

Theorem I. [17] *Let S be defined as in* Theorem D *and $n(\geq 8)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)>\frac{2}{n-1}$. $E_f(S,\infty)=E_g(S,\infty)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ then $f\equiv g$.*

It is to be observed that the smallest cardinality $n=7$ corresponding to the set S is obtained by Lahiri [11]. We now give the following example which establishes the fact that the set S in *Theorems D-I* can not be replaced by any arbitrary set containing six distinct elements.

Example 2.4. *Let $f(z)=\sqrt{\alpha\beta\gamma}\,e^{z}$ and $g(z)=\sqrt{\alpha\beta\gamma}\,e^{-z}$ and $S=\{\alpha\sqrt{\beta},\alpha\sqrt{\gamma},\beta\sqrt{\alpha},\beta\sqrt{\gamma},\gamma\sqrt{\alpha},\gamma\sqrt{\beta}\}$, where α, β and γ are three nonzero distinct complex numbers. Clearly $E_f(S,\infty)=E_g(S,\infty)$ but $f\not\equiv g$.*

So it remains an open problem for investigations whether the degree of the equation defining S can be reduced to six and at the same time the conditions over ramification indexes can be weakened further ?

In this direction recently Bhoosnurmath and Dyavanal [1] proved the following.

Theorem J. [1] *Let S be defined as in* Theorem D *and $n(\geq 5)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)>\frac{2}{n-1}$ and $\Theta(\infty;g)>\frac{2}{n-1}$, $E_f(S,\infty)=E_g(S,\infty)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ and $N(r,0;f\mid=1)=S(r,f)$, $N(r,0;g\mid=1)=S(r,g)$ then $f\equiv g$.*

The purpose of the paper is to continue the investigations of further improvement, generalisation and extensions of *Theorem J*. Our result also supplement *Theorem G* to a large extent.

Following theorem is the main result of the paper.

Theorem 2.5. *Let S be given as in* Theorem D. *Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S,m)=E_g(S,m)$, $E_f(\{\infty\},\infty)$ $=E_g(\{\infty\},\infty)$ and $N(r,0;f\mid=1)=S(r,f)$, $N(r,0;g\mid=1)=S(r,g)$ and $\delta(0;f)+\Theta(\infty;f)+\delta(0;g)+\Theta(\infty;g)>\frac{4}{n-1}$. If*

(i) *$m\geq 2$ and $n\geq 5$;*
(ii) *or if $m=1$ and $n\geq 6$;*
(iii) *or if $m=0$ and $n\geq 10$*

then $f\equiv g$.

We now explain some definitions and notations which are used in the paper.

Definition 2.6. *Let f and g be two non-constant meromorphic functions such that f and g share $(a,0)$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the reduced counting function of those a-points of f and g where $p>q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p=q=1$, by $\overline{N}_E^{(2}(r,a;f)$ the reduced counting function of those a-points of f and g where $p=q\geq 2$. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$. In a similar manner we can define $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a\in\mathbb{C}\cup\{\infty\}$. When f and g share (a,m), $m\geq 1$ then $N_E^{1)}(r,a;f)=N(r,a;f\mid=1)$.*

Definition 2.7. *We denote by $\overline{N}(r,a;f\mid=k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k, where $k\geq 2$ is an integer.*

Definition 2.8. [9, 10] *Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$. and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$

3. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined as follows.

$$F=\frac{f^{n-1}(f+a)}{-b},G=\frac{g^{n-1}(g+a)}{-b}. \tag{3.1}$$

Henceforth we shall denote by H and V the following two functions

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{3.2}$$

Lemma 3.1. [14] *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$where $a_n\neq 0$ and $b_m\neq 0$ Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 3.2. [18] *If F, G are two non-constant meromorphic functions such that they share (1,0) and $H\not\equiv 0$ then*

$$N_E^{1)}(r,1;F\mid=1)=N_E^{1)}(r,1;G\mid=1)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 3.3. *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root, n (≥ 3) is an integer and F, G be given by (3.1). If for two non-constant meromorphic functions f and g $E_f(S,0) = E_g(S,0)$, $E_f(\infty,0) = E_g(\infty,0)$ and $H \not\equiv 0$ then*

$$\begin{aligned} N(r,H) \;\leq\; & \overline{N}(r,0,f) + \overline{N}(r,0;g) + \overline{N}_*(r,\infty;f,g) + \overline{N}(r,0;nf + a(n-1)) \\ & + \overline{N}(r,0;ng + a(n-1)) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;f^{'}) + \overline{N}_0(r,0;g^{'}), \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ is the reduced counting function of those zeros of $f^{'}$ which are not the zeros of f and $(F-1)$ and $\overline{N}_0(r,0;g^{'})$ is similarly defined.

Proof. Since $E_f(S,0) = E_g(S,0)$ and $E_f(\infty,0) = E_g(\infty,0)$ it follows that F and G share $(1,0)$ and $(\infty,0)$. We have from (3.1) that

$$F^{'} = [nf + (n-1)a]\, f^{n-2} f^{'}$$

and

$$G^{'} = [ng + (n-1)a]\, g^{n-2} g^{'}.$$

We can easily verify that possible poles of H occur at (i) zeros of f and g, (ii) zeros of $nf + a(n-1)$ and $ng + a(n-1)$, (iii) common poles of f and g with different multiplicities, (iv) common 1-points of F and G with different multiplicities, (v) zeros of $f^{'}$ which are not the zeros of $f(F-1)$, (v) zeros of $g^{'}$ which are not zeros of $g(G-1)$. Since H has only simple poles, the lemma follows from above. This proves the lemma. □

Lemma 3.4. *([11], Lemma 5) If f, g share $(\infty,0)$ then for $n(\geq 2)$*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where a,b are finite nonzero constants.

Lemma 3.5. *Let f, g be two non-constant meromorphic functions such that $\delta(0;f) + \Theta(\infty;f) + \delta(0;g) + \Theta(\infty;g) > \frac{4}{n-1}$ then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies $f \equiv g$, where n (≥ 3) is an integer and a is a nonzero finite constant.*

Proof. Let

$$f^{n-1}(f+a) \equiv g^{n-1}(g+a) \tag{3.3}$$

and suppose $f \not\equiv g$. We consider two cases:

Case I Let $y = \frac{g}{f}$ be a constant. Then from (3.3) it follows that $y \neq 1$, $y^{n-1} \neq 1$, $y^n \neq 1$ and $f \equiv -a\frac{1-y^{n-1}}{1-y^n}$, a constant, which is impossible.

Case II Let $y = \frac{g}{f}$ be non-constant. Then

$$f \equiv -a\frac{1-y^{n-1}}{1-y^n} \equiv a\left(\frac{y^{n-1}}{1+y+y^2+\ldots+y^{n-1}} - 1\right). \tag{3.4}$$

From (3.4) we see by *Lemma 3.1* that

$$T(r,f) = (n-1)T(r,y) + S(r,y).$$

We first note that the zeros of $1 + y + y^2 + \ldots + y^{n-2}$ contributes to the zeros of both f and g. In addition to this the poles of y contributes to the zeros of f

and since $g = fy$ the zeros of y contributes to the zeros of g. So from (3.4) we see that

$$\sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \overline{N}(r, \infty; y) \leq \overline{N}(r, 0; f), \sum_{k=1}^{n-1} \overline{N}(r, u_k; y) \leq \overline{N}(r, \infty; f)$$

where $u_k = exp\ (\frac{2k\pi i}{n})$ for $k = 1, 2, \ldots, n-1$ and $v_j = exp\ (\frac{2j\pi i}{n-1})$ for $j = 1, 2, \ldots, n-2$.

By the second fundamental theorem we get

$$\begin{aligned}
(2n-4)\ T(r, y) &\leq \overline{N}(r, \infty; y) + \sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \sum_{k=1}^{n-1} \overline{N}(r, u_k; y) + S(r, y) \\
&\leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, y) \\
&\leq (2 - \delta(0; f) - \Theta(\infty; f) + \varepsilon)\ T(r, f) + S(r, y) \\
&= (n-1)\ (2 - \delta(0; f) - \Theta(\infty; f) + \varepsilon)\ T(r, y) + S(r, y)
\end{aligned}$$

i.e.,

$$\frac{2n-4}{n-1}\ T(r, y) \leq (2 - \delta(0; f) - \Theta(\infty; f) + \varepsilon)\ T(r, y) + S(r, y), \tag{3.5}$$

where $0 < 2\varepsilon < \delta(0; f) + \Theta(\infty; f) + \delta(0; g) + \Theta(\infty; g)$.

Again noting that $\sum\limits_{j=1}^{n-2} \overline{N}(r, v_j; y) + \overline{N}(r, 0; y) \leq \overline{N}(r, 0; g)$, by the second fundamental theorem we get

$$\begin{aligned}
&(2n-3)\ T(r, y) \\
\leq\ & \overline{N}(r, \infty; y) + \overline{N}(r, 0; y) + \sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \sum_{k=1}^{n-1} \overline{N}(r, u_k; y) + S(r, y) \\
\leq\ & \overline{N}(r, \infty; y) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + S(r, y) \\
\leq\ & \overline{N}(r, \infty; y) + (n-1)\,(2 - \delta(0; g) - \Theta(\infty; g) + \varepsilon)\ T(r, y) + S(r, y)
\end{aligned}$$

i.e.,

$$\frac{2n-4}{n-1}\ T(r, y) \leq (2 - \delta(0; g) - \Theta(\infty; g) + \varepsilon)\ T(r, y) + S(r, y), \tag{3.6}$$

Adding (3.5) and (3.6) we get

$$\left(\frac{4n-8}{n-1} - 4 + \delta(0; f) + \Theta(\infty; f) + \delta(0; g) + \Theta(\infty; g) - 2\varepsilon\right)\ T(r, y) \leq S(r, y),$$

which is a contradiction.

Hence $f \equiv g$ and this proves the lemma. □

Lemma 3.6. [13] *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid< k) + k\overline{N}(r, 0; f \mid\geq k) + S(r, f).$$

4. Proof of the theorem

Proof of Theorem 2.5. We know from the assumption that the zeros of $z^n + az^{n-1} + b$ are simple and we denote them by w_j, $j = 1, 2, \ldots n$. Let F, G be given by (3.1) and (3.2). Since $E_f(S,m) = E_g(S,m)$ and $E_f(\infty,\infty) = E_g(\infty,\infty)$ it follows that F, G share $(1,m)$ and (∞,∞) and so $\overline{N}_*(r,\infty; f,g) = 0$. Also we note that by the assumption of the theorem $2\overline{N}(r,0;f) \le N(r,0;f) + S(r,f)$ and $2\overline{N}(r,0;g) \le N(r,0;g) + S(r,g)$

Case 1. Suppose that $H \not\equiv 0$.

Subcase 1.1. $m \ge 1$. While $m \ge 2$, using *Lemma 3.6* we note that

$$\begin{aligned} & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\ge 2) + \overline{N}_*(r,1;F,G) \qquad (4.1)\\ \le\ & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\ge 2) + \overline{N}(r,1;G \mid\ge 3)\\ \le\ & \overline{N}_0(r,0;g^{'}) + \sum_{j=1}^{n}\{\overline{N}(r,\omega_j;g \mid= 2) + 2\overline{N}(r,\omega_j;g \mid\ge 3)\}\\ \le\ & N(r,0;g^{'} \mid g \ne 0) + S(r,g) \le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g). \end{aligned}$$

Hence using (4.1), *Lemmas 3.1*, *3.2* and *3.3* we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned} & nT(r,f) \qquad (4.2)\\ \le\ & \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\ge 2) - N_0(r,0;f^{'})\\ & +S(r,f)\\ \le\ & 2\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,0;nf + a(n-1))\\ & +\overline{N}(r,0;ng + a(n-1)) + \overline{N}(r,1;G \mid\ge 2) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;g^{'})\\ & +S(r,f) + S(r,g)\\ \le\ & 2\left\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\right\} + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + T(r,f) + T(r,g)\\ & +S(r,f) + S(r,g)\\ \le\ & N(r,0;f) + N(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + T(r,f) + T(r,g)\\ & +S(r,f) + S(r,g)\\ \le\ & (6 - \delta(0;f) - \Theta(\infty;f) - \delta(0;g) - \Theta(\infty;g) + \varepsilon)\, T(r) + S(r). \end{aligned}$$

In a similar way we can obtain

$$T(r,g) \le (6 - \delta(0;f) - \Theta(\infty;f) - \delta(0;g) - \Theta(\infty;g) + \varepsilon)\, T(r) + S(r). \qquad (4.3)$$

Combining (4.2) and (4.3) we see that

$$(n - 6 + \delta(0;f) + \Theta(\infty;f) + \delta(0;g) + \Theta(\infty;g) - \varepsilon)\, T(r) \le S(r). \qquad (4.4)$$

Since $\varepsilon > 0$, (4.4) leads to a contradiction for $n \geq 5$.
While $m = 1$, using *Lemma 3.6*, (4.1) changes to

$$\begin{aligned} & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) \qquad (4.5)\\ \leq\ & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_L(r,1;G) + \overline{N}(r,1;F \mid\geq 3)\\ \leq\ & \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \frac{1}{2}\sum_{j=1}^{n}\{N(r,\omega_j;f) - \overline{N}(r,\omega_j;f)\}\\ \leq\ & \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \frac{1}{2}\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\} + S(r,f) + S(r,g). \end{aligned}$$

So using (4.5), *Lemmas 3.2* and *3.3* and proceeding as in (4.2) we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned} & nT(r,f) \qquad (4.6)\\ \leq\ & \frac{5}{2}\overline{N}(r,0;f) + \frac{3}{2}\overline{N}(r,\infty;f) + 2\overline{N}(r,0;g) + \overline{N}(r,\infty;g) + T(r,f) + T(r,g)\\ & +S(r,f) + S(r,g)\\ \leq\ & \frac{5}{4}\{N(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + N(r,0;g) + T(r,f) + T(r,g)\\ & +S(r,f) + S(r,g)\\ \leq\ & N(r,0;f) + \overline{N}(r,\infty;f) + N(r,0;g) + \overline{N}(r,\infty;g) + \frac{3}{2}T(r,f) + \frac{5}{4}T(r,g)\\ & +S(r,f) + S(r,g)\\ \leq\ & \left(6\frac{3}{4} - \delta(0;f) - \delta(0;g) - \Theta(\infty;f) - \Theta(\infty;g) + \varepsilon\right)T(r) + S(r). \end{aligned}$$

Similarly we can obtain

$$\begin{aligned} & nT(r,g) \qquad (4.7)\\ \leq\ & \left(6\frac{3}{4} - \delta(0;f) - \delta(0;g) - \Theta(\infty;f) - \Theta(\infty;g) + \varepsilon\right)T(r) + S(r). \end{aligned}$$

Combining (4.6) and (4.7) we see that

$$\left(n - 6\frac{3}{4} + \delta(0;f) + \Theta(\infty;f) + \delta(0;g) + \Theta(\infty;g) - \varepsilon\right)\ T(r) \leq S(r). \qquad (4.8)$$

Since $\varepsilon > 0$, (4.8) leads to a contradiction for $n \geq 6$.
Case 2. $m = 0$. Using *Lemma 3.6* we note that

$$\begin{aligned} & \overline{N}_0(r,0;g^{'}) + \overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) \qquad (4.9)\\ \leq\ & \overline{N}_0(r,0;g^{'}) + \overline{N}_E^{(2}(r,1;G) + \overline{N}_L(r,1;G) + \overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F)\\ \leq\ & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F)\\ \leq\ & N(r,0;g^{'} \mid g \neq 0) + \overline{N}(r,1;G \mid\geq 2) + 2\overline{N}(r,1;F \mid\geq 2)\\ \leq\ & 2\{N(r,0;g^{'} \mid g \neq 0) + N(r,0;f^{'} \mid f \neq 0)\}\\ \leq\ & 2\{\overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + \overline{N}(r,\infty;f)\} + S(r,f) + S(r,g) \end{aligned}$$

Hence using (4.9), *Lemmas 3.2* and *3.3* we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned}
& n\,T(r,f) \qquad (4.10)\\
\leq\; & \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + N_E^{1)}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G)\\
& + \overline{N}_E^{(2}(r,1;F) - N_0(r,0;f^{'}) + S(r,f)\\
\leq\; & 2\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + T(r,f) + T(r,g) + \overline{N}_E^{(2}(r,1;F)\\
& + 2\overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) + \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g)\\
\leq\; & 4\overline{N}(r,0;f) + 3\overline{N}(r,\infty;f) + 3\overline{N}(r,0;g) + 2\overline{N}(r,\infty;g) + T(r,f) + T(r,g)\\
& + S(r,f) + S(r,g)\\
\leq\; & 2N(r,0;f) + \frac{5}{2}\overline{N}(r,\infty;f) + \frac{3}{2}N(r,0;g) + \frac{5}{2}\overline{N}(r,\infty;g) + T(r,f) + T(r,g)\\
& S(r,f) + S(r,g)\\
\leq\; & \frac{3}{2}\{N(r,0;f) + \overline{N}(r,\infty;f) + N(r,0;g) + \overline{N}(r,\infty;g)\} + \frac{5}{2}T(r,f) + 2T(r,g)\\
& + S(r,f) + S(r,g)\\
\leq\; & \left\{10\frac{1}{2} - \frac{3}{2}\left(\delta(0;f) + \Theta(\infty;f) + \delta(0;g) + \Theta(\infty;g)\right) + \varepsilon\right\} T(r) + S(r).
\end{aligned}$$

In a similar manner we can obtain

$$\begin{aligned}
& nT(r,g) \qquad (4.11)\\
\leq\; & \left\{10\frac{1}{2} - \frac{3}{2}\left(\delta(0;f) + \Theta(\infty;f) + \delta(0;g) + \Theta(\infty;g)\right) + \varepsilon\right\} T(r) + S(r).
\end{aligned}$$

Combining (4.10) and (4.11) we see that

$$\begin{aligned}
& \left(n - 10\frac{1}{2} + \frac{3}{2}\left(\delta(0;f) + \Theta(\infty;f) + \delta(0;g) + \Theta(\infty;g)\right) - \varepsilon\right\} T(r) \qquad (4.12)\\
\leq\; & S(r).
\end{aligned}$$

Since $\varepsilon > 0$, (4.12) leads to a contradiction for $n \geq 10$.
Case 2. $H \equiv 0$. On integration we get from (3.2)

$$\frac{1}{F-1} \equiv \frac{A}{G-1} + B, \tag{4.13}$$

where A, B are constants and $A \neq 0$. From (4.13) we obtain

$$F \equiv \frac{(B+1)G + A - B - 1}{BG + A - B}. \tag{4.14}$$

Clearly (4.14) together with *Lemma 3.1* yields

$$T(r,f) = T(r,g) + O(1). \tag{4.15}$$

Subcase 2.1. Suppose that $B \neq 0, -1$. Since F and G share (∞,∞), it follows from (4.13) and (4.14) that ∞ is a Picard exceptional value of both f and g.

If $A-B-1\neq 0$, from (4.14) we obtain

$$\overline{N}(r,\frac{B+1-A}{B+1};G)=\overline{N}(r,0;F).$$

From above, *Lemma 3.1* and the second fundamental theorem we obtain

$$\begin{aligned} nT(r,g) &< \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{B+1-A}{B+1};G)+S(r,g)\\ &\leq \overline{N}(r,0;g)+\overline{N}(r,0;g+a)+\overline{N}(r,0;f)+\overline{N}(r,0;f+a)+S(r,g)\\ &\leq 2T(r,f)+2T(r,g)+S(r,g), \end{aligned}$$

which in view of (4.15) implies a contradiction as $n\geq 5$. Thus $A-B-1=0$ and hence (4.15) reduces to

$$F\equiv\frac{(B+1)G}{BG+1}.$$

From this we have

$$\overline{N}(r,\frac{-1}{B};G)=\overline{N}(r,\infty;f).$$

Again by *Lemma 3.1*, (4.15) and the second fundamental theorem we have

$$\begin{aligned} nT(r,g) &< \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{-1}{B};G)+S(r,g)\\ &\leq \overline{N}(r,0;g)+\overline{N}(r,0;g+a)+S(r,g)\\ &\leq 2T(r,g)+S(r,g), \end{aligned}$$

which in view of (4.15) leads to a contradiction since $n\geq 5$.
Subcase 2.2. Suppose that $B=-1$.
From (4.14) we obtain

$$F\equiv\frac{A}{-G+A+1}. \tag{4.16}$$

If $A+1\neq 0$, from (4.17) we obtain

$$\overline{N}(r,A+1;G)=\overline{N}(r,\infty;f).$$

So using the same argument as used in the above subcase we can again obtain a contradiction. Hence $A+1=0$ and we have from (4.17) that $FG\equiv 1$ that means $f^{n-1}(f+a)g^{n-1}(g+a)\equiv b^2$, which is impossible by *Lemma 3.4*.
Subcase 2.3. Suppose that $B=0$.
From (4.14) we obtain

$$F\equiv\frac{G+A-1}{A}. \tag{4.17}$$

If $A-1\neq 0$, from (4.17) we obtain

$$\overline{N}(r,1-A;G)=\overline{N}(r,0;F).$$

Using (4.15), *Lemma 3.1* and the second fundamental theorem we obtain

$$\begin{aligned}
& nT(r,g) \\
< \; & \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,1-A;G)+S(r,g) \\
\leq \; & \overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\overline{N}(r,0;g+a)+\overline{N}(r,0;f)+\overline{N}(r,0;f+a)+S(r,g) \\
\leq \; & \frac{1}{2}\{N(r,0;f)+N(r,0;g)\}+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f+a) \\
& +S(r,f)+S(r,g) \\
\leq \; & \frac{5}{2}T(r,f)+\frac{3}{2}T(r,f)+S(r,g)\leq 4T(r,g)+S(r,g),
\end{aligned}$$

which is a contradiction for $n \geq 5$. So $A = 1$ and hence $F \equiv G$ that is $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$. Now the theorem follows from *Lemma 3.5*. $\square$

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