

ON CLASS OF BI-NONLOCAL PROBLEMS INVOLVING FRACTIONAL $p(\cdot, \cdot)$ -LAPLACIAN OPERATORS

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ABSTRACT. This work aims to obtain the existence and multiplicity of solutions for a bi-nonlocal problem involving the $p(\cdot, \cdot)$ -Laplacian operator. Under specific assumptions, the paper uses the Mountain Pass Theorem combined with the Ekeland variational approach to establish the existence of two non-trivial solutions for the problem. The existence of infinitely many solutions is proved using the Symmetric Mountain Pass Theorem and the Ljusternik-Schnirelman Theorem. Finally, we study a $p(\cdot, \cdot)$ -Laplacian problem with non-variational form.

1. INTRODUCTION

Non-local problems are widely used to represent various problems in concrete fields, including population dynamics, thin obstacle problems, phase transitions, minimal surfaces, and game theory (see, e.g., [3, 9, 19, 26, 27]). On the other hand, there is plenty of motivation to study problems involving variable exponent growth conditions given their flexibility in representing the different types of phenomena found in the study of elastic mechanics [31], electrorheological fluids [1], or image restoration [10]. In this sense, the work of Kaufmann, Rossi, and Vidal, published in [17], paved the way by providing significant results on fractional Sobolev spaces with variable exponents of the form $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$. Their contributions include proving the compactness of these spaces in variable-exponent Lebesgue spaces and establishing the existence of solutions for non-local problems involving the fractional $p(\cdot, \cdot)$ -Laplacian. Concerning the basic features of fractional Sobolev spaces, we can consult [11].

Our main objective in the present article is to study the bi-nonlocal fractional $p(\cdot, \cdot)$ -Laplacian equation with the Neumann boundary condition. In detail, we

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are interested in the problem below:

$$\begin{cases} M(\int_Q \frac{|w(x)-w(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx dy)(-\Delta_{p(\cdot,\cdot)})^s w(x) + |w(x)|^{\bar{p}(x)-2} w(x) = K(\int_\Omega F(x, w) dx) f(x, w) & \text{in } \Omega, \\ M(\int_Q \frac{|w(x)-w(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx dy) \mathcal{N}_{s,p(\cdot,\cdot)} w(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (\text{P})$$

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded open domain and $Q := \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$, $N \geq 2$, $s \in (0, 1)$ and $p : \overline{Q} \rightarrow (1, +\infty)$ is a bounded continuous function, such that

$$1 < p^- = \inf_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x, y) \leq p(x, y) \leq p^+ = \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x, y) < +\infty, \quad (1.1)$$

$$p(y, x) = p(x, y), \text{ for all } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N; \quad \text{that is, } p \text{ is symmetric.} \quad (1.2)$$

We set

$$\bar{p}(x) := p(x, x) \text{ for all } x \in \mathbb{R}^N.$$

The operator $(-\Delta_{p(\cdot,\cdot)})^s$ mentioned above is the fractional $p(\cdot, \cdot)$ -Laplacian operator determined by the expression:

$$(-\Delta_{p(\cdot,\cdot)})^s w(x) = p.v \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^{p(x,y)-2} (w(x) - w(y))}{|x - y|^{N+sp(x,y)}} dy, \text{ for all } x \in \mathbb{R}^N,$$

where, the acronym $p.v$ is frequently employed to refer to the principal value. Furthermore, $\mathcal{N}_{s,p(\cdot,\cdot)}$ is defined by:

$$\mathcal{N}_{s,p(\cdot,\cdot)} w(x) = \int_\Omega \frac{|w(x) - w(y)|^{p(x,y)-2} (w(x) - w(y))}{|x - y|^{N+sp(x,y)}} dy, \text{ for all } x \in \mathbb{R}^N \setminus \Omega.$$

The p -Laplacian operator has already been integrated into the enlargement of the analysis of the Kirchhoff equations

$$\begin{cases} -M(\int_\Omega |\nabla w|^p dx) \Delta_p w = f(x, w, \nabla w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

(see [12, 16, 20, 21, 30]). However, many studies have extended the case of constants to cover the $p(x)$ -Laplacian operator, for instance, in [14], based on mapping theory and Mountain-Pass geometry, Fan presented the non-local Dirichlet problem $p(x)$ -Laplacian with non-variational form

$$\begin{cases} -\mathcal{A}(w) \operatorname{div}(|\nabla w|^{p(x)-2} \nabla w) = \mathcal{B}(w) f(x, w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

and nonlocal $p(x)$ -Laplacian Dirichlet problem with variational form

$$\begin{cases} -a(\int_\Omega \frac{|\nabla w|^{p(x)}}{p(x)} dx) \Delta_{p(x)} w = b(\int_\Omega F(x, w) dx) f(x, w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively and has obtained the existence of solutions, where $\mathcal{A}$, $\mathcal{B}$ are two functionals defined on $W_0^{1,p(x)}(\Omega)$, and $F(x, t) = \int_0^t f(x, s) ds$. Likewise, several researchers extended the classical case of integers to include the fractional one. For instance, in [15], the authors approved the existence of infinitely many solutions without considering the Ambrosetti–Rabinowitz condition by applying variational methods, particularly the Symmetric Mountain Pass theorem.

$$\begin{cases} M(\int_{\Omega \times \Omega} \frac{|w(x)-w(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx dy)(-\Delta_{p(x,\cdot)})^s w(x) = f(x, w) & \text{in } \Omega, \\ w = 0 & \text{in } \partial\Omega, \end{cases}$$

where M and f are continuous functions satisfying certain growth conditions.

Inspired by the above papers, we aim to study for the first time $p(\cdot, \cdot)$ -Kirchhoff equation with the Neumann boundary condition. To establish the existence and multiplicity of solutions, we assume that f , M and K verify certain specific conditions.

Concerning the Kirchhoff coefficient in (P), we suppose that, $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying the following conditions:

(M_0): For every $\kappa > 0$ there exists $m_0 = m_0(\kappa) > 0$ such that

$$M(s) \geq m_0, \text{ for all } s \geq \kappa.$$

(M_1): There are $\gamma \in (0, 1)$ and $s_0 > 0$ such that

$$\hat{M}(s) \geq \gamma M(s)s, \text{ for all } s \geq s_0 > 0, \text{ where } \hat{M}(s) = \int_0^s M(t) dt.$$

(M_2): There is $\alpha > 1$ such that

$$\limsup_{s \rightarrow 0^+} \frac{\hat{M}(s)}{s^\alpha} < +\infty.$$

A classic model of M , presented by Kirchhoff in 1883, is defined as

$$M(s) = a + bs^{\alpha-1}, \quad a, b \geq 0, \quad a + b > 0, \quad s \geq 0,$$

and

$$\begin{cases} \alpha \in (1, +\infty) & \text{if } b > 0, \\ \alpha = 1 & \text{if } b = 0, \end{cases}$$

Recall that Kirchhoff problems are degenerate if, for each $s > 0$, $M(s) > 0$ and $M(0) = 0$. On the contrary, they are classified as non-degenerate when $M(s) > 0$ for each $s \geq 0$, and this occurs, for instance when, $a > 0$ and $b \geq 0$.

For the function $K : \mathbb{R} \rightarrow \mathbb{R}$, we consider that it is continuous and verifying the assumptions (K_0) and (K_1) such that:

(K_0): There exist $\beta_0 \in (0, 1)$ and $s_0 > 0$ such that

$$0 < \beta_0 \hat{K}(s) \leq K(s)s, \text{ for all } s \geq s_0 > 0, \text{ where } \hat{K}(s) = \int_0^s K(t) dt.$$

(K_1): There is $\beta_1 > 0$ such that

$$\limsup_{s \rightarrow 0} \frac{\hat{K}(s)}{|s|^{\beta_1}} < +\infty.$$

While, the continuous function f fulfills the following assumptions:

$$(H_1) : |f(x, s)| \leq C(1 + |s|^{q(x)-1}), \text{ for all } (x, s) \in \Omega \times \mathbb{R},$$

where $C > 0$, $\bar{p}(x) < q(x) < p_s^*(x) := \frac{N\bar{p}(x)}{N-s\bar{p}(x)}$.

(H_2): The Ambrosetti-Rabinowitz (A-R) condition holds:

$$F(x, s) = \int_0^s f(x, t) dt,$$

is θ -super-homogeneous at infinity, i.e. there exists $s_0 > 0$ such that $0 < \theta F(x, s) \leq sf(x, s)$, for all $|s| > s_0$, $x \in \Omega$ where $\theta > \frac{p^+}{\gamma\beta_0}$.

(H_3): The following holds uniformly for $x \in \Omega$:

$$\lim_{s \rightarrow 0} \frac{f(x, s)}{|s|^{\bar{p}(x)-2}s} = 0.$$

(H_4): $f(x, -s) = -f(x, s)$, for all $(x, s) \in \Omega \times \mathbb{R}$.

(H_5): There exist an open subset Ω_0 of Ω and $r > 0$ such that

$$\liminf_{s \rightarrow 0} \frac{F(x, s)}{|s|^r} > 0 \text{ uniformly for } x \in \Omega_0, \text{ with } r\beta_0 \leq p^-.$$

(H_6): There is $\beta \in C(Q, (1, +\infty))$, verifies (1.1) and (1.2) such that

$$\lim_{s \rightarrow +\infty} \frac{F(x, s)}{|s|^{\beta(x)}} > 0, \text{ with } \beta^- > \frac{p^+}{\gamma\beta_0}.$$

Now we can reveal our principal results.

Theorem 1. *Let the function $p \in C(\bar{\Omega})$ such that $p^+ < \beta_1 p^-$. Assume that (M_0) -(M_1), (K_0) -(K_1) and (H_1) -(H_3) are verified. Then the problem (P) has at least two nontrivial weak solutions.*

If in addition, f satisfies conditions (H_4) and (H_6) . Then the problem (P) has a sequence of solutions $(\pm w_j)$ such that $\psi(\pm w_j) \rightarrow \infty$ as $j \rightarrow \infty$.

Theorem 2. (i) *Let the function $p \in C(\bar{\Omega})$ such that $\beta_1 q^+ < p^-$. Assume that (M_0) -(M_2), (K_0) -(K_1) and (H_1) -(H_5) are satisfied. Then the problem (P) has a sequence of solutions $(\pm w_j)$ such that $\psi(\pm w_j) < 0$.*

(ii) *Under the assumptions (M_0) -(M_2), (H_1) , (H_5) and the following condition:*

$$(H_7): \lim_{|s| \rightarrow +\infty} [F(x, s) - \lambda_1 \mu \frac{|s|^{p^-}}{\bar{p}(x)}] = -\infty, \text{ uniformly for almost every } x \in \Omega,$$

$$\text{where } \lambda_1 := \inf_{w \in E \setminus \{0\}} \frac{\int_Q \frac{|w(x)-w(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy + \int_{\Omega} \frac{|w|^{\bar{p}(x)}}{\bar{p}(x)} dx}{\int_{\Omega} \frac{|w|^{\bar{p}(x)}}{\bar{p}(x)} dx} > 0 \text{ and } \mu := \min\{\frac{m_0}{2}, 1\} > 0.$$

The problem

$$\begin{cases} M(\int_Q \frac{|w(x)-w(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx dy)(-\Delta_{p(\cdot,\cdot)})^s w(x) + |w(x)|^{\bar{p}(x)-2} w(x) = f(x, w) & \text{in } \Omega, \\ M(\int_Q \frac{|w(x)-w(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dx dy) \mathcal{N}_{s,p(\cdot,\cdot)} w(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (\text{Q})$$

has a nontrivial weak solution.

Example 1.1. (1) The following functions indicate that our conditions for Theorem 1 are satisfied:

- $M(s) = a + bs$ for all $s \geq 0$, with $a > 0$ and $b \geq 0$.
- $K(s) = |s|^{\beta_1-2} s$, for all $s \in \mathbb{R}$, where $\beta_1 > 1$.
- $f(x, s) = |s|^{q(x)-2} s$, for all $s \in \mathbb{R}$, where $\bar{p}(x) < q(x) < p_s^*(x)$, for all $x \in \bar{\Omega}$.

(2) An example of our conditions being satisfied for the first assertion of the Theorem 2 is provided by the next functions:

- $M(s) = bs^{\alpha-1}$ for all $s > 0$, with $\alpha > 0$ and $b > 0$.
- $K(s) = |s|^{\beta_1-2} s$, for all $s \in \mathbb{R}$, where $\beta_1 > 1$.
- $f(x, s) = |s|^{q(x)-2} s$, for all $s \in \mathbb{R}$, where $\bar{p}(x) < q(x) < p_s^*(x)$, for all $x \in \bar{\Omega}$.

The remaining sections of the paper are structured in this manner. In the second section, we present certain initial results relating to Lebesgue and fractional Sobolev spaces, and we recall the essential tools for the sections coming up. The validity of the compactness condition is confirmed in the third section. While, the fourth and fifth sections are reserved for proving Theorem 1 and 2, respectively. Finally, the sixth section is devoted to studying the general (non-variational) form of the problem (P).

2. PRELIMINARY RESULTS

In this section, we provide a brief overview of Lebesgue spaces, including their definitions and some basic properties. Interested readers can consult ([4, 18, 25, 26]), and the references therein. Additionally, we discuss certain essential features of fractional Sobolev spaces with variable exponents.

2.1. Variable-exponent Lebesgue spaces. Let Ω be an open bounded subset of $\mathbb{R}^N$. In this subsection, we present certain needed features of variable exponent Lebesgue spaces. We take the set

$$C_+(\bar{\Omega}) = \{\alpha \in C(\bar{\Omega}) \mid \alpha(x) > 1, \text{ for all } x \in \bar{\Omega}\}.$$

For all $\alpha \in C_+(\bar{\Omega})$, we define:

$$\alpha^+ = \sup_{x \in \bar{\Omega}} \alpha(x) \quad \text{and} \quad \alpha^- = \inf_{x \in \bar{\Omega}} \alpha(x),$$

such that

$$1 < \alpha^- \leq \alpha(x) \leq \alpha^+ < +\infty.$$

For any $\alpha \in C_+(\overline{\Omega})$, the generalized Lebesgue space is defined as

$$L^{\alpha(x)}(\Omega) = \left\{ w : w \text{ is a measurable real-valued function, } \int_{\Omega} |w(x)|^{\alpha(x)} dx < +\infty \right\}.$$

and it is equipped a norm as given below (the Luxemburg norm)

$$|w|_{L^{\alpha(\cdot)}} = \inf \left\{ \mu > 0 \mid \int_{\Omega} \left| \frac{w(x)}{\mu} \right|^{\alpha(x)} dx \leq 1 \right\}.$$

Thus $(L^{\alpha(\cdot)}(\Omega), |\cdot|_{L^{\alpha(\cdot)}})$ becomes a separable reflexive Banach space.

Let $\beta \in C_+(\overline{\Omega})$ such that, $\frac{1}{\alpha(\cdot)} + \frac{1}{\beta(\cdot)} = 1$. We then obtain the next Hölder-type inequality:

Lemma 2.1. *If $v \in L^{\alpha(\cdot)}$ and $w \in L^{\beta(\cdot)}$; then:*

$$\int_{\Omega} |vw| dx \leq 2|v|_{L^{\alpha(\cdot)}} |w|_{L^{\beta(\cdot)}}.$$

A key element in the manipulation of generalized Lebesgue spaces is represented by the modular of the space $L^{\alpha(x)}(\Omega)$, which is defined as follows:

$$\begin{aligned} \rho_{\alpha(\cdot)} : L^{\alpha(x)}(\Omega) &\rightarrow \mathbb{R} \\ w &\rightarrow \int_{\Omega} |w(x)|^{\alpha(x)} dx. \end{aligned}$$

Lemma 2.2. *Let $w \in L^{\alpha(\cdot)}(\Omega)$, then we get*

- $|w|_{L^{\alpha(\cdot)}(\Omega)} < 1$ ($rp := 1, > 1$) $\Leftrightarrow \rho_{\alpha(\cdot)}(w) < 1$ ($rp := 1, > 1$).
- $|w|_{L^{\alpha(\cdot)}(\Omega)} > 1 \Rightarrow |w|_{L^{\alpha(\cdot)}(\Omega)}^{\alpha^-} \leq \rho_{\alpha(\cdot)}(w) \leq |w|_{L^{\alpha(\cdot)}(\Omega)}^{\alpha^+}$.
- $|w|_{L^{\alpha(\cdot)}(\Omega)} < 1 \Rightarrow |w|_{L^{\alpha(\cdot)}(\Omega)}^{\alpha^+} \leq \rho_{\alpha(\cdot)}(w) \leq |w|_{L^{\alpha(\cdot)}(\Omega)}^{\alpha^-}$.

2.2. Variable-exponent fractional Sobolev spaces. The definition and some results on fractional Sobolev spaces with variable exponents are provided in this subsection, which originally appeared in [8, 17].

Let $p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be a Lipschitz continuous bounded function verifies (1.1) and (1.2).

We define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$X = W^{s,p(x,y)}(\Omega) = \left\{ w \in L^{\bar{p}(x)}(\Omega) \mid \int_{\Omega \times \Omega} \frac{|w(x) - w(y)|^{p(x,y)}}{\mu^{p(x,y)} |x - y|^{N+sp(x,y)}} dx dy < +\infty, \text{ for some } \mu > 0 \right\}.$$

The space X is a separable, reflexive and Banach space if it is equipped with the norm

$$\|w\|_X = |w|_{L^{\bar{p}(x)}} + [w]_{s,p(x,y)}$$

where $[.]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[w]_{s,p(x,y)} := [w]_{s,p(x,y),\Omega} = \inf \left\{ \mu > 0 \mid \int_{\Omega \times \Omega} \frac{|w(x) - w(y)|^{p(x,y)}}{\mu^{p(x,y)} |x - y|^{N+sp(x,y)}} dx dy \leq 1 \right\}.$$

Let $w : \mathbb{R}^N \rightarrow \mathbb{R}$ be a measurable function. We set

$$\|w\|_E := |w|_{L^{\bar{p}(x)}} + [w]_{s,p(x,y),Q}, \quad \text{where } E = \{w : \mathbb{R}^N \rightarrow \mathbb{R} \mid \|w\|_E < +\infty\}.$$

Proposition 2.1. *Let hypotheses (1.1) and (1.2) be satisfied. Then, $(E, \|\cdot\|_E)$ is a reflexive Banach space.*

Proof. The demonstration is analogous to that of [[7], Proposition 3.1].

Proposition 2.2. *Let hypotheses (1.1) and (1.2) be satisfied. Then, for any $h \in C^1(\bar{\Omega})$ with $1 < h(x) < p_s^*(x)$ for all $x \in \bar{\Omega}$, there exists a constant $c > 0$ such that $|w|_{L^{h(\cdot)}(\Omega)} \leq c\|w\|_E$ for all $w \in E$. Moreover, this embedding is compact.*

Proof. It is obvious that $\|w\|_X \leq \|w\|_E$, for all $w \in E$. Therefore, $X \hookrightarrow L^{h(\cdot)}(\Omega)$. So we achieve the intended result.

Note that the norm $\|\cdot\|_E$ is equivalent on E to $\|\cdot\| = \inf \{\mu > 0 \mid \rho(\frac{u}{\mu}) \leq 1\}$, where the modular $\rho : E \rightarrow \mathbb{R}$ is defined by

$$\rho(w) = \int_Q \frac{|w(x) - w(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy + \int_{\Omega} |w(x)|^{\bar{p}(x)} dx.$$

Later on in reasoning, this lemma will be useful.

Lemma 2.3. *Let assumptions (1.1) and (1.2) be verified and $w \in E$. Then the following hold:*

- (i) $\|w\| < 1$ ($rp := 1, > 1$) $\iff \rho(w) < 1$ ($rp := 1, > 1$).
- (ii) $\|w\| < 1 \implies \|w\|^{p^+} \leq \rho(w) \leq \|w\|^{p^-}$.
- (iii) $\|w\| > 1 \implies \|w\|^{p^-} \leq \rho(w) \leq \|w\|^{p^+}$.

Proof. The demonstration is analogous to that of [5].

Now, we present the functional $\mathcal{I} : E \rightarrow \mathbb{R}$, given by

$$\mathcal{I}(w) = \int_Q \frac{|w(x) - w(y)|^{p(x,y)}}{p(x,y)|x - y|^{N+sp(x,y)}} dx dy + \int_{\Omega} \frac{|w(x)|^{\bar{p}(x)}}{\bar{p}(x)} dx.$$

Lemma 2.4. [[6], Lemma 3.1] *Suppose that condition (1.1) and (1.2) holds. Then, the functional $\mathcal{I}$ is well-defined and convex on E . Furthermore, $\mathcal{I} \in C^1(E, \mathbb{R})$ with the derivative expressed as follows*

$$\begin{aligned} \langle \mathcal{L}(w), v \rangle &= \langle \mathcal{I}'(w), v \rangle = \int_Q \frac{|w(x) - w(y)|^{p(x,y)-2} (w(x) - w(y)) (v(x) - v(y))}{|x - y|^{N+sp(x,y)}} dx dy \\ &\quad + \int_{\Omega} |w(x)|^{\bar{p}(x)-2} w(x) v(x) dx. \end{aligned}$$

Lemma 2.5. [[8], Lemma 4.2] *Suppose that condition (1.1) and (1.2) holds. Then, the following assertions hold:*

- (i) $\mathcal{L}$ is a bounded and strictly monotone operator.
- (ii) $\mathcal{L}$ is a mapping of type (S_+) , that is, if $w_k \rightharpoonup w$ in E and $\limsup_{k \rightarrow +\infty} \langle \mathcal{L}(w_k) - \mathcal{L}(w), w_k - w \rangle \leq 0$, then $w_k \rightarrow w$ in E .
- (iii) $\mathcal{L}$ is a homeomorphism.

The following statement refers to the non-local equivalent of the divergence Theorem also called integration by parts formula.

Proposition 2.3 ([7]). *Let hypotheses (1.1) and (1.2) be satisfied. Let w and v be bounded C^2 -functions in $\mathbb{R}^N$. Then,*

(1)

$$\int_{\Omega} (-\Delta_{p(\cdot,\cdot)})^s w(x) dx = - \int_{\mathbb{R}^N \setminus \Omega} \mathcal{N}_{s,p(\cdot,\cdot)} w(x) dx.$$

(2)

$$\begin{aligned} \frac{1}{2} \int_Q \frac{|w(x) - w(y)|^{p(x,y)-2} (w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+sp(x,y)}} dx dy &= \int_{\Omega} v(-\Delta_{p(\cdot,\cdot)})^s w dx \\ &+ \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_{s,p(\cdot,\cdot)} w dx. \end{aligned}$$

Definition 2.1. *Let $(E, \|\cdot\|)$ be a Banach space and $J \in C^1(E)$. We state that J verifies the Palais-Smale condition at the level $c \in \mathbb{R}$ (" $(PS)_c$ condition" for short) if any sequence $w_n \in E$ verifying*

$$J(w_n) \rightarrow c \text{ and } J'(w_n) \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

has a convergent subsequence.

3. EXISTENCE AND MULTIPLICITY RESULTS

Definition 3.1. *We say that $u \in E$ is a weak solution of problem (P), if*

$$\begin{aligned} &\frac{1}{2} M(\sigma_{p(x,y)}(u)) \int_Q \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp(x,y)}} dx dy \\ &+ \int_{\Omega} |u(x)|^{\bar{p}(x)-2} u(x) v(x) dx - K \left(\int_{\Omega} F(x, u) dx \right) \int_{\Omega} f(x, u) v dx = 0, \text{ for all } v \in E. \end{aligned}$$

$$\text{Where, } \sigma_{p(x,y)}(u) = \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{2p(x,y)|x - y|^{N+sp(x,y)}} dx dy.$$

The energy functional $\psi : E \rightarrow \mathbb{R}$ associated with the problem (P) is giving by:

$$\psi(u) = \hat{M}(\sigma_{p(x,y)}(u)) + \int_{\Omega} \frac{|u(x)|^{\bar{p}(x)}}{\bar{p}(x)} dx - \hat{K} \left(\int_{\Omega} F(x, u) dx \right) := \Phi(u) - \varphi(u), \text{ for all } u \in E.$$

$$\text{Where } \Phi(u) = \hat{M}(\sigma_{p(x,y)}(u)) + \int_{\Omega} \frac{|u(x)|^{\bar{p}(x)}}{\bar{p}(x)} dx \text{ and } \varphi(u) = \hat{K} \left(\int_{\Omega} F(x, u) dx \right).$$

Standard arguments (see, [5], [14]) and the continuity of M and K imply that ψ is well-defined and of C^1 -class on E . Moreover, we have

$$\begin{aligned} \langle \psi'(u), v \rangle &= \frac{1}{2} M(\sigma_{p(x,y)}(u)) \int_Q \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp(x,y)}} dx dy \\ &+ \int_{\Omega} |u(x)|^{\bar{p}(x)-2} u(x) v(x) dx - K \left(\int_{\Omega} F(x, u) dx \right) \int_{\Omega} f(x, u) v dx, \text{ for all } u, v \in E. \end{aligned}$$

Thus the weak solutions of (P) are the critical points of ψ .

- Proposition 3.1.** (i) ψ is sequentially weakly lower semi-continuous (s.w.l.s.c).
(ii) $\varphi' : E \rightarrow E^*$ is sequentially weakly-strongly continuous.
(iii) For any open set $V \subset E \setminus \{0\}$, with $\bar{V} \subset E \setminus \{0\}$, mapping $\psi' : \bar{V} \rightarrow E^*$ is of type (S_+) .

Proof.

- (i) From [[8], Lemma 2.4] the operator $u \rightarrow \sigma'_{p(x,y)}(u)$ is strictly monotone. From [[29], Proposition 25.10], $\sigma_{p(x,y)}$ is strictly convex. So, let $(u_n) \subset E$ be a sequence such that $u_n \rightarrow u$ in E . Then, by the convexity of $\sigma_{p(x,y)}$, we have

$$\sigma_{p(x,y)}(u_n) - \sigma_{p(x,y)}(u) \geq \langle \sigma'_{p(x,y)}(u_n), u_n - u \rangle.$$

Hence we obtain

$$\sigma_{p(x,y)}(u) \leq \liminf_{n \rightarrow +\infty} \sigma_{p(x,y)}(u_n).$$

So the map $u \rightarrow \sigma_{p(x,y)}(u)$ is (s.w.l.s.c). However, as $t \rightarrow \hat{M}(t)$ is a continuous and increasing function, we obtain

$$\liminf_{n \rightarrow +\infty} \Phi(u_n) = \liminf_{n \rightarrow +\infty} \hat{M}(u_n) + \liminf_{n \rightarrow +\infty} \int_{\Omega} \frac{|u_n(x)|^{\bar{p}(x)}}{\bar{p}(x)} dx \geq \hat{M}(u) + \int_{\Omega} \frac{|u(x)|^{\bar{p}(x)}}{\bar{p}(x)} dx = \Phi(u).$$

Thus, the functional Φ is (s.w.l.s.c). Moreover φ is (s.w.l.s.c) ([14]), and thus ψ is (s.w.l.s.c).

- (ii) ([14], Proposition 3.1).

- (iii) Let $\bar{V} \subset E \setminus \{0\}$. Assume that $(u_n) \subset \bar{V}$ such that $u_n \rightharpoonup u$ in E and $\limsup_{n \rightarrow +\infty} \Phi'(u_n - u) \leq 0$, then there is positive constants c_1 and c_2 such that $c_2 \leq \sigma_{p(x,y)}(u_n) \leq c_1$, and therefore there is positive constants c_3 and c_4 such that $c_4 \leq M(\sigma_{p(x,y)}(u_n)) \leq c_3$. It follows from $\limsup_{n \rightarrow +\infty} \Phi'(u_n)(u_n - u) \leq 0$ that, $\limsup_{n \rightarrow +\infty} \sigma'_{p(x,y)}(u_n)(u_n - u) \leq 0$. From ([8], Lemma 2.4), $\sigma'_{p(x,y)}$ is of type (S_+) , we obtain $u_n \rightarrow u$ in E . Moreover, since φ' is sequentially weakly-strongly continuous, the mapping $\psi' : \bar{V} \rightarrow E^*$ is of type (S_+) .

From now on, C_i and c_i , $i = 1, 2, \dots$ are positive constants whose exact value may change from line to line.

Lemma 3.1. Under assumptions (M_0) -(M_1), (K_0) and (H_1) -(H_2), the functional ψ satisfies the $(PS)_c$ condition for any $c \neq 0$.

Proof. There are two steps to take.

Step 1: We demonstrate that (u_n) is bounded in E . Let $(u_n) \subset E$ be a $(PS)_c$ sequence for any $c \neq 0$. When $\|u\|$ is sufficiently large, we can use (M_1) to get

$$\begin{aligned} \Phi(u) &= \hat{M}(\sigma_{p(x,y)}(u)) + \int_{\Omega} \frac{|u(x)|^{\bar{p}(x)}}{\bar{p}(x)} dx \\ &\geq M(\sigma_{p(x,y)}(u))\sigma_{p(x,y)}(u) + \frac{1}{p^+} \int_{\Omega} |u(x)|^{\bar{p}(x)} dx \\ &\geq \frac{\gamma}{p^+} \Phi'(u) \cdot u. \end{aligned}$$

From (H_2) , there exist $C_1 > 0$ such that

$$-C_1 \leq \theta \int_{\Omega} F(x, u) dx \leq \int_{\Omega} f(x, u)u dx + C_1, \text{ for all } u \in E,$$

thus, for all $\varepsilon \in (0, \theta)$, there exist $A_\varepsilon \geq s_0 > 0$ such that

$$(\theta - \varepsilon) \int_{\Omega} F(x, u) dx \leq \int_{\Omega} f(x, u)u dx, \text{ if } \int_{\Omega} F(x, u) dx \geq A_\varepsilon. \quad (3.1)$$

We may suppose that $A_\varepsilon \geq \frac{C_1}{\theta}$. It should be pointed out that in this instance, the inequality $\int_{\Omega} F(x, u) dx \geq A_\varepsilon$ is equivalent to $|\int_{\Omega} F(x, u) dx| \geq A_\varepsilon$, because $\int_{\Omega} F(x, u) dx \geq -\frac{C_1}{\theta}$, for all $u \in E$.

We claim that there exist $C_\varepsilon \geq 0$ such that

$$\varphi'(u)u - \beta_0(\theta - \varepsilon)\varphi(u) \geq -C_\varepsilon, \text{ for all } u \in E. \quad (3.2)$$

Indeed, if $|\int_{\Omega} F(x, u) dx| \leq A_\varepsilon$, then it is immediately clear that (3.2) is valid.

When $|\int_{\Omega} F(x, u) dx| \geq A_\varepsilon$, i.e. $\int_{\Omega} F(x, u) dx \geq A_\varepsilon$, it results from (3.1) and (K_0) that

$$\beta_0(\theta - \varepsilon)\varphi(u) = \beta_0(\theta - \varepsilon)\hat{K}\left(\int_{\Omega} F(x, u) dx\right) \leq (\theta - \varepsilon)K\left(\int_{\Omega} F(x, u) dx\right) \int_{\Omega} F(x, u) dx \leq \varphi'(u) \cdot u.$$

Thus, (3.2) is proven. Now let $(u_n) \subset E \setminus \{0\}$, $\psi'(u_n) \rightarrow 0$ and $\psi(u_n) \rightarrow c$ with $c \neq 0$. Since $p^+ < \beta_0\theta\gamma$, there exist $\varepsilon > 0$ sufficiently small so that $p^+ < \beta_0(\theta - \varepsilon)\gamma$.

Therefore, for n large enough we get

$$\begin{aligned} \beta_0(\theta - \varepsilon)c + 1 + \|u_n\| &\geq \beta_0(\theta - \varepsilon)\psi(u_n) - \psi'(u_n)u_n \\ &\geq \left(\beta_0(\theta - \varepsilon) - \frac{p^+}{\gamma}\right)\Phi(u_n) + \left(\frac{p^+}{\gamma}\Phi(u_n) - \Phi'(u_n) \cdot u_n\right) - \beta_0(\theta - \varepsilon)\varphi(u_n) \\ &\quad + \varphi'(u_n)u_n \\ &\geq \left(\beta_0(\theta - \varepsilon) - \frac{p^+}{\gamma}\right)\Phi(u_n) - C_\varepsilon \\ &\geq \left(\beta_0(\theta - \varepsilon) - \frac{p^+}{\gamma}\right)\frac{\gamma}{p^+}\Phi'(u_n) \cdot u_n - C_\varepsilon \\ &= \left(\frac{\beta_0(\theta - \varepsilon)\gamma}{p^+} - 1\right)\left(\frac{1}{2}M(\sigma_{p(x,y)}(u_n))\int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy\right. \\ &\quad \left.+ \int_{\Omega} |u_n(x)|^{\bar{p}(x)} dx\right) - C_\varepsilon \\ &\geq \min\left(\frac{m_0}{2}, 1\right)\left(\frac{\beta_0(\theta - \varepsilon)\gamma}{p^+} - 1\right)\rho(u_n) - C_\varepsilon \\ &\geq \min\left(\frac{m_0}{2}, 1\right)\left(\frac{\beta_0(\theta - \varepsilon)\gamma}{p^+} - 1\right)\|u_n\|^{p^-} - C_\varepsilon \\ &\geq c\|u_n\|^{p^-} - C_\varepsilon. \end{aligned}$$

With c is a positive constant. According to the preceding inequalities, (u_n) is bounded in E , due to $p^- > 1$.

Step 2: We now assert that the subsequence of (u_n) is strongly convergent. Setting $V = \{w \in E \mid \|w\| > \varepsilon\}$. As $\psi(u_n) \rightarrow c \neq 0$ and $\psi(0) = 0$, there exists $\varepsilon > 0$ small enough such that for n sufficiently large, $\|u_n\| > \varepsilon$, then $u_n \in V$ for n sufficiently large. We can take a subsequence of (u_n) , still given by (u_n) such that $u_n \in V$ and $u_n \rightharpoonup u$, for the reason that (u_n) is bounded. The condition $\psi'(u_n) \rightarrow 0$ implies $\psi'(u_n)(u_n - u) \rightarrow 0$. Since $\psi' : \bar{V} \rightarrow E^*$ is of (S_+) -type, we have $u_n \rightarrow u$ in $\bar{V}$.

4. PROOF OF THEOREM 1

Under appropriate assumptions, the Mountain Pass Theorem, the Symmetric Mountain Pass Theorem, and the Ekeland variational principle are used to discuss the existence of non-trivial weak solutions for (P) in the section that follows. This means we must establish two lemmas.

Lemma 4.1. *Suppose that assumptions (M_1) , (K_0) and (H_1) – (H_2) are fulfilled. Hence for each $w \in E \setminus \{0\}$ $\psi(sw) \rightarrow -\infty$ as $s \rightarrow +\infty$.*

Proof. Let $w \in E \setminus \{0\}$. From (M_1) and for $s \geq 1$, we may easily assert that $\hat{M}(s) \leq \hat{M}(1)s^{\frac{1}{\gamma}}$. Then

$$\begin{aligned} \Phi(sw) &= \hat{M}(\sigma_{p(x,y)}(sw)) + \int_{\Omega} \frac{|sw|^{\bar{p}(x)}}{\bar{p}(x)} dx \\ &\leq \hat{M}(1)(\sigma_{p(x,y)}(sw))^{\frac{1}{\gamma}} + \frac{1}{p^-} \int_{\Omega} |sw|^{\bar{p}(x)} dx \\ &\leq c_1 s^{\frac{p^+}{\gamma}} + c_2 s^{p^+}. \end{aligned}$$

By conditions (H_1) and (H_2) , we have

$$\int_{\Omega} F(x, sw) dx \geq c_3 s^{\theta},$$

From (K_0) we have that $\hat{K}(t) \geq c_1 t^{\beta_0}$, for t large enough, thus

$$\varphi(sw) = \hat{K}\left(\int_{\Omega} F(x, sw) dx\right) \geq c_4 s^{\beta_0 \theta},$$

hence for any $w \in E \setminus \{0\}$,

$$\psi(sw) \leq c_1 s^{\frac{p^+}{\gamma}} + c_2 s^{p^+} - c_4 s^{\beta_0 \theta}.$$

Since, $\beta_0 \theta > \frac{p^+}{\gamma} > p^+$, we deduce that $\psi(sw) \rightarrow -\infty$ as $s \rightarrow +\infty$.

Lemma 4.2. *Assume that conditions (M_0) – (M_1) , (K_1) , (H_1) and (H_3) are verified. Hence, there exist positive numbers a, R such that $\psi(u) \geq a > 0$ with $\|u\| = R$.*

Proof. Given $\|u\|$ small enough, we get:

$$\int_{\Omega} F(x, u) dx \leq \varepsilon \int_{\Omega} |u|^{\bar{p}(x)} dx \leq C_2 \|u\|^{p^-}.$$

Therefore

$$\varphi(u) = \hat{K} \left(\int_{\Omega} F(x, u) dx \right) \leq C_3 \|u\|^{\beta_1 p^-}.$$

Thus,

$$\begin{aligned} \psi(u) &\geq \frac{m_0 \gamma}{2p^+} \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy + \frac{1}{p^+} \int_{\Omega} |u|^{\bar{p}(x)} dx - \varphi(u) \\ &\geq \frac{\min \left\{ \frac{m_0 \gamma}{2}, 1 \right\}}{p^+} \|u\|^{p^+} - C_3 \|u\|^{\beta_1 p^-}. \end{aligned}$$

Since, $\beta_1 p^- > p^+$, so there exists two constants a, R such that $\psi(u) \geq a > 0$ with $\|u\| = R$.

We then deduce that ψ has a critical point $e_1 \in E$ which is a weak solution for problem (P), by using Lemmas 3.1, 4.1, 4.2 and the Mountain pass theorem. Moreover, e_1 satisfies

$$\psi(e_1) \geq a > 0, \quad (3.3)$$

which implies that e_1 is non-trivial.

Via the Ekeland variational principle, we will demonstrate the existence of a second weak solution $e_2 \neq e_1$.

By lemma 4.1, there exist $w \in E$ such that $\psi(sw) < 0$ for $s > 0$ large enough, and according to lemma 4.2, we get $\inf_{u \in \partial B(0, R)} (\psi(u)) > 0$. As indicated in the proof of lemma 4.2, for $u \in B(0, R)$ we have

$$\psi(u) \geq \frac{\min \left\{ \frac{m_0 \gamma}{2}, 1 \right\}}{p^+} \|u\|^{p^+} - C_3 \|u\|^{\beta_1 p^-}.$$

Therefore $-\infty < \bar{c} = \inf_{u \in \overline{B(0, R)}} (\psi(u)) < 0$.

Let $\delta > 0$, such that $0 < \delta < (\inf_{u \in \partial B(0, R)} (\psi(u)) - \inf_{u \in B(0, R)} (\psi(u)))$. We may conclude from the preceding facts that functional $\psi : \overline{B(0, R)} \rightarrow \mathbb{R}$, is lower bounded and $\psi \in C^1(\overline{B(0, R)}, \mathbb{R})$. As a result of applying the Ekeland principle, we deduce that there is $u_\delta \in \overline{B(0, R)}$, such that

$$\bar{c} \leq \psi(u_\delta) \leq \bar{c} + \delta \quad \text{and} \quad \psi(u_\delta) < \psi(u) + \delta \|u - u_\delta\|, \quad \text{for all } u \neq u_\delta.$$

Given that $\psi(u_\delta) \leq \inf_{u \in \overline{B(0, R)}} (\psi(u)) + \delta \leq \inf_{u \in B(0, R)} (\psi(u)) + \delta < \inf_{u \in \partial B(0, R)} (\psi(u))$, we can deduce that

$u_\delta \in B(0, R)$. Now, we define $J_\delta : \overline{B(0, R)} \rightarrow \mathbb{R}$, by $J_\delta(u) = \psi(u) + \delta \|u - u_\delta\|$. Obviously, u_δ is a minimum of J_δ . Consequently, for any $v \in B(0, R)$ and for $s > 0$ large enough, we have

$$\frac{(J_\delta(u_\delta + sv) - J_\delta(u_\delta))}{s} \geq 0, \quad \text{that is,} \quad \frac{(\psi(u_\delta + sv) - \psi(u_\delta))}{s} + \delta \|v\| \geq 0.$$

By allowing s to trend to infinity, we get $\psi'(u_\delta)(v) + \delta \|v\| \geq 0$. This means that $\|\psi'(u_\delta)\| \leq \delta$. By the preceding reasoning, we conclude the existence of a sequence $(u_n) \in B(0, R)$, such that

$$\psi(u_n) \rightarrow \bar{c} \quad \text{and} \quad \psi'(u_n) \rightarrow 0. \quad (3.4)$$

Since $(u_n) \in B(0, R)$, it follows that (u_n) is bounded in E . So, up to a subsequence, there exist $e_2 \in E$ such that (u_n) converges weakly to $e_2 \in E$. We can therefore conclude that $u_n \rightarrow e_2$ strongly in E , based on the proof of Lemma 3.1. Since $\psi \in C^1(E, \mathbb{R})$, we have $\psi'(u_n) \rightarrow \psi'(e_2)$, as $n \rightarrow +\infty$. Hence, from (3.4), we deduce that

$$\psi'(e_2) = 0, \quad \|e_2\| < R, \quad \text{and} \quad \psi(e_2) < 0. \quad (3.5)$$

This signifies that e_2 is non-trivial solution for problem (P). And then, by adding (3.3) and (3.5), we get $\psi(e_2) < 0 < \psi(e_1)$. Thus the problem (P) has at least two non-trivial weak solutions.

Now, by using the Symmetric Mountain Pass Theorem [29], we will show that the problem (P) has a sequence of solutions $(\pm u_j)$ such that $\psi(\pm u_j) \rightarrow \infty$ as $j \rightarrow \infty$. We first need the next lemma.

Lemma 4.3. *Assume that assumptions (M_1) , (K_0) and (H_6) are satisfied. Then for any finite-dimentional subspace $\tilde{E} \subset E$, $\psi(u) \rightarrow -\infty$, $\|u\| \rightarrow +\infty$, $u \in \tilde{E}$.*

Proof. Assume on the contrary that there is a sequence $u_n \in \tilde{E}$ such that

$$\|u_n\| \rightarrow +\infty, \quad n \rightarrow +\infty \quad \text{and} \quad \psi(u_n) \geq l, \quad \text{for all } n \in \mathbb{N},$$

whith $l \in \mathbb{R}$ is a fixed constant independent on $n \in \mathbb{N}$. It is obvious that

$$\Phi(u_n) \leq c_1 \|u_n\|^{\frac{p^+}{\gamma}}$$

(H_6) implies that for s large enough, one can find $c_2 > 0$ such that

$$F(x, s) \geq c_2 |s|^{\bar{\beta}(x)}.$$

Thus

$$\int_{\Omega} F(x, u_n) \, dx \geq c_2 \int_{\Omega} |u_n|^{\bar{\beta}(x)} \, dx \geq c_2 |u_n|_{\bar{\beta}(x)}^{\beta^-}.$$

Since $\dim \tilde{E} < +\infty$, all norms on $\tilde{E}$ are equivalent. Hence, there exist $d_1, d_2 > 0$ such that

$$d_1 \|u_n\| \leq |u_n|_{\bar{\beta}(x)} \leq d_2 \|u_n\|, \quad \forall u \in \tilde{E}.$$

Therefore,

$$\int_{\Omega} F(x, u_n) \, dx \geq d_3 \|u_n\|^{\beta^-}.$$

Where $d_3 > 0$. And by (K_0) we have $\hat{K}(s) \geq c_3 s^{\beta_0}$, then

$$\varphi(u_n) = \hat{K} \left(\int_{\Omega} F(x, u_n) \, dx \right) \geq c_3 \left(\int_{\Omega} F(x, u_n) \, dx \right)^{\beta_0} \geq d_4 \|u_n\|^{\beta_0 \beta^-}.$$

Where d_4 is a positives constants. Hence

$$l \leq \psi(u_n) \leq \left(\frac{\hat{M}(1)}{(2p^-)^{\frac{1}{\gamma}}} + \frac{1}{p^-} \right) \|u_n\|^{\frac{p^+}{\gamma}} - d_4 \|u_n\|^{\beta_0 \beta^-} \rightarrow -\infty.$$

Which is a contradiction. Consequently, by hypothesis (H_4) , ψ is an even functional. Since $\psi(0) = 0$, based on lemmas 3.1, 4.2, 4.3 and the Symmetric Mountain Pass Theorem, we obtain the existence of an unbounded sequence of weak solutions to problem (P). Thus, we have established Theorem 1.

5. PROOF OF THEOREM 2

In this section, we aim to prove that problems (P) and (Q) has a sequence of solutions (u_j) such that $\psi(u_j) < 0$ and a nontrivial weak solution respectively, by applying the Ljusternik–Schnirelman Theorem [29] and the Direct Method in Calculus of Variations. To achieve this, we have a requirement to prove two lemmas.

Lemma 5.1. *Under conditions (K_1) , (M_0) – (M_1) , (H_1) and (H_3) , the functional ψ is coercive.*

Proof. For all $\|u\|$ large enough, by 2.3 we have that

$$\Phi(u) = \hat{M}(\sigma_{p(x,y)}(u)) + \int_{\Omega} \frac{|u|^{\bar{p}(x)}}{\bar{p}(x)} dx \geq C_1 \|u\|^{p^-} \text{ and } \int_{\Omega} F(x, u) dx \leq C_2 \|u\|^{q^+}.$$

Therefore,

$$\varphi(u) = \hat{K} \left(\int_{\Omega} F(x, u) dx \right) \leq C_3 \|u\|^{\beta_1 q^+}.$$

Thus

$$\psi(u) \geq C_1 \|u\|^{p^-} - C_3 \|u\|^{\beta_1 q^+}.$$

Since $p^- > \beta_1 q^+$, the above inequality implies that ψ is coercive.

Lemma 5.2. *Suppose that conditions (K_0) , (M_2) and (H_5) are satisfied. Then for every $\omega \in C_0^\infty(\Omega_0)$, there exist $\lambda_\omega > 0$ such that $\psi(\lambda_\omega \omega) < 0$.*

Proof. Take $\omega \in C_0^\infty(\Omega_0) \setminus \{0\}$. Then, if $\lambda > 0$ is small enough, we have that

$$\Phi(\lambda\omega) = \hat{M}(\sigma_{p(x,y)}(\lambda\omega)) + \int_{\Omega} \frac{|\lambda\omega|^{\bar{p}(x)}}{\bar{p}(x)} dx \leq c_1 \lambda^{\alpha p^-} + c_2 \lambda^{p^-} \leq c_3 \lambda^{p^-}.$$

(H_5) and (K_0) imply that

$$\int_{\Omega} F(x, \lambda\omega) dx \geq c_4 \lambda^r \text{ and } \varphi(\lambda\omega) \geq c_5 \lambda^{r\beta_0}.$$

Therefore,

$$\psi(\lambda\omega) \leq c_3 \lambda^{p^-} - c_5 \lambda^{r\beta_0}.$$

Hence $\psi(\lambda\omega) < 0$.

Condition (H_4) implies that the functional ψ is even. Since ψ is coercive, by step 2 in the proof of lemma 3.1, ψ satisfies $(P.S)_c$ condition for any $c \neq 0$. As we are already aware, $\psi(0) = 0$ and $\inf_{u \in E} \psi(u) > -\infty$. Thus, by using the Ljusternik–Schnirelman Theorem [29], to establish the second Theorem, it just needs to show that for any positive m , corresponds a symmetric closed set $U_m \subset E$, for which the Krasnoselskii genus $\gamma(U_m) \geq m$ and $\sup_{u \in U_m} \psi(u) < 0$. Noting that $C_0^\infty(\Omega_0)$ is an infinite-dimensional subspace of E , for an m -dimensional subspace $H_m \subset C_0^\infty(\Omega_0)$, for every m . Set $S_{m-1} = \{u \in H_m \mid \|u\| = 1\}$. From lemma 5.2 we know that, for every $\omega \in S_{m-1}$, there exists $\lambda_\omega > 0$ such that $\psi(\lambda_\omega \omega) < 0$. Since S_{m-1} is compact, so there is a positive constant $\mu > 0$ such that $\psi(\mu\omega) < 0$ for all $\omega \in S_{m-1}$. Set $U_m = \mu S_{m-1}$, then $\gamma(U_m) = m$ and $\sup_{u \in U_m} \psi(u) < 0$. The proof of (i) is complete.

We now finish this section by showing (ii) of Theorem 2. Thus, based on (H_7) we can see that, for any $N > 0$, there is $R_N > 0$ such that

$$G(x, s) \leq -M, \forall |s| \geq R_N, \text{ a.e } x \in \Omega. \quad (5.1)$$

Where $G(x, s) = F(x, s) - \lambda_1 \mu \frac{|s|^{p^-}}{\bar{p}(x)}$. By proposition 3.1 and condition (H_1) , we get that J is weakly lower semicontinuous. Next, we show the coercivity of the energy functional $J : E \rightarrow \mathbb{R}$ associated to problem (Q) such that $J(u) = \hat{M}(\sigma_{p(x,y)}(u)) + \int_{\Omega} \frac{|u(x)|^{\bar{p}(x)}}{\bar{p}(x)} dx - \int_{\Omega} F(x, u) dx$, for all $u \in E$. Otherwise, there are $L \in \mathbb{R}$ and $(u_n) \subset E$ such that

$$\|u_n\| \rightarrow \infty \text{ and } J(u_n) \leq L.$$

Putting $w_n := \frac{u_n}{\|u_n\|}$, so $\|w_n\| = 1$. For a subsequence, we can then suppose that for $w \in E$, we have $w_n \rightharpoonup w$ in E , $w_n \rightarrow w$ strongly in $L^{\bar{p}(x)}(\Omega)$, $w_n(x) \rightarrow w(x)$ for almost every $x \in \Omega$. We then use (5.1) to get

$$\begin{aligned} L &\geq J(u_n) \\ &\geq \frac{m_0}{2p^+} \int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy + \frac{1}{p^+} \int_{\Omega} |u_n|^{\bar{p}(x)} dx - \int_{\Omega} F(x, u_n) dx \\ &\geq C_1 \|u_n\|^{p^-} - \lambda_1 \mu \int_{\Omega} \frac{1}{\bar{p}(x)} |u_n|^{p^-} dx - \int_{\Omega} G(x, u_n) dx \\ &\geq C_1 \|u_n\|^{p^-} - \lambda_1 \mu \int_{\Omega} \frac{1}{\bar{p}(x)} |u_n|^{p^-} dx + N_1, \end{aligned} \quad (5.2)$$

where $N_1 \in \mathbb{R}$. By proceeding to the limit and dividing (5.2) by $\|u_n\|^{p^-}$, we find that

$$C_1 - \lambda_1 \mu \int_{\Omega} \frac{1}{\bar{p}(x)} |w|^{p^-} dx \leq 0,$$

thus, $w \neq 0$. Therefore $|D| > 0$ with $D = \{x \in \Omega : w(x) \neq 0\}$, then $|u_n(x)| \rightarrow +\infty$ for a.e $x \in D$. Furthermore,

$$\begin{aligned} \lambda_1 \int_A \frac{1}{\bar{p}(x)} |u|^{p^-} dx &\leq \lambda_1 \int_A \frac{1}{\bar{p}(x)} |u|^{\bar{p}(x)} dx \\ &\leq \lambda_1 \int_{\Omega} \frac{1}{\bar{p}(x)} |u|^{\bar{p}(x)} dx \\ &\leq \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y) |x - y|^{N+sp(x,y)}} dx dy + \int_{\Omega} \frac{|u|^{\bar{p}(x)}}{\bar{p}(x)} dx. \end{aligned} \quad (5.3)$$

Where, $A = \{x \in \Omega : |u| \geq 1\}$. From (H_7) and the above inequalities we deduce

$$\begin{aligned}
L &\geq \frac{m_0}{2} \int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy + \int_\Omega \frac{1}{\bar{p}(x)} |u_n|^{\bar{p}(x)} dx - \lambda_1 \mu \int_\Omega \frac{1}{\bar{p}(x)} |u_n|^{p^-} dx \\
&\quad - \int_\Omega G(x, u_n) dx \\
&\geq \mu \left[\int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy + \int_\Omega \frac{1}{\bar{p}(x)} |u_n|^{\bar{p}(x)} dx \right] - \lambda_1 \mu \int_\Omega \frac{1}{\bar{p}(x)} |u_n|^{p^-} dx \\
&\quad - \int_\Omega G(x, u_n) dx \\
&\geq \mu \left[\int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dx dy + \int_\Omega \frac{1}{\bar{p}(x)} |u_n|^{\bar{p}(x)} dx \right] - \lambda_1 \mu \int_A \frac{1}{\bar{p}(x)} |u_n|^{p^-} dx \\
&\quad - \lambda_1 \mu \int_{\Omega \setminus A} \frac{1}{\bar{p}(x)} |u_n|^{p^-} dx - \int_{B_{R_1}} G(x, u_n) dx - \int_{\Omega \setminus B_{R_1}} G(x, u_n) dx \\
&\geq -\lambda_1 \mu \int_{\Omega \setminus A} \frac{1}{\bar{p}(x)} |u_n|^{p^-} dx - \int_{B_{R_1}} G(x, u_n) dx - \int_{\Omega \setminus B_{R_1}} G(x, u_n) dx,
\end{aligned}$$

where $B_{R_1} = \{x \in \Omega : |u| < R_1\}$, with R_1 is large enough, which is a contradiction, thus J is coercive. Therefore, by the direct method in calculus of variations, there exists $\hat{u}$ a critical point which minimizes J on E . Now, we show that $\hat{u}$ is nontrivial weak solution of problem (Q) . Indeed, fix $\phi_0 \in E$, with $\|\phi_0\| = 1$, then by lemma 5.2, and for $t > 0$ is small enough, it is known that

$$J(t\phi_0) \leq c_3 t^{p^-} - c_4 t^r < 0, \quad (5.4)$$

because $r < p^-$. Consequently, by (5.4) the critical point of J which we have established above is nontrivial. This means that $\hat{u}$ is a nontrivial weak solution of problem (Q) . The proof of the second Theorem is closed.

6. THE GENERAL CASE

In this section we consider the bi-nonlocal $p(.,.)$ -Laplacian problem with general (non-variational) form

$$\begin{cases} \mathcal{A}(w)(-\Delta_{p(.,.)}^s w(x) + |w(x)|^{\bar{p}(x)-2} w(x) = \mathcal{B}(w)f(x, w) & \text{in } \Omega, \\ \mathcal{A}(w)\mathcal{N}_{s,p(.,.)} w(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (T)$$

Definition 6.1. We say that $w \in E$ is a weak solution of problem (T) if

$$\begin{aligned}
\frac{1}{2} \mathcal{A}(w) \int_Q \frac{|w(x) - w(y)|^{p(x,y)-2} (w(x) - w(y))(v(x) - v(y))}{|x-y|^{N+sp(x,y)}} dx dy + \int_\Omega |w|^{\bar{p}(x)-2} w v dx \\
= \mathcal{B}(w) \int_\Omega f(x, w) v dx, \quad \text{for all } v \in E.
\end{aligned}$$

Furthermore define the functionals T_1 , T_2 , $L_{p(\cdot, \cdot)}$, D and $N_f : E \rightarrow E^*$ respectively by:

$$\begin{aligned} \langle T_1(w), v \rangle &= \frac{1}{2} \mathcal{A}(w) \int_Q \frac{|w(x) - w(y)|^{p(x,y)-2} (w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+sp(x,y)}} dx dy \\ &\quad + \int_{\Omega} |w|^{\bar{p}(x)-2} w v dx, \\ \langle T_2(w), v \rangle &= \mathcal{B}(w) \int_{\Omega} f(x, w) v dx, \\ \langle L_{p(\cdot, \cdot)}(w), v \rangle &= \frac{1}{2} \int_Q \frac{|w(x) - w(y)|^{p(x,y)-2} (w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+sp(x,y)}} dx dy, \\ \langle D(w), v \rangle &= \int_{\Omega} |w|^{\bar{p}(x)-2} w v dx, \\ \langle N_f(w), v \rangle &= \int_{\Omega} f(x, w) v dx. \end{aligned}$$

$\forall w, v \in E$. Then $T_1(w) = \mathcal{A}(w)L_{p(\cdot, \cdot)}(w) + D(w)$ and $T_2(w) = \mathcal{B}(w)N_f(w)$ for $w \in E$. It is clear that $w \in E$ is a solution of (T) if and only if $T(w) = T_1(w) - T_2(w) = 0$.

Lemma 6.1. *Assume that $\mathcal{A}$ satisfies the following assumption:*

$(A_1) : \mathcal{A} : E \rightarrow [0, +\infty)$ is continuous and bounded on any bounded subset of E , $\mathcal{A}(w) > 0$ for all $w \in E \setminus \{0\}$, and for any bounded sequence $(w_n) \subset E$ for which $\mathcal{A}(w_n) \rightarrow 0$, w_n must converge strongly to 0 in E .

Then the mapping $T_1 : E \rightarrow E^$ is continuous and bounded, and is of type (S_+) .*

Proof. Since $L_{p(\cdot, \cdot)}$, D and $\mathcal{A}$ are continuous and bounded, so is T . Thus, we focus on the (S_+) property. Let $(w_n) \subset E$ such that $w_n \rightharpoonup w$ in E and $\limsup_{n \rightarrow +\infty} \langle T_1(w_n), w_n - w \rangle \leq 0$, so we remain to prove that any subsequence of (w_n) has a strongly convergent subsequence. Let (w_{n_k}) be a subsequence of (w_n) . Then, one can find a subsequence of (w_{n_k}) , always denoted by (w_{n_k}) , such that $\mathcal{A}(w_{n_k}) \rightarrow c$. When $c = 0$, It then follows directly from assumption (A_1) that $w_{n_k} \rightarrow 0$ in E . Now let $c \neq 0$. Then by condition (A_1) , we have that $c > 0$ and

$$\begin{aligned} c \limsup_{k \rightarrow +\infty} \langle L_{p(\cdot, \cdot)}(w_{n_k}), w_{n_k} - w \rangle &+ \limsup_{k \rightarrow +\infty} \langle D(w_{n_k}), w_{n_k} - w \rangle \\ &= \limsup_{k \rightarrow +\infty} \langle T_1(w_{n_k}), w_{n_k} - w \rangle \leq 0 \end{aligned}$$

which implies $\limsup_{k \rightarrow +\infty} \langle \mathcal{L}(w_{n_k}), w_{n_k} - w \rangle \leq 0$. From Lemma 2.2.3 we have that $\mathcal{L}$ is of type (S_+) , so we get $w_{n_k} \rightarrow w$ in E .

Lemma 6.2 ([14]). *Assume that f satisfies (H_1) . If we suppose that $\mathcal{B}$ verify the following condition :*

(B_1) *The functional $\mathcal{B} : E \rightarrow \mathbb{R}$ is continuous and bounded on any bounded subset of E .*

Then the mapping $T_2 : E \rightarrow E^$ is completely continuous.*

Proof. The proof is similar to [14].

Proposition 6.1 ([14]). *Let assumptions (A_1) , (B_1) and (H_1) being satisfying. Then the mapping $T : E \rightarrow E^*$ is continuous and bounded, and is of type (S_+) .*

Theorem 3. *Let assumptions (A_1) , (B_1) and (H_1) hold and assume that the following conditions are verified:*

(A₂) There are constants $M > 0$ and $c_1 > 0$ such that

$$\mathcal{A}(w) \geq c_1 \text{ for } w \in E, \text{ with } \|w\| \geq M.$$

(B₂) There are constants $\tau \in \mathbb{R}$, $M > 0$ and $c_2 > 0$ such that

$$|\mathcal{B}(w)| \leq c_2 \|w\|^\tau \text{ for } w \in E, \text{ with } \|w\| \geq M.$$

If $p^- > \tau + q^+$, then (T) has at least one solution. Moreover, the mapping $T : E \rightarrow E^$ is surjective, and then the equation $T(w) = h$ has at least one solution, for each $h \in E^*$.*

Proof. For $\|w\|$ large enough, we have that

$$\begin{aligned} \langle T(w), w \rangle &= \mathcal{A}(w) - \langle L_{p(\cdot)}(w), w \rangle + \langle D(w), w \rangle - \mathcal{B}(w) - \langle N_f(w), w \rangle \\ &\geq c_3 \|w\|^{p^-} - c_4 \|w\|^{\tau+q^+} \\ &\geq c_5 \|w\|^{p^-} > 0. \end{aligned}$$

Thus, by the degree theory for (S_+) type mappings, for $R > 0$ large enough, we have $\deg(T, B(0, R), 0) = 1$, and consequently, there exists $w \in B(0, R)$ such that $T(w) = 0$, that is, (T) has at least one solution $w \in B(0, R)$.

In the other hand, since $p^- - 1 > 0$, we get

$$\lim_{\|w\| \rightarrow \infty} \frac{\langle T(w), w \rangle}{\|w\|} \geq \lim_{\|w\| \rightarrow \infty} \|w\|^{p^- - 1} = +\infty$$

that is, the mapping T is coercive. In addition, T is continuous and is of type (S_+) , so its pseudomonotone. Thus, by the surjection Theorem for the pseudomonotone mappings, the mapping T is surjective.

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