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ATOMIC SOLUTION FOR BOTH ORDINARY AND FRACTIONAL ABSTRACT CAUCHY PROBLEM IN TENSOR PRODUCT FORM

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Abstract. In this paper, we discuss the atomic solutions of the abstract Cauchy problem in two variables. The proposed abstract Cauchy problem is assumed to be non-homogeneous and in tensor product form. We utilize some results from the theory of tensor product of Banach spaces to obtain solutions in the atom operator form. The approach is applied for both ordinary and α-conformable fractional derivatives.

1. Introduction

Mathematical modeling is a powerful tool for handling both non-natural and natural dynamical systems as it enables us to study hidden connections between the physical quantities which are included within these systems. Moreover, it leads to reasonable interpretation for the dynamical behaviour of these systems over various situations. In any mathematical modeling, the process, mainly, includes the construction of differential equations and their corresponding solutions then, finally checking whether, or not, the model reveals an adequate prediction of the system behaviour.

In general, finding analytical or numerical solutions of differential equations is not straightforward, classes for which we have general analytic methods or even numerical approaches of solutions are quite limited. However, it is possible to investigate the subject of differential equations from a purely mathematical perspective. It is known that, for any differential equation, if the range of an unknown function is included inside some abstract space, then the differential equation is said to be an abstract differential equation. The most famous abstract differential equation is the abstract Cauchy problem ACP which can be presented, in its general form, as follows:

$$\begin{aligned} &\frac{d\phi}{ds} = L\phi(s), \qquad s \geq 0, \\ &\phi(0) = m, \end{aligned} \tag{1.1}$$

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where $L : D(L) \subseteq M \to M$ is a linear operator of an applicable class such that $Rang(\phi) \subseteq Dom(L)$, $m \in M$ is given and $\phi : [0, \infty) \to M$ is an unknown function. Such problems have many applications and can be employed to handle some complex dynamical systems. In spite of the fact that, the abstract Cauchy problems have a standing investigate history that has profound roots in time, a common hypothesis that controls the solutions of such problems has not, however, been confirmed yet. Early studies, principally, were conducted by employing the theory of semigroup of linear operators and the Laplace transform approach [2], [3], [4] and [5], [6], [7].

Lately, in 2010, modern strategy has been suggested [8], [10], [11], [9] to manipulate specific types of ordinary and fractional abstract Cauchy problems. The new approach is carried out by utilizing the theory of atoms operators in Banach spaces and the obtained solutions in such cases are called atomic solutions.

Before we introduce our main result that discusses the atomic solutions of tensor product abstract Cauchy problem in two variables for both ordinary and fractional orders, we start with the some definitions and theorems.

2. Preliminaries

In the following, we derive introductory material that are needed for the main part of the current paper.

2.1. **Fractional Derivative.**

Definition 2.1. [12] Let $\alpha \in (0, 1)$, and $u : I \subseteq (0, \infty) \to \mathbb{R}$. For $x \in I$, let

$$D^\alpha u(x) = \lim_{\epsilon \to 0} \frac{u(x + \epsilon x^{1-\alpha}) - u(x)}{\epsilon}. \tag{2.1}$$

If the limit exists, then it is called the α-conformable fractional derivative of u at x. For α-conformable fractional derivative, the following differentiation rules hold:

$$\begin{array}{l} (i)\ D^\alpha(c_1 u + c_2 \upsilon) = c_1 D^\alpha(u) + c_2 D^\alpha(\upsilon), \text{ for all } c_1, c_2 \in \mathbb{R}, \\ (ii)\ D^\alpha(k) = 0, \text{ for all constant functions } f(x) = k, \\ (iii)\ D^\alpha(u\upsilon) = u D^\alpha(\upsilon) + \upsilon D^\alpha(u), \\ (iv)\ D^\alpha(\frac{u}{\upsilon}) = \frac{\upsilon D^\alpha(u) - u D^\alpha(\upsilon)}{\upsilon^2},\ \upsilon(x) \neq 0. \end{array} \tag{2.2}$$

Moreover, the α-conformable fractional derivatives of some elementary functions are

$$\begin{array}{l} (i)\ D^\alpha(x^p) = p x^{p-\alpha}, \\ (ii)\ D^\alpha(\sin(\frac{1}{\alpha} x^\alpha)) = \cos(\frac{1}{\alpha} x^\alpha), \\ (iii)\ D^\alpha(\cos(\frac{1}{\alpha} x^\alpha)) = -\sin(\frac{1}{\alpha} x^\alpha), \\ (iv)\ D^\alpha(e^{\frac{1}{\alpha} x^\alpha}) = e^{\frac{1}{\alpha} x^\alpha}. \end{array} \tag{2.3}$$

It should be noted that, in (2.3), if $\alpha = 1$, then we get the classical ordinary derivatives [15], [13], [14]. Also, respectively, $D_s^\alpha u$ and $D_t^\alpha u$, denote the partial α-conformable fractional derivative of $u(s, t)$ with respect to s and t.

Definition 2.2. [12] The α-fractional integral of a function f starting from $\mathbf{a} \geq 0$ is denoted by $I_{\alpha}^{\mathbf{a}}(u)(x)$ such that

$$I_{\alpha}^{\mathbf{a}}(u)(x) = I_{1}^{\mathbf{a}}\left(x^{\alpha-1}u\right) = \int_{\mathbf{a}}^{x} \frac{u(t)}{t^{1-\alpha}}\, dt, \qquad \alpha \in (0,1). \tag{2.4}$$

2.2. **Atomic Solution.**

Definition 2.3. Let M and N be any two Banach spaces and M^* be the dual space of M. For $m \in M$ and $n \in N$, the operator $L : M^* \to N$, defined by

$$Lm^* = m^*(m)n = \langle m, m^* \rangle\, n, \tag{2.5}$$

is a bounded one rank linear operator (range of L is the span of n). We write $m \otimes n$ for such operator L and it is called an atom.

For example, let $M = N = C\,[0,1]$ under the L^{∞} norm and set up M^* to be the space of all regular Borel measures on $[0,1]$. Hence, if $m^* \in M^*$, then $m^*(h) = \int\limits_{0}^{1} h\ dm^*$ for any $h \in C\,[0,1]$. Moreover, let $m \in M$, $n \in N$, and $m^* \in M^*$, then $(m \otimes n)\, m^* = m^*(m)n = n\int\limits_{0}^{1} m(t)\ dm^*(t)$. Further, $m \otimes n$ is an atom such that $m \otimes n : M^* \to N$. Also, indeed, we have $\|(m \otimes n)(m^*)\| = |m^*(m)|\, \|n\| \leq \|m^*\|\, \|m\|\, \|n\|$ for all $m^* \in M^*$and hence, $m \otimes n$ is a bounded linear operator.

The reason behind naming the previous operator, L, an "atom" is because of the nuclear space which consists of the space $M \otimes N$ together with the projection norm. In Banach space related theories such as tensor product and best approximation, atoms operators are basic tools by which many functional analysis approached can be constructed [17] to handle Banach abstract differential equations. For more related work on these concepts, we suggest [18], [19], [20], and [16].

Definition 2.4. [21] Let G and H be Banach spaces and $\Psi : Dom(\Psi) \subseteq G \to H$ be linear operator, then Ψ is said to be of exponential order if

$$\begin{array}{l} (i)\ x \in Dom(\Psi) \text{ then } \{x,\, \Psi x,\, \Psi^2 x,\, \cdots\} \subseteq Dom(\Psi), \\ (ii)\ \sum_{j=1}^{\infty} \frac{t^j}{j!} \left\|\Psi^j x\right\| < \infty, \forall x \in Dom(\Psi). \end{array} \tag{2.6}$$

Clearly every bounded linear operator is of exponential order.

We will write $e^{t\Psi}x$ for $\sum_{j=1}^{\infty} \frac{t^j}{j!} \Psi^j x$.

Moreover, one of the known results, see [22], that we need in our current paper is as follows.

Lemma 2.5. [22] *Let $m_1 \otimes n_1$ and $m_2 \otimes n_2$ be two nonzero atoms in $M \otimes N$ such that $m_1 \otimes n_1 + m_2 \otimes n_2 = m_3 \otimes n_3$. Then either $m_1 = m_2 = m_3$ or $n_1 = n_2 = n_3$.*

To prove Lemma (1), assume two atoms, namely, $m_1 \otimes n_1$ and $m_2 \otimes n_2$ such that $m_1 \otimes n_1 = m_2 \otimes n_2$. Then by definition (3), for any $m^* \in M^*$, we have $\langle m_1, m^* \rangle\, n_1 = \langle m_2, m^* \rangle\, n_2$. Hence, $n_1 = n_2$. Similarly, one can prove $m_1 = m_2$.

3. Main Results

In this section, we find the atomic solution, that is a solution of the form $u(s) \otimes \upsilon(t)$, to the abstract Cauchy problem in two variables. We consider the abstract Cauchy problem to be non-homogeneous and in tensor product form. The approach will be conducted for both ordinary and α-conformable fractional derivatives orders.

3.1. **Tensor Product Abstract Cauchy Problem of Ordinary Derivative Order.**

Theorem 3.1. *Consider the following tensor product abstract Cauchy problem*

$$u'(s) \otimes \upsilon'(t) = Au(s) \otimes B\upsilon(t) + f(s) \otimes g(t), \tag{3.1}$$

where X and Y are Banach spaces, $u : [0, \infty) \to X$, $\upsilon : [0, \infty) \to Y$, $A : Dom(A) \subseteq X \to X$, $Rang(u) \subseteq Dom(A)$, $B : Dom(B) \subseteq Y \to Y$, $Rang(\upsilon) \subseteq Dom(B)$, A, B are closed operators of exponential order, and both $f : [0, \infty) \to X$, $g : [0, \infty) \to Y$ are given. Moreover, let us assume $u(0) = x_\circ$ and $\upsilon(0) = y_\circ$. Then (3.1) has the following two atomic solutions of the form $u(s) \otimes \upsilon(t)$ such that

$$u(s) \otimes \upsilon(t) = e^{sA} x_\circ \otimes \left[e^{tB} y_\circ + \int_0^t e^{(t-\omega)B} g(\omega)\, d\omega \right], \tag{3.2}$$

and

$$u(s) \otimes \upsilon(t) = \left[e^{tA} x_\circ + \int_0^s e^{(s-\theta)A} f(\theta)\, d\theta \right] \otimes e^{tB} y_\circ. \tag{3.3}$$

Proof. Clearly, equation (3.1) indicates that the sum of two atoms is an atom. Hence, by lemma (1), we have the following two cases:

$$\begin{array}{l} \text{Case } (i)\ u'(s) = Au(s) = f(s), \\ \text{Case } (ii)\ \upsilon'(t) = B\upsilon(t) = g(t). \end{array} \tag{3.4}$$

Now, case (i) has the following three situations:

$$\begin{array}{l} (a)\ u'(s) = Au(s), \\ (b)\ u'(s) = f(s), \\ (c)\ Au(s) = f(s). \end{array} \tag{3.5}$$

Therefore, case (i) admits an atomic solution provided that $Au(s) = f(s)$. But, from situation (a), namely, $u'(s) = Au(s)$, we have

$$u(s) = e^{sA} x_\circ, \text{ where } u(0) = x_\circ. \tag{3.6}$$

Hence,

$$f(s) = Ae^{sA} x_\circ. \tag{3.7}$$

Now, substituting both (3.6) and (3.7) into (3.1) yields

$$Ae^{sA} x_\circ \otimes [\upsilon'(t) - B\upsilon(t) - g(t)] = 0, \tag{3.8}$$

equivalently,

$$\upsilon'(t) = B\upsilon(t) + g(t), \tag{3.9}$$

which is the classical abstract Cauchy problem where the solution is given by

$$v(t) = e^{tB} y_\circ + \int_0^t e^{(t-\omega)B} g(\omega)\, d\omega, \text{ where } v(0) = y_\circ. \tag{3.10}$$

Thus, the first atomic solution corresponding to case (i), can be obtained by considering both (3.6) and (3.10) to give

$$u(s) \otimes v(t) = e^{sA} x_\circ \otimes \left[e^{tB} y_\circ + \int_0^t e^{(t-\omega)B} g(\omega)\, d\omega \right]. \tag{3.11}$$

Similarly, because of the symmetry of equation (3.1), the second atomic solution corresponding to case (ii) has the form

$$u(s) \otimes v(t) = \left[e^{tA} x_\circ + \int_0^s e^{(s-\theta)A} f(\theta)\, d\theta \right] \otimes e^{tB} y_\circ. \tag{3.12}$$

$\square$

3.2. **Tensor product abstract Cauchy problem of α-conformable fractional derivative order.** In this section we find the atomic solution of the α-conformable fractional version of (3.1).

Theorem 3.2. *Consider the α-conformable fractional version of* (3.1) *that is*

$$D_s^\alpha u(s) \otimes D_t^\alpha v(t) = Au(s) \otimes Bv(t) + f(s) \otimes g(t), \ \alpha \in (0,1). \tag{3.13}$$

Then (3.13) *has the following two atomic solutions of the form $u(s) \otimes v(t)$ such that*

$$u(s) \otimes v(t) = e^{\frac{s^\alpha}{\alpha} A} x_\circ \otimes \left[e^{\frac{t^\alpha}{\alpha} B} y_\circ + \int_0^t \frac{1}{\omega^{1-\alpha}} e^{(\frac{t^\alpha}{\alpha} - \frac{\omega^\alpha}{\alpha})B} g(\omega)\, d\omega \right], \tag{3.14}$$

and

$$u(s) \otimes v(t) = \left[e^{\frac{s^\alpha}{\alpha} A} x_\circ + \int_0^s \frac{1}{s^{1-\alpha}} e^{(\frac{s^\alpha}{\alpha} - \frac{\theta^\alpha}{\alpha})A} f(\theta)\, d\theta \right] \otimes e^{\frac{t^\alpha}{\alpha} B} y_\circ. \tag{3.15}$$

Proof. Clearly, equation (3.13) indicates that the sum of two atoms is an atom. Hence, by Lemma (1), we have the following two cases:

$$\begin{array}{l} \text{Case } (i) \ D_s^\alpha u(s) = Au(s) = f(s), \\ \text{Case } (ii) \ D_t^\alpha v(t) = Bv(t) = g(t). \end{array} \tag{3.16}$$

Now, case (i) has the following three situations:

$$\begin{array}{l} (a) \ D_s^\alpha u(s) = Au(s), \\ (b) \ D_s^\alpha u(s) = f(s), \\ (c) \ Au(s) = f(s). \end{array} \tag{3.17}$$

Therefore, case (i) admits an atomic solution provided that $Au(s) = f(s)$. But, from situation (a), namely, $D_s^\alpha u(s) = Au(s)$ we have

$$u(s) = e^{\frac{s^\alpha}{\alpha} A} x_\circ, \text{ where } u(0) = x_\circ. \tag{3.18}$$

Hence,

$$f(s) = Ae^{\frac{s^\alpha}{\alpha} A} x_\circ. \tag{3.19}$$

Now, substituting both (3.18) and (3.19) into (3.13) yields

$$Ae^{\frac{s^\alpha}{\alpha}A}x_\circ \otimes [D_t^\alpha \upsilon(t) - B\upsilon(t) - g(t)] = 0, \tag{3.20}$$

equivalently,

$$D_t^\alpha \upsilon(t) = B\upsilon(t) + g(t), \tag{3.21}$$

which is the classical α-conformable fractional abstract Cauchy problem where the solution is given by

$$\upsilon(t) = e^{\frac{t^\alpha}{\alpha}B}y_\circ + \int_0^t \frac{1}{\omega^{1-\alpha}} e^{(\frac{t^\alpha}{\alpha}-\frac{\omega^\alpha}{\alpha})B} g(\omega)\, d\omega, \text{ where } \upsilon(0) = y_\circ. \tag{3.22}$$

Thus, the first atomic solution corresponding to case (i), can be obtained by considering both (3.18) and (3.22) as follows

$$u(s) \otimes \upsilon(t) = e^{\frac{s^\alpha}{\alpha}A}x_\circ \otimes \left[e^{\frac{t^\alpha}{\alpha}B}y_\circ + \int_0^t \frac{1}{\omega^{1-\alpha}} e^{(\frac{t^\alpha}{\alpha}-\frac{\omega^\alpha}{\alpha})B} g(\omega)\, d\omega \right]. \tag{3.23}$$

Similarly, because (3.13) is symmetric with respect to $u(s)$ and $\upsilon(t)$, the second atomic solution corresponding to case (ii) has the form

$$u(s) \otimes \upsilon(t) = \left[e^{\frac{s^\alpha}{\alpha}A}x_\circ + \int_0^s \frac{1}{s^{1-\alpha}} e^{(\frac{s^\alpha}{\alpha}-\frac{\theta^\alpha}{\alpha})A} f(\theta)\, d\theta \right] \otimes e^{\frac{t^\alpha}{\alpha}B}y_\circ. \tag{3.24}$$

□

4. Conclusion

This paper studies the atomic solutions of a non-homogeneous abstract Cauchy problem in two variables for both ordinary as well as α-conformable fractional derivatives orders. Atomic operators were utilized to achieve the atomic solutions.

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