Gulf Journal of Mathematics
Vol 2, Issue 3 (2014) 1-18

POSTULATION IN PROJECTIVE SPACES OF A UNION OF A LOW DIMENSIONAL SCHEME AND GENERAL RATIONAL CURVES OR GENERAL UNIONS OF LINES

EDOARDO BALLICO[1]

ABSTRACT. Let $Y \subset \mathbb{P}^r$, $r \geq 3$, be a scheme with $\dim(Y) \leq r-2$. We prove the existence of an integer d_0 such that for all $d \geq d_0$ a general union X of Y and either a general degree d rational curve or (if $r \neq 3$) d lines has maximal rank, i.e. for each $t \in \mathbb{N}$ X gives the expected number of conditions to the set of all degree t hypersurfaces of $\mathbb{P}^r$. We also consider unions of Y and a general curve with a prescribed genus and high degree.

1. INTRODUCTION

Let $X \subset \mathbb{P}^r$ be a closed subscheme. We recall that X is said to have *maximal rank* if for all integers $t \geq 0$ either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$, i.e. if for all integers $t \geq 0$ the restriction map $\rho_{X,t} : H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(\mathcal{O}_X(t))$ is either injective or surjective. In many interesting cases the integer $h^0(\mathcal{O}_X(t))$ is known. Hence if we also know that X has maximal rank, then we know that $h^0(\mathcal{I}_X(t)) = \max\{0, \binom{r+t}{r} - h^0(\mathcal{O}_X(t))\}$, i.e. we know the number of conditions that X imposes to the set of all degree t hypersurfaces of $\mathbb{P}^r$ and this number is the " expected " one.

For all closed subschemes $Y \subset \mathbb{P}^r$ with $\dim(Y) \leq r-2$ and all integers $d \geq 0$ let $R(r, Y, d)$ (resp. $L(r, Y, d)$) be the set of all schemes $X \subset \mathbb{P}^r$ which are the disjoin union of Y and a smooth rational d curve (resp. Y and the disjoint union of d lines); our convention says that the $\emptyset$ is the only rational curve of degree 0. For all integers $g \geq 0$, $r \geq 3$ and $d \geq 2g+r$ let $Z(r, Y, d, g)$ denote the set of all disjoint unions $X \subset \mathbb{P}^r$ of Y and a smooth curve of degree d and genus g. Set $Z(r, d) := R(r, \emptyset, d)$, $L(r, d) := L(r, \emptyset, d)$ and $Z(r, d, g) := Z(r, \emptyset, d, g)$. The conditions " $r \geq 3$ and $d \geq 2g+r$ " implies that the set $Z(r, d, g)$ of all smooth curves $E \subset \mathbb{P}^r$ with degree d and genus g is parametrized by an irreducible variety and hence we are allowed to say the sentence " the general element of $E \in Z(r, Y, d, g)$ ". Fix any $E \in Z(r, d, g)$. Since $d \geq 2g+r$, we have $h^1(\mathcal{O}_E(1)) = 0$ and $h^0(\mathcal{O}_E(t)) = td + 1 - g$ for all $t > 0$. If either E is general or $r = 3$, then E spans $\mathbb{P}^r$. We even know the Hilbert function of a general $E \in Z(r, d, g)$ ([16], [5], [6], [7]) (E has maximal rank), but we will not need it,

Date: Received: Feb 21, 2014; Accepted: Mar 28, 2014.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 14N05; 14H99.
Key words and phrases. postulation; Hilbert function; unions of lines; rational curves.

because we will always have $d \gg 2g+r$ and for these triples (r,d,g) the maximal rank of E is easily proved.

Let $\mathrm{Hilb}^p(\mathbb{P}^r)$ be the set of all closed subschemes $Y \subset \mathbb{P}^r$ with p as Hilbert polynomial. Grothendieck's proved that $\mathrm{Hilb}^p(\mathbb{P}^r)$ is (in a natural way) a projective algebraic scheme. The regularity of all $Y \in \mathrm{Hilb}^p(\mathbb{P}^r)$ is bounded ([18, Theorem at page 101]). G. Gotzmann found the optimal upper bound for the regularity. By Gotzmann's Regularity theorem ([13, 78, Satz (2.9)] and [17, Lemma C.23]), there exists a number x only depending on p and easily obtained from the coefficients of p, called the *Gotzmann number* of p , for which the ideal sheaf $\mathcal{I}_Y$ of each scheme $Y \in \mathrm{Hilb}^p(\mathbb{P}^r)$ is x-regular (in the sense of Castelnuovo-Mumford regularity [18, Definition at page 99]). In particular there is an integer d_1 such that $h^1(\mathcal{I}_Y(t)) = 0$ and $h^i(\mathcal{O}_Y(t)) = 0$ for all $i > 0$, all $t \ge d_1$ and all $Y \in \mathrm{Hilb}^p(\mathbb{P}^r)$. In particular we have $h^0(\mathcal{O}_Y(t)) = p(t)$ for every integer $t \ge d_1$. Hence for each $Y \in \mathrm{Hilb}^p(\mathbb{P}^r)$, each integer $d > 0$, each integer $t \ge d_1$, each $A \in Z(r,Y,d)$ and $B \in L(r,Y,d)$ we have $h^0(\mathcal{O}_A(t)) = p(t)+td+1$ and $h^0(\mathcal{O}_B(t)) = p(t)+(t+1)d$.

In this paper we prove the following results (the case $\deg(p) = 0$ of Theorem 1.1 improves [3]).

Theorem 1.1. *Fix integers r,m such that $m \ge 0$, $r \ge 4$ and $r \ge m+2$. Let p be a degree m admissible polynomial. Let $Y \subset \mathbb{P}^r$ be a closed subscheme with p as its Hilbert polynomial. Then there is an integer d_0 (depending only on r and p) with the following property. Fix any integer $d \ge d_0$. Let $X \subset \mathbb{P}^r$ be a general union of Y and d lines. Then X has maximal rank.*

Theorem 1.2. *Fix integers r,m such that $m \ge 0$, $r \ge 3$ and $r \ge m+2$. Let p be a degree m admissible polynomial. Let $Y \subset \mathbb{P}^r$ be a closed subscheme with p as its Hilbert polynomial. Then there is an integer d_0 (depending only on r and p) with the following property. Fix any integer $d \ge d_0$. Let $X \subset \mathbb{P}^r$ be a general union of Y and a general degree d rational curve of $\mathbb{P}^r$. Then X has maximal rank.*

Theorem 1.3. *Fix integers r,m,g such that $m \ge 0$, $r \ge 3$, $g \ge 0$, and $r \ge m+2$. Let p be a degree m admissible polynomial. Let $Y \subset \mathbb{P}^r$ be a closed subscheme with $p(t)$ as its Hilbert polynomial. Then there is an integer $d_0 \ge 2g+r$ (depending only on r, g and p) with the following property. Fix any integer $d \ge d_0$. Let $X \subset \mathbb{P}^r$ be a general union of Y and a general degree d smooth curve $E \subset \mathbb{P}^r$ with genus g. Then X has maximal rank.*

Of course, Theorem 1.1 is a particular case of Theorem 1.3. We point out that statements like Theorem 1.3 may easily proved using statements like Theorem 1.1 if in Theorem 1.3 we don't look for a " good'" upper bound for the integer d_0.

Non-asymptotic results are harder (see [1], [2] and [4] for some of them).

2. Preliminaries

We use the convention that $\emptyset$ is the only scheme with dimension -1. Let $\mathbb{K}$ be an algebraically closed base field. For any integral curve $E \subset \mathbb{P}^n$, $n \ge 3$, with $\deg(E) \ge 2$ (resp. $\deg(E) \ge 1$) a *2-secant line* (resp. a 1-secant line) D of E

is a line $D \subset \mathbb{P}^n$ such that $\sharp(E \cap D) = 2$ (resp. $\sharp(E \cap D) = 1$), $D \cap E \subset E_{\mathrm{reg}}$ and at each $P \in E \cap D$ the tangent line $T_P E$ of E at P is different from D. If E is irreducible, then the set of all 2-secant lines of E is parametrized by an irreducible variety of dimension 2. This variety is non-empty in characteristic zero and in most cases in positive characteristic (i.e. if E is not very strange in the sense of [21]; this is always true for smooth curves and in particular for the general rational curve of degree d of $\mathbb{P}^n$). Let p'' be the Hilbert polynomial of the scheme $Y \cap H \cap M$ in M; we have $p''(x) = p(x) - 2p(x-1) + p(x-2)$ and in particular $p'' \equiv 0$ if $\dim(Y) \le 1$.

We usually write " rational curve " instead of " smooth rational curve ". Each $Z(r,d)$, $d > 0$, is an irreducible variety. If $d \ge r$, then a general $X \in Z(r,d)$ is non-degenerate. Let $Z'(r,d)$ be the closure of $Z'(r,d)$ in the Hilbert scheme of $\mathbb{P}^r$. For any integer $e > 0$, any closed subscheme $W \subset \mathbb{P}^r$ and any $T \in |\mathcal{O}_{\mathbb{P}^r}(e)|$ let $\mathrm{Res}_T(W)$ denote the residual scheme of W with respect to T, i.e. the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_W : \mathcal{I}_T$ as its ideal sheaf. For each $t \in \mathbb{Z}$ we have the following exact sequence, which is often called the Castelnuovo's sequence:

$$0 \to \mathcal{I}_{\mathrm{Res}_T(W)}(t-e) \to \mathcal{I}_W(t) \to \mathcal{I}_{W \cap T,T}(t) \to 0$$

Remark 2.1. Let $Y \subset \mathbb{P}^r$ be a closed subscheme with $\dim(Y) < r$. Fix an integer $e \in \{1,2\}$ and take a general $T \in |\mathcal{O}_{\mathbb{P}^r}(e)|$. Since $\mathcal{O}_{\mathbb{P}^r}(e)$ is very ample and T is general, T does not contain the support of any component of Y, not even any embedded component, by the prime avoidance lemma. Hence $\mathrm{Res}_T(Y) = Y$.

Remark 2.2. Let $U \subset \mathbb{P}^n$, $n \ge 3$, be a reducible conic. Call P the singular point of U. Let $V \subseteq \mathbb{P}^n$ be any 3-dimensional linear subspace containing U. Let Z be the closed subscheme of V with $(\mathcal{I}_{P,V})^2$ as its ideal sheaf. The scheme $U \cup Z$ is called a sundial ([10]). U is the support of $U \cup Z$, because $U = (U \cup Z)_{\mathrm{red}}$. P is the support of the nilradical of the sheaf $\mathcal{O}_{U \cup Z}$. The scheme $U \cup Z$ is a flat limit of a family of pairs of disjoint lines ([15, Example 2.1.1], [10]).

For all integer-valued polynomials $p \in \mathbb{Q}[x]$ and all positive integers r, t define the integers $a_{r,p,t}$, $b_{r,p,t}$, $c_{r,p,t}$, and $d_{r,p,t}$ by the relations

$$p(t) + ta_{r,p,t} + 1 + b_{r,p,t} = \binom{r+t}{t}, 0 \le b_{r,b,t} \le t-1 \tag{2.1}$$

$$p(t) + (t+1)c_{r,p,t} + d_{r,p,t} = \binom{r+t}{t}, 0 \le d_{r,b,t} \le t \tag{2.2}$$

From (2.1) and (2.2) we get the following relations.

$$p'(t) + a_{r,p,t-1} + t(a_{r,p,t} - a_{r,p,t-1}) + b_{r,p,t} - b_{r,p,t-1} = \binom{r+t-1}{r-1} \tag{2.3}$$

$$p'(t) + c_{r,p,t-1} + (t+1)(c_{r,p,t} - c_{r,p,t-1}) + d_{r,p,t} - d_{r,p,t-1} = \binom{r+t-1}{r-1} \tag{2.4}$$

If t is at least the Gotzmann's number, d_1, of p, then $p(t) \ge 0$ and equality holds if and only if $p(t) \equiv 0$. Hence $c_{r,p,t} \le c_{r,0,t}$ and $a_{r,p,t} \le a_{r,0,t}$ if $t \ge d_1$. We also have $p'(t) = p(t) - p(t-1) \ge 0$ if $t > d_1$. From (2.3) and (2.4) we also get $a_{r,p,t} - a_{r,p,t-1} \ge a_{r,0,t} - a_{r,0,t-1} - 1$ and $c_{r,p,t} - c_{r,p,t-1} \ge a_{r,0,t} - a_{r,0,t-1} - 1$ if $t > d_1$, because $b_{r,p,t} - b_{r,p,t-1} - b_{r,0,t} + b_{r,p,t-1} < 2t$ and $d_{r,p,t} - d_{r,p,t-1} - d_{r,0,t} + d_{r,p,t-1} < 2t+2$.

Remark 2.3. Fix integers $d \ge r \ge 3$. Fix a hyperplane $H \subset \mathbb{P}^r$. If $r = 3$, then fix a smooth quadric surface Q. Let $C \subset \mathbb{P}^r$ be a smooth rational curve of degree d. Let N_C be the normal bundle of C in $\mathbb{P}^r$. N_C is a rank $r-1$ vector bundle on C and $\deg(N_C) = (r+1)d - 2$. Since $C \cong \mathbb{P}^1$, N_C is isomorphic to a direct sum of $r-1$ line bundles of degree $a_1 \ge \cdots \ge a_{r-1}$. If C is general, then N_C is rigid, i.e., $a_1 - a_r \le 1$, i.e., $a_1 = \lceil ((r+1)d-2)/(r-1) \rceil$ and $a_{r-1} = \lfloor ((r+1)d-2)/(r-1) \rfloor$ ([20], [22]). Hence $h^1(C, N_C(-1)) = 0$. Hence for a general $S \subset H$ with $\sharp(S) = d$ there is $A \in Z(r,d)$ with $S = A \cap H$ ([19, Theorem 1.5]). Now assume $r = 3$. In this case we get $a_1 = a_2 = 2d-1$ and hence $h^1(C, N_C(-2)) = h^0(C, N_C(-2)) = 0$ for a general $C \in Z(3,d)$. Hence for a general $B \subset Q$ with $\sharp(B) = 2d$ there is $A \in Z(3,d)$ such that $B = A \cap C$ ([19, Theorem 1.5]).

3. A reformulation of the theorems

It is easy to check that to prove Theorem 1.1 (resp. Theorem 1.2, resp. Theorem 1.3) for the integer r and the scheme Y with Hilbert polynomial p it is sufficient to prove the following statements.

Lemma 3.1. *Fix r and the admissible polynomial p. There is an integer t_0 with the following property. Fix any integer $t \ge t_0$ and any integer $d \ge 0$. Let $E \subset \mathbb{P}^r$ be a general union of d lines. Then either $h^1(\mathcal{I}_{Y\cup E}(t)) = 0$ or $h^0(\mathcal{I}_{Y\cup E}(t)) = 0$.*

Lemma 3.2. *Fix r and the admissible polynomial p. There is an integer t_0 with the following property. Fix any integer $t \ge t_0$ and any integer $d \ge 0$. Let $E \subset \mathbb{P}^r$ be a general rational curve of degree d. Then either $h^1(\mathcal{I}_{Y\cup E}(t)) = 0$ or $h^0(\mathcal{I}_{Y\cup E}(t)) = 0$.*

Lemma 3.3. *Fix r, the admissible polynomial p and $g \in \mathbb{N}$. There is an integer t_0 with the following property. Fix any integer $t \ge t_0$ and any integer $d \ge 2g + r$. Let $E \subset \mathbb{P}^r$ be a general smooth curve of degree d and genus g. Then either $h^1(\mathcal{I}_{Y\cup E}(t)) = 0$ or $h^0(\mathcal{I}_{Y\cup E}(t)) = 0$.*

Remark 3.4. Fix r and the admissible polynomial p. In the set-up of Theorem 1.3 also fix an integer $g \ge 0$. Suppose you want to prove Lemma 3.1 (resp. Lemma 3.2, resp. Lemma 3.3) for the integer t. It is sufficient to find a disjoint union $E \subset \mathbb{P}^r$ of $c_{r,p,t}$ lines (resp. a rational curve $E \subset \mathbb{P}^r$ of degree $a_{r,p,t}$, resp. a smooth genus curve $E \subset \mathbb{P}^r$ of degree $a_{r,p,t,g}$) and a disjoint union $E' \subset \mathbb{P}^r$ of $c_{r,p,t}+1$ lines (resp. a rational curve $E' \subset \mathbb{P}^r$ of degree $a_{r,p,t}+1$, resp. a smooth genus curve $E' \subset \mathbb{P}^r$ of degree $a_{r,p,t,g}+1$) such that $E \cap Y = E' \cap Y = \emptyset$, $h^1(\mathcal{I}_{Y\cup E}(t)) = 0$ and $h^0(\mathcal{I}_{Y\cup E'}(t)) = 0$. If $d_{r,p,t} = 0$ (resp. $b_{r,b,t} = 0$, resp. $b_{r,p,t,g} = 0$), then we do not need to find E'. By the semicontinuity theorem for the cohomology ([14, III.12.8]) instead of finding E or E' it is sufficient to find a scheme G or G' in the closure of $L(r,d)$ or of $Z(r,d)$ or of $Z(r,d,g)$ in the Hilbert scheme of $\mathbb{P}^r$ such that $Y \cap G = Y \cap G' = \emptyset$, $h^1(\mathcal{I}_{Y\cup G}(t)) = 0$ and $h^0(\mathcal{I}_{Y\cup G'}(t)) = 0$.

4. Proofs of Theorems 1.2 and 1.3 in $\mathbb{P}^3$

Fix a polynomial $p \in \mathbb{Z}[x]$ with $\deg(p) \le 1$ and write $p(x) = a_0x + a_1$. Since a general rational curve has maximal rank ([16]), we may assume $p \ne 0$, i.e. $(a_0, a_1) \ne (0,0)$. If $a_0 = 0$, then p is admissible if and only if $a_1 \ge 0$; a_1 is the Gotzmann's number of the constant polynomial p. If $a_1 \ne 0$, then p is admissible if and only if $a_0 > 0$ and $a_1 \ge -a_0(a_0-3)/2$; in this case $a_0(a_0-1)/2 + a_1$ is the Gotzmann's number of p ([12, Example 1.2]). Fix $Y \in \mathrm{Hilb}^p(\mathbb{P}^3)$.

From (2.1) and (2.2) we get

$$2a_0 + 2a_{3,p,t-2} + t(a_{3,p,t} - a_{3,p,t-2}) + b_{3,p,t} - b_{3,p,t-2} = (t+1)^2 \tag{4.1}$$

$$2a_0 + 2c_{3,p,t-2} + (t+1)(c_{3,p,t} - c_{3,p,t-2}) + d_{3,p,t} - d_{3,p,t-2} = (t+1)^2$$

Fix an integer $t \ge d_1 + 2$.

Remark 4.1. We have $c_{3,0,t} = (t+3)(t+2)/6$, $d_{3,0,t} = 0$ if $t \equiv 0,1 \pmod 3$ and $c_{3,0,t} = (t^2+5t+4)/6$, $d_{3,0,t} = (t+1)/3$ if $t \equiv 2 \pmod 3$. First assume $a_0 \ge a_1$. In this case if $t \gg 0$, say $t > 3(a_0 - a_1)$, we have $c_{3,p,t} = (t+3)(t+2)/6 - a_0$, $d_{3,p,t} = a_0 - a_1$ if $t \equiv 0,1 \pmod 3$ and $c_{3,p,t} = (t^2+5t+4)/6 - a_0$, $d_{3,p,t} = (t+1)/3 + a_0 - a_1$ if $t \equiv 2 \pmod 3$. Now assume $a_0 < a_1$. In this case if $t \gg 0$, say $t > 3(a_1 - a_0)$, we have $c_{3,p,t} = (t+3)(t+2)/6 - a_0 - 1$, $d_{3,p,t} = t + 1 + a_0 - a_1$ if $t \equiv 0,1 \pmod 3$ and $c_{3,p,t} = (t^2+5t+4)/6 - a_0$, $d_{3,p,t} = (t+1)/3 + a_0 - a_1$ if $t \equiv 2 \pmod 3$.

Remark 4.2. Write $t = 6k + e$ with $0 \le e \le 5$. We have $a_{3,0,6k} = 6k^2 + 6k + 1$, $b_{3,0,6k} = 5k$, $a_{3,0,6k+1} = 6k^2 + 8k + 4$, $b_{3,0,6k+1} = 0$, $a_{3,0,6k+2} = 6k^2 + 10k + 6$, $b_{3,0,6k+2} = 3k + 1$, $a_{3,0,6k+3} = 6k^2 + 12k + 6$, $b_{3,0,6k+3} = 2k + 1$, $a_{3,0,6k+4} = 6k^2 + 14k + 8$, $b_{3,0,6k+4} = 3k + 2$, $a_{3,0,6k+5} = 6k^2 + 16k + 11$, $b_{3,0,6k+5} = 0$ ([16]). First assume $a_1 > 0$ and t very large, say $t > 7a_1$. We have $a_{3,p,6k} = 6k^2 + 6k + 1 - a_0$, $b_{3,p,6k} = 5k - a_1$, $a_{3,p,6k+1} = 6k^2 + 8k + 3 - a_0$, $b_{3,p,6k+1} = 6k + 1 - a_1$, $a_{3,p,6k+2} = 6k^2 + 10k + 6 - a_0$, $b_{3,0,6k+2} = 3k + 1 - a_1$, $a_{3,p,6k+3} = 6k^2 + 12k + 6 - a_0$, $b_{3,0,6k+3} = 2k + 1 - a_1$, $a_{3,0,6k+4} = 6k^2 + 14k + 8 - a_0$, $b_{3,0,6k+4} = 3k + 2 - a_1$, $a_{3,0,6k+5} = 6k^2 + 16k + 10 - a_0$, $b_{3,0,6k+5} = 6k + 5 - a_1$. If $a_1 \le 0$, then $a_{3,p,t} = a_{3,p,t} - a_0$ and $b_{3,p,t} = b_{3,0,t} + a_1$. Hence $a_{3,p,t} - a_{3,p,t} \ge 2 + t/2$ if, say, $t \ge 12 + \max\{d_1, 12a_0\}$. In the set-up of Theorem 1.3 we get the same values just taking $a_1 - g$ instead of a_1.

Fix an integer ρ such that $(\rho - 2)/4$ is at least the Gotzmann's number of Y and $t \equiv \rho \pmod 2$.

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface such that $Q \cap Y$ has dimension zero if $\dim(Y) = 1$ and it is empty if $\dim(Y) = 0$.

Lemma 4.3. *Fix an integer $t \ge \rho$ such that $t - \rho$ is even. Let $E \subset \mathbb{P}^3$ be a general rational curve of degree $a_{3,0,t} - a_{3,0,\rho}$. Then $h^1(\mathcal{I}_{Y \cup E}(t)) = 0$.*

Proof. If $t = \rho$, then $E = \emptyset$ and hence the lemma is true. Hence we may assume $t \ge \rho + 2$ and that the lemma is true for the integer $t' := t - 2$. Let $E' \subset Q$ be a general rational curve of degree $a_{3,0,t-2} - a_{3,0,\rho}$. If $t \ne \rho + 2$, then $\deg(E') > 2$. Hence E' spans $\mathbb{P}^3$ and $E' \cap Q$ is a general union of $2(a_{3,0,t-2} -$

$a_{3,0,\rho}$) points of Q (Remark 2.3). If $t=\rho+2$, then $E'=\emptyset$ and hence even in this case $E'\cap Q$ is a general union of $2(a_{3,0,t-2}-a_{3,0,\rho})$ points of Q. Let $F\subset Q$ be a general union curve of type $(1,a_{3,0,t-2}-a_{3,0\rho}-1)$ of Q with the only restriction that if $E'\neq\emptyset$, then F contains a point of $E'\cap Q$. Since $E'\cup F$ is a flat limit of a family of smooth rational curves of degree $a_{3,0,t}-a_{3,0,\rho}$, it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup E'\cup F}(t))=0$. Hence it is sufficient to prove that $h^1(Q,\mathcal{I}_{Y\cap Q\cup F\cup(E'\cap Q)}(t))=0$, i.e. $h^1(Q,\mathcal{I}_{Y\cap Q\cup(E'\cap(Q\setminus F)}(t)(-F))=0$. The values of the integers $a_{3,0,z}$, $z=\rho,t-2,t$, show that $a_{3,0,t-2}-a_{3,0,\rho}\geq t/4$. Since ρ is at least 4 times the Gotzmann's number of Y and $t\geq\rho$, we have $h^1(Q,\mathcal{I}_{Y\cap Q}(t)(-F))=0$. Hence to conclude it is sufficient to check that $2a_0+\epsilon+t(a_{3,0,t-2}-a_{3,0,\rho})+1\leq(t+1)^2$, where $\epsilon=2(a_{3,0,t-2}-a_{3,0,\rho})-1$ if $E'\neq\emptyset$ and $\epsilon=0$ if $E'=\emptyset$. By (4.1) for $Y=\emptyset$ we have $2a_{3,0,t-2}+(t+1)\deg(F)+b_{3,0,t}-b_{3,0,t-2}=(t+1)^2$. Since $b_{3,0,t}<t$, it is sufficient to check that $2a_{3,0,\rho}>2a_0$. This is obvious, because $a_0<\rho$. □

Let η be an integer such that $\eta\geq\rho$. If $a_0>0$ we also assume $\eta>12a_1+\rho$ and $\eta>12a_0+\rho$. Set $a_{\eta,\eta}:=a_{3,0,\eta}-a_{3,0,\rho}$ and $b_{3,p,\eta,\eta}:=0$. For all integers $x\geq\eta+2$ such that $x\equiv\eta\pmod 2$ set $a_{x,\eta}:=\max\{0,(x-\eta)/2\}$. For all integers $x\geq\eta$ define the integers $a_{3,p,x,\eta}$ and $b_{3,p,x,\eta}$ by the relations

$$p(x)+xa_{3,p,x,\eta}+1+b_{3,p,x,\eta}+a_{x,\eta}=\binom{x+3}{3},\ 0\leq b_{3,p,x,\eta}\leq x-1$$

We have

$$\begin{aligned}&p(x)-p(x-2)+2a_{3,p,x-2,\eta}+x(a_{3,p,x,\eta}-a_{3,p,x-2,\eta})\\&+b_{3,p,x,\eta}-b_{3,p,x,\eta}+a_{x,\eta}-a_{x-2,\eta}=(x+1)^2\end{aligned}\tag{4.2}$$

and $-2\leq a_{x,\eta}-a_{x-2,\eta}\leq 0$ (more precisely, we have $a_{x-2,\eta}\geq 0$, $a_{x,\eta}=0$ if $a_{x-2,\eta}\leq 1$ and $a_{x,\eta}=a_{x-2,\eta}-2$ if $a_{x-2,\eta}\geq 2$).

Lemma 4.4. *For all integers $x\geq\eta$ with $x\equiv\eta\pmod 2$ we have $a_{3,p,x,\eta}\leq a_{3,p,x}\leq a_{3,0,x}$. We have $a_{3,p,x,\eta}=a_{3,p,x}$ and $b_{3,p,x,\eta}=b_{3,p,x}$ for all $x\geq\eta+2a_{\eta,\eta}$.*

Proof. The first inequality is true, because $a_{x,\eta}\geq 0$. The second inequality is true, because $p(x)\geq 0$, since $x\geq\rho$. The last assertion is true by the definitions of the integers $a_{3,p,x,\eta}$, $a_{3,p,x}$, $b_{3,p,x,\eta}$ and $b_{3,p,x}$, because $a_{x,\eta}=0$ for all $x\geq\eta+2a_{\eta,\eta}$. □

Lemma 4.5. *Assume $x\geq\eta+2$. Then:*

(a) $a_{3,0,x}-a_{3,0,\rho}-(x-\eta)/2\leq a_{3,p,x,\eta}$.
(b) $a_{3,p,x,\eta}-a_{3,p,x-2,\eta}\geq a_{3,0,x}-a_{3,0,x-2}-1\geq 4+x/2$.
(c) $a_{3,p,x,\eta}-a_{3,p,x-2,\eta}\leq x-2$.
(d) *Assume $a_0>0$, i.e. assume $\dim(Y)=1$. Then $a_{3,p,x,\eta}-a_{3,p,x-2,\eta}\leq x-2a_0-2$.*

Proof. Since $a_{3,p,\eta,\eta}=a_{3,0,x}-a_{3,0,\rho}$, the equality in part (a) is true if $x=\eta$. Now assume $x\geq\eta+2$. Since $x\geq 10a_0$ and $a_{3,p,x-2,\eta}\leq a_{3,0,x}$, (4.2) first with p and η and then with $p\equiv 0$ and $\eta=0$ gives $a_{3,p,x,\eta}-a_{3,p,x-2,\eta}\geq a_{3,0,x}-a_{3,0,x-2}-1$, concluding the proof of part (a) and proving the first inequality of part (b). The second inequality in part (b) is true by the explicit values of the integers $a_{3,x,0}$ and $a_{3,x-2}$ ([5, III.2]).

Part (c) follows from (4.2) and the relations $a_{3,p,x-2,\eta} \ge x$, $p(x) - p(x-2) = 2a_0 \ge 0$, $a_{x,\eta} - a_{x-2,\eta} \ge -1$, $b_{3,p,x-2,\eta} \ge 0$ and $b_{3,p,x,\eta} \le x-1$.

Now assume $a_0 > 0$. From (a) we get $a_{3,p,x-2,\eta} \ge 2xa_0 + 2$, by our restrictions on η. Hence part (d) follows from (4.2), because $p(x) - p(x-2) = 2a_0$. □

Among the many papers concerning the postulation of curves in $\mathbb{P}^3$ there is [5] which contains one statement (called $R(n,g)$ in [5]), which we are able to adapt to our need (see [5, IV.2 and Lemmas VI.4, VI.5]). For all integers $z > 0$, $r \ge 3$ let $Z'(r,z)$ be the closure in the Hilbert scheme of $\mathbb{P}^r$ of the set of all rational curves of degree z. It is well-known that if $A \in Z'(r,z)$ and D is a 1-secant line of A, then $A \cup D \in Z'(r,z+1)$ ([16], [5], [23]).

We have $a_{3,p,\eta,\eta} = a_{3,0,\eta} - a_{3,0,\rho}$ and $b_{3,0,\eta,\eta} = 0$. For all integers $x \ge \eta+2$ with $x \equiv \eta \pmod 2$ consider the following assertion R_x:

R_x, $x \ge \eta+2$, $x \equiv \eta \pmod 2$: For every integer y such that $0 \le 2y \le b_{3,p,x,\eta}$ there is a quintuple (A, D, D', S, S') with the following properties:

(1) $A \in Z'(3, a_{3,p,x,\eta})$ and A intersects transversally Q;
(2) D and D' are disjoint lines 1-secant to A and contained in Q, $Y \cap (D \cup D') = \emptyset$;
(3) S and S' are finite sets, $S \subset D \setminus D \cap Y$, $S' \subset D' \cap D' \cap Y$, $\sharp(S) = b_{3,p,x,\eta} - y$, $\sharp(S') = y$; if $\pi : Q \to D$ is the natural projection, then $\pi(S')$ is contained in $S \cap \pi(Y \cap (Q \setminus D \cup D'))$;
(4) $A = C \cup D_1 \cup \dots \cup D_z$, $z := a_{3,p,x,\eta} - a_{3,p,x-2,\eta}$, with C a smooth rational curve of degree $a_{3,p,x-2}$, each D_i a 1-secant line of C and $D_i \cap D_j = \emptyset$ for all $i \ne j$;
(5) $h^1(\mathcal{I}_{Y \cup A \cup S \cup S'}(x)) = 0$.

Since $p(x) + xa_{3,p,x,\eta} + 1 + b_{3,p,x,\eta} + a_{x,\eta} = \binom{x+3}{3}$ and $\sharp(S \cup S') = b_{3,p,x,\eta}$, the last condition of R_x is equivalent to the condition $h^0(\mathcal{I}_{Y \cup A \cup S \cup S'}(x)) = a_{x,\eta}$.

Lemma 4.6. *$R_{\eta+2}$ is true.*

Proof. Fix an integer y such that $0 \le 2y \le b_{3,p,\eta+2,\eta}$. By part (a) of Lemma 4.5 there is a smooth rational curve $B \subset \mathbb{P}^3$ such that $\deg(B) = a_{3,p,\eta,\eta} \ge \eta$, $B \cap Y = \emptyset$ and $h^1(\mathcal{I}_{Y \cup B}(\eta)) = 0$, i.e. $h^0(\mathcal{I}_{Y \cap B}(\eta)) = a_{\eta,\eta}$. Hence $h^i(\mathcal{I}_{Y \cup B \cup U}(\eta)) = 0$ for a general $U \subset \mathbb{P}^3$ such that $\sharp(U) = a_{\eta,\eta}$. Since U is general, we have $(Y \cup B) \cap U = \emptyset$. Fix $P \in U$. For a general set U we may assume the existence of a line R such that $P \in R$, $R \cap U = \{P\}$ and R is 1-secant to B. Hence the general line through P and intersecting B is 1-secant to B. Take a general such a line R'. A general line of $\mathbb{P}^3$ is disjoint from Y. Hence for a general B and a general U we may also assume that $(R \cup R') \cap Y = \emptyset$. There is a smooth quadric surface Q' such that $R \cup R' \subset Q'$. Call Q a general smooth quadric containing $R \cup R'$. For general U we may also assume that either $Y \cap Q = \emptyset$ (case $\dim(Y) = 0$) or that $\dim(Y \cap Q) = 0$ (case $\dim(Y) = 1$). First deforming U and then deforming B among the rational curves containing the points $R \cap B$ and $R' \cap B$ (both with fixed R, R', Q) we may assume that B is transversal to Q and that $B \cap (Q \setminus (R \cup R')$ is formed by $2a_{3,p,\eta,\eta} - 2$ general points of Q (Remark 2.3). Call $|\mathcal{O}_Q(0,1)|$ the ruling of Q containing R. Set $Z := 2P \subset \mathbb{P}^3$. The scheme $R \cup R' \cup Z$ is a sundial (Remark 2.2). Since $R \cup R' \cup Z$ is a flat limit of a family of two disjoint lines, one of them containing the point $R \cap B$ and the other one the point $R' \cap B$, we have $B \cup R \cup R' \cup Z \in Z'(3, a_{3,p,\eta,\eta} + 2)$. Part (a) of Lemma 4.5 gives $a_{3,p,\eta+2,\eta} - a_{3,p,\eta,\eta} \geq 2$. Let $R_1, \dots, R_z$, $z := a_{3,p,\eta+2,\eta-2}$, be general lines of type $(0,1)$ on Q. Since $R_i \cap R_j = \emptyset$ for all $i \neq j$, R_i is 1-secant to R and $R_i \cap B = \emptyset$ for all i, we have $W := B \cup R \cup R' \cup Z \cup R_1 \cup \cdots \cup R_z \in Z'(3, a_{3,p,\eta+2,\eta})$. Set $F := R \cup R' \cup Z \cup R_1 \cup \cdots \cup R_z$. For general lines R_i we have $W \cap Y = \emptyset$. Fix lines D, D' of Q of type $(0,1)$ containing exactly a point of $B \cap (Q \setminus R \cup R')$, $D \neq D'$, and call $\pi : Q \to D$ the projection. For general B we may assume that $\pi_{B\cap Q}$ is injective and that $\pi(B \cap Q) \cap \pi(Q \cap Y) = \emptyset$. Fix $S' \subset D'$ such that $\pi(S') \subset \pi(B \cap (Q \setminus R \cup R'))$ and $\sharp(S') = y$ (this is possible, because $\sharp(B \cap Q) = 2\deg(B) \geq y + 2$. Fix $S \subset D$ such that $\sharp(S) = b_{3,p,\eta+2,\eta} - 2$ and $\pi(S') \subseteq S$ (notice that y of the points of S are fixed; we may assume that the other points of S are general in D). Since $B \cup W$ is a flat limit of a family of curves which are unions of B and $a_{3,p,\eta+2,\eta} - a_{3,p,\eta,\eta}$ lines, to prove $R_{\eta+2}$ it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup B\cup W\cup S\cup S'}(\eta+2)) = 0$. Since $\mathrm{Res}_Q(Y \cup B \cup W \cup S \cup S') = Y \cup B \cup \{P\}$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Q(Y\cup B\cup W\cup S\cup S')}(\eta)) = 0$, it is sufficient to check that $h^1(Q, \mathcal{I}_{Q\cap(Y\cup B\cup W\cup S\cup S')}(\eta+2)) = 0$. By (4.2) we have $h^0(\mathcal{O}_{Q\cap(Y\cup B\cup W\cup S\cup S')}(\eta+2)) = (\eta+3)^2 = h^0(\mathcal{O}_Q(\eta+2))^2$. Hence it is sufficient to prove that $h^1(Q, \mathcal{I}_{Y\cap Q\cup(B\cap(Q\setminus W)}(\eta+2,\eta+2))(-F)) = 0$. We first need to check that $h^1(Q, \mathcal{I}_{Y\cap Q}(\eta+2,\eta+2)(-F)) = 0$. This true by part (d) of Lemma 4.5, because $\deg(Y \cap Q) = 2a_0$. We also need to check that $h^1(Q, \mathcal{I}_{Y\cap Q\cup S\cup S'}(\eta+2,\eta+2)(-F)) = 0$; this is true because $\sharp(S) \leq \eta+1$ and in parts (c) and (d) of Lemma 4.5 we have the term -2 in the right hand sides of the inequalities. □

Lemma 4.7. *R_x is true for all integers $x \geq \eta + 2$*

Proof. By Lemma 4.6 we may assume $x \geq \eta + 4$ and that R_{x-2} is true. Fix an integer y such that $0 \leq 2y \leq b_{3,p,x,\eta}$.

(a) In this step we assume $b_{3,p,x-2,\eta} \geq a_{3,p,x,\eta} - a_{3,p,x-2,\eta}$. This is the case handled in [5, Lemma VI.4]; here we use part (b) of Lemma 4.5 (which substitutes [5, Lemma VII.5] and (when $\dim(Y) = 1$), part (d) Lemma 4.5 (which substitutes [5, Lemma VII.6]).

(b) Now assume $b_{3,p,x,\eta} < a_{3,p,x,\eta} - a_{3,p,x-2,\eta}$. This is the case handled in [5, Lemma VI.5] (it is easier than case (a)). □

Proof of Theorem 1.2 for $r = 3$ and the admissible polynomial p: We need to check Lemma 3.2 and Remark 3.4. We have $a_{x,\eta} = 0$ for all $x \geq \eta + 2a_{\eta,\eta}$. As quite standard in this set-up (e.g. as in [5, §V]), we may take as t_0 the maximum of the integers $\eta + 2a_{\eta,\eta}$ for the two congruence classes modulo 2 of the integer η. In this way we get the h^1-part of Remark 3.4. Now we check the h^0-part of Remark 3.4. If $b_{3,p,t} = 0$, then $h^i(\mathcal{I}_{Y\cup W}(t)) = 0$, $i = 0, 1$, for a general $W \in Z(3, a_{3,p,t})$ and hence the h^0-part of Remark 3.4 is true in this case. Now assume $b_{r,p,t} > 0$. We need to prove that $h^0(\mathcal{I}_{Y\cup U}(t)) = 0$ for a general $U \in Z(3, a_{3,p,t} + 1)$. First assume $b_{3,p,t-2} \leq a_{3,p,t} - a_{3,p,t-2} + 1$. Take (A, D, D', S, S') with $y = 0$ and use that we get an $h^0(Q, \mathcal{I}_T(t)) = 0$ for a suitable t whose one-dimensional part is the union of a curve of type $(1,0)$ and $a_{3,p,t}-a_{3,p,t-2}$ lines of type $(0,1)$. Now assume $b_{3,p,t-2} > a_{3,p,t}-a_{3,p,t-2}+2$. Take (A,D,D',S,S') for $y = b_{3,p,t-2}-a_{3,p,t}+a_{3,p,t-2}+1$. Then we add in Q two lines of type $(1,0)$ (D and D') and $a_{3,p,t}-a_{3,p,t-2}-1$ lines of type $(0,1)$. □

Proof of Theorem 1.3 in $\mathbb{P}^3$*:* We may assume $g > 0$. Fix $Y \subset \mathbb{P}^3$ with p as its Hilbert polynomial. It is sufficient to prove Lemma 3.4 for $r = 3$ and the scheme Y. Take η as in the proof of Theorem 1.2 for Y. Let z_0 be a positive integer such that $z_0 \ge \eta + 2 + 2g$, $a_{z_0,\eta} = 0$ and Lemma 3.3 is true for Y and all $t \ge z_0$. Hence $a_{z,\eta} = 0$ for all $z \ge z_0$. We fix a large integer t for which we want to prove Lemma 3.4 (we need at least $t \ge z_0 + 2g$). Increasing by one if necessary z_0 we may assume $t \equiv z_0 \pmod 2$. Set $g(z_0) := 0$. For all all integers $z \ge z_0+2$ such that $z \equiv z_0 \pmod 2$ set $g(z) := \min\{g, g(z_0)+(z-z_0)/2\}$. Define the integers a_{3,p,g,z,z_0} and b_{3,p,g,z,z_0} by the relations

$$p(z) + za_{3,p,g,z,z_0} + 1 - g(z) + b_{3,p,g,z,z_0} = \binom{z+3}{3}$$

If $z \ge z_0 + 2$ we have $0 \le g(z) - g(z-2) \le 1$ and

$$2a_0 + 2a_{3,p,g,z,z_0} + z(a_{3,p,g,z,z_0} - a_{3,p,g,z-2,z_0}) + g(z-2) - g(z) + b_{3,p,g,z,z_0} - b_{3,p,g,z,z_0}$$

Call $R_z(g)$ the following assertion:

$R_z(g)$, $z \ge z_0$, $z \equiv z_0 \pmod 2$: For all integer y such that $0 \le 2y \le b_{3,p,g,z,z_0}$ there is a quintuple (A,D,D',S,S') with the following properties:

(1) A is a curve of degree a_{3,p,g,z,z_0} and arithmetic genus $g(y)$, $A \cap Y = \emptyset$ and A intersects transversally Q;
(2) D is a 2-secant line of A, D' is a 1-secant to A, $D \cup D' \subset Q$, $D \cap D' = \emptyset$, $Y \cap (D \cup D') = \emptyset$;
(3) S and S' are finite sets, $S \subset D \backslash D \cap Y$, $S' \subset D' \cap D' \cap Y$, $\sharp(S) = b_{3,p,g,z,z_0} - y$, $\sharp(S') = y$; if $\pi : Q \to D$ is the natural projection, then $\pi(S')$ is contained in $S \cap \pi(Y \cap (Q \setminus D \cup D'))$;
(4) $A = C \cup D_1 \cup \cdots \cup D_w$, $w := a_{3,p,g,z,z_0} - a_{3,p,g,z-2,z_0}$, with C a smooth curve of degree $a_{3,p,g,z-2,z_0}$ and genus $g(z)$, each D_i a 1-secant line of C and $D_i \cap D_j = \emptyset$ for all $i \ne j$;
(5) $h^i(\mathcal{I}_{Y\cup A\cup S\cup S'}(z)) = 0$, $i = 0,1$.

Since $a_{\eta,z-2} = a_{\eta,z} = 0$ and $0 \le g(z) - g(z-2) \le 1$, it is easy to see that Lemma 4.5 holds for the integers a_{3,p,g,z,z_0} and b_{3,p,g,z,z_0}; alternatively one Remark 2.3. Since $R_{z_0}(g)$ is true and the proof that $R_{z-2}(g) \Longrightarrow R_z(g)$ is similar to the proof of Lemma 4.7, we get $R_z(g)$ for all $z \ge z_0$, $z \equiv z_0 \pmod 2$ and then Lemma 3.3 using Remark 3.4. In [5, Lemma VI.1] the authors described how to increase at each step the arithmetic genus by the maximal amount possible for non-special curves; in [5, Lemmas VI.2 and VI.3] the authors described how to increases the genus by an unknown (non-maximal) value. □

The next lemma will be used to prove the case $r = 4$ of Theorem 1.1. We will also use it in the proofs of Theorem 1.2 and 1.3 (Lemma 5.1), although for them a weaker form would be sufficient.

Lemma 4.8. *Fix an admissible polynomial p with $\deg(p) \le 1$. Then there is an integer $\psi > 0$ with the following property. Fix $Y \subset \mathbb{P}^3$ with p as its Hilbert polynomial and fix any integer $t \ge \psi$. Let $A \subset \mathbb{P}^3$ be a general union of $c_{3,0,t} - c_{3,0,\psi}$ disjoint lines. Then $h^1(\mathcal{I}_{Y\cup A}(t)) = 0$.*

Proof. Take a large integer ψ bigger than 4 times the Gotzmann's number of p and such that for which for all integers $y \ge \psi - 2$ the integers $c_{3,p,y}$ and $d_{3,p,y}$ are as in Remark 5.3. Notice that for all integers $t \ge \psi - 1$ we get a statement similar to Lemma 4.5 for the integers $c_{3,p,t}$ and $d_{3,p,t}$. For all integers $t \ge \psi - 1$ call $A(t)$ the statement of the lemma if $t \equiv \psi$ and a similar statement with $c_{3,0,t} - c_{3,0,\psi-1}$ if $t \equiv \psi - 1 \pmod 2$. Notice that if $A(t)$ is true for some $t \ge \psi$ with $t - \psi$ odd, then Lemma 4.8 is true for the integer t, because $c_{3,0,t} - c_{3,0,\psi} \ge c_{3,0,t} - c_{3,0,\psi-1}$. Fix $a \in \{\psi - 1, \psi\}$ and set $\alpha(t) := \binom{t+3}{3} - [p(t) + (t+1)(c_{3,0,t} - c_{3,0,a}] = (t+1)c_{3,0,a} + d_{3,0,t} - p(t)$. For large a we have $\alpha(t) \ge \alpha(t-2)$, because $p(t) - p(t-2) + |d_{3,0,t} - d_{3,0,t-2}| \le 2a_0 + t$. Hence we may prove that $A(t)$ implies $A(t+2)$ just adding $c_{3,0,t+2} - c_{3,0,t-2}$ lines of type $(0,1)$ on Q, without even having to check the different case for the congruence classes of t modulo 3 as in [15, §2]. □

5. Proofs in $\mathbb{P}^r$, $r \ge 4$

Fix the admissible polynomial p with $\deg(p) \le r-2$ and set $p'(x) = p(x) - p(x-1)$ and $p''(x) = p'(x) - p'(x-1)$ (p' is the first difference polynomial of p and p'' is the second difference polynomial of p). Let $Y \subset \mathbb{P}^r$ be closed subscheme with p as its Hilbert polynomial. Let $H \subset \mathbb{P}^r$, $r \ge 4$, be a hyperplane. We always assume that $\mathrm{Res}_H(Y) = Y$ and that $\dim(H \cap Y) = \dim(Y) - 1$ (this is true, e.g., if H is general by Remark 2.1). The scheme $Y \cap H$ has p' as its Hilbert polynomial.

5.1. **Proof of Theorem 1.2.** In this subsection we also assume that Theorem 1.2 is true in $H \cong \mathbb{P}^{r-1}$ for the Hilbert polynomial p' of $Y \cap H$ (this is possible, because the case $r = 3$ is proved in section 4). It is sufficient to prove Lemma 3.2 for Y and the integer r. We assume that Lemma 3.2 is true in H for the scheme $Y \cap H$ and call t_1 an integer such that for all integers $t \ge t_1$ and $d \ge 0$ either $h^0(\mathcal{I}_{Y\cap H\cup W}(t)) = 0$ or $h^1(\mathcal{I}_{Y\cap H\cup W}(t)) = 0$ for a general rational curve $W \subset H$ such that $\deg(W) = d$. If $r \ge 5$ we also assume that t_1 works in $\mathbb{P}^{r-2}$ for the admissible polynomial p''.

Lemma 5.1. *Fix an integer $r \ge 4$ and an integer t. Assume $t \ge r - 3 + \rho$ and $t \ge \psi + 2$, where ψ is the integer appearing on Lemma 4.8. Let $E \subset \mathbb{P}^r$ be a general union of t lines. Then $h^1(\mathcal{I}_{Y\cup E}(t)) = 0$.*

Proof. Let $N \subset \mathbb{P}^r$ be a general 3-plane. Let q be the Hilbert polynomial of the scheme $Y \cap N$. Since $t \ge \psi + 2$ and $c_{3,0,x} = \lfloor (x+3)(x+2)/6 \rfloor$ for all x, we have $t \le c_{3,0,t} - c_{3,0,\psi}$. Hence $h^1(N, \mathcal{I}_{Y\cap N\cup E}(t)) = 0$ (Lemma 4.8). Take a general flag $H_0 \supset H_1 \supset \cdots \supset H_{r-3}$ with $\dim(H_i) = r - i$ for all i and $N = H_{r-3}$. Each scheme $Y \cap H_i$ has as Hilbert polynomial the i-th difference function of p. Our assumption on t implies $h^i(H_i, \mathcal{I}_{Y\cap H_i}(t-1-i)) = 0$. Using $r-3$ Castelnuovo's sequences we get $h^1(\mathcal{I}_{Y\cup E}(t)) = 0$. Hence the lemma follows from the semicontinuity theorem for cohomology. $\square$

Lemma 5.2. *Assume $r = 4$. Fix integers $t \ge t_1 + 4$ and d, e such that $d > 0$, $0 \le e \le t-1$ and $p'(t) + td + 1 + (t+1)e + 3t \le \binom{t+3}{3}$. Let $A \subset H$ be a general rational curve of degree d, $B \subset H$ be a general union of e lines and $L \subset H$ a general 2-secant line of A. Then $h^1(H, \mathcal{I}_{Y\cap H\cup A\cup B\cup L}(t)) = 0$.*

Proof. Write $p'(x) = a_0x + a_1$.

(a) Assume that $d \ge a_{3,p',t-4}$. Let $U \subset H$ be a general rational curve of degree $a_{3,p',t-4}$. By the inductive assumption we have $h^0(H, \mathcal{I}_{Y\cap H\cup U}(t-4)) = 0$ and $h^0(\mathcal{I}_{Y\cap H\cup U}(t-4)) = b_{3,p',t-4}$. Fix a general 2-secant line $D \subset H$ of U and let $Q \subset H$ be a general quadric surface containing D. Q is smooth, since U and D are general, D does not contain the support of any associated component of $Y\cap H$ (not even an embedded one). Since Q is general among the quadric surfaces containing D. Hence $\dim(Y\cap Q) \le 0$ and $\deg(Y\cap Q) = 2a_0$. Call $|\mathcal{O}_Q(0,1)|$ the ruling of Q containing D. First assume $e \ge a_{3,p',t-2} - a_{3,p',t-4} - 2$. Let $E \subset Q$ be a general union of $a_{3,p',t-2} - a_{3,p',t-4} - 2$ lines of type $(0,1)$. We have $(t+1)(a_{3,p',t-2} - a_{3,p',t-4} - 1) - 2 \le t(a_{3,p',t-2} - a_{3,p',t-4})$, because $a_{3,p',t-2} - a_{3,p',t-4} \le t-2$ (part (c) of Lemma 4.5). Since $2a_0 + 2a_{3,p',t-4} + (t-2)(a_{3,p',t-2} - a_{3,p',t-4}) + b_{3,p',t-2} - b_{3,p',t-4} = (t-1)^2$ by (4.1), we get $h^0(\mathcal{O}_{Q\cap(Y\cup U\cup D\cup E)}(t-2)) \le (t-1)^2$. As in Lemma 4.3 we have $h^1(Q, \mathcal{I}_{Y\cap Q}(t-2)(-E)) = 0$. By Remark 2.3 we may assume that $U\cap Q$ is a general union of $2a_{3,p',t-4}$ points of Q. As in [5, §VIII], we may control the postulation of $(U\cup D)\cap Q$. Hence $h^1(Q, \mathcal{I}_{Y\cap Q}(t-2)(-E)) = 0$. A Castelnuovo's sequence gives $h^1(H, \mathcal{I}_{Y\cap H\cup U\cup D\cup E}(t-2)) = 0$. Then we deform E to a general union $E' \subset H$ of $a_{3,p',t-2} - a_{3,p',t-4} - 2$. By the semicontinuity theorem we have $h^1(H, \mathcal{I}_{Y\cap H\cup U\cup D\cup E'}(t-2)) = 0$. Take a general smooth quadric Q'. Set $e' := e - (a_{3,p',t-2} - a_{3,p',t-4} - 2)$ and $d' := d - a_{3,p',t-4}$. Since $a_{3,p',t} - a_{3,p',t-4} \ge t-3$ (part (b) Lemma 4.5), we have $e \le (a_{3,p',t-2} - a_{3,p',t-4} - 2) + (a_{3,p',t} - a_{3,p',t-2} - 1)$. Since $d + e \le a_{4,p,t} - 3$, we have $d' + e' \le a_{3,p',t} - a_{3,p',t-2} - 1$. Let $F' \subset Q'$ be a general union of $d' + e'$ lines of type $(0,1)$ of Q', with the only restriction that d' of them contain a point of $U\cap Q'$. We have $h^0(\mathcal{O}_{Q'\cap(Y\cup Q\cup F')}(t)) = 0$. Since $E'\cap Q'$ is general in Q' and $h^0(Q', \mathcal{O}_{Q'\cap Y\cup F'\cup E'\cap U}(t)) \le (t+1)^2$ as above we get $h^1(Q', \mathcal{I}_{Y\cap Q'\cup F'\cup(E'\cup U)\cap Q'}(t)) = 0$.

Now assume $e \le a_{3,p',t-2} - a_{3,p',t-4} - 3$. We add in Q the line D, e general lines of type $(0,1)$ and $\min\{a_{3,p',t-2} - a_{3,p',t-4} - 2 - e, d - a_{3,p',t-4}\}$ lines of type $(0,1)$ each of them meeting U at one point (call T the union of the latter $a_{3,p',t-2} - a_{p',t-4} - 2 - e$ lines). Before making the last step we smooth $U\cup T$ to a smooth rational curve U_1, and move D during the smoothing to a 2-secant line of U_1.

(b) Now assume $d < a_{3,p',t-4}$. Instead of U we take a general rational curve $U' \subset H$ with $\deg(U') = d$. We again take as D a general 2-secant of U' and as Q a general smooth quadric containing D. In Q and Q' we add no line intersecting U', except D. $\square$

Lemma 5.3. *Assume $r \ge 5$. Fix integers $t > t_1$ and d, e such that $d > 0$, $0 \le e \le t-1$ and $p'(t) + td + 1 + (t+1)e + 3t \le \binom{r+t-1}{r-1}$. Let $A \subset H$ be a general rational curve of degree d, $B \subset H$ be a general union of e lines and $L \subset H$ a general 2-secant line of A. Then $h^1(H, \mathcal{I}_{Y\cap H\cup A\cup B\cup L}(t)) = 0$.*

Proof. Let $M \subset H$ be a general hyperplane.

(a) Assume $d \ge a_{r-1,p',t-1}$. Let $W \subset H$ be a general rational curve of degree $a_{r-1,p',t-1}$. The inductive assumption gives $h^1(H, \mathcal{I}_{Y\cap H\cup W}(t-1)) = 0$ and $h^0(\mathcal{I}_{Y\cap H\cup W}(t-1)) = b_{r-1,p',t-1}$. For a general W the set $W\cap M$ is a general subset of M with cardinality $a_{r-1,p',t-1}$ (Remark 2.3). Let $E \subset M$ be a general union of a smooth rational curve E_1 of degree $d - a_{r-1,p',t-1}$ (if $d > a_{r-1,p',t-1}$, i.e. if $E_1 \ne \emptyset$, then impose that E_1 is general among the rational curves of M containing a prescribed point, P, of $W \cap M$), a general union E_2 of e lines of M and a line D spanned by two of the points of $W \cap M$. Since $W \cap M$ is general, E may be considered as a general union of a general rational curve of degree $d - a_{r-1,p',t-1}$ and $e+1$ disjoint lines. Recall that $p''(x) = p'(x) - p'(x-1)$. Since

$$p''(t) + a_{r-1,p',t-1} + t(a_{r-1,p',t} - a_{r-1,p',t-1}) + b_{r-1,p',t} - b_{r-1,p',t-1} = \binom{r+t-2}{r-2},$$

we may use induction on r to get first $h^1(M, \mathcal{I}_{Y\cap M\cup E}(t)) = 0$ (here the starting point of the induction is given by Lemma 5.2, at each inductive step we use that $a_{r-1,p',t} - a_{r-1,p',t-1} < a_{r-2,p'',t}$ for large t) and then $h^1(M, \mathcal{I}_{Y\cap M\cup W\cap M\cup E}(t)) = 0$. A Castelnuovo's sequence gives $h^1(H, \mathcal{I}_{Y\cap H\cup W\cup E}(t)) = 0$.

(b) Now assume $d < a_{r-1,p',t-1}$. Instead of W we take a general rational curve $W' \subset H$ with $\deg(W') = d$ and take $E = E_2 \cup D \subset M$ with E_2 union of e general lines and D the line spanned by two of the points of $W' \cap M$. $\square$

(i) To prove Theorem 1.2 we need to prove Lemma 3.2 and use Remark 3.4. We use induction on the integer r, the starting point of the induction being the case $r = 3$ proved in section 4. We easily see (as in the case $p = 0$, i.e. $Y = \emptyset$) the existence of an integer $t_2 \ge t_1$ such that for every $t \ge t_2$ we have $h^1(\mathcal{I}_{Y\cup W}(t)) = 0$ for a general $W \in Z(r, a_{r,0,t} - a_{r,0,t_1})$ (we use Remark 2.3 and Lemmas 5.2 and 5.3 ignoring the secant line D). Set $\eta := t_2$ and $a_{\eta,\eta} := \binom{r+\eta}{r} - p(\eta) - \eta(a_{r,0,t} - a_{r,0,t_1}) - 1$. We have $a_{\eta,\eta} = \eta a_{r,0,\eta} + b_{r,0,\eta} - p(\eta)$. For all integers $t \ge \eta$ set $a_{t,\eta} := \max\{0, a_{\eta,\eta} - t + \eta\}$. For all integers $t \ge \eta$ define the integers $a_{r,p,t,\eta}$ and $b_{r,p,t,\eta}$ by the relations

$$p(t) + ta_{r,p,t,\eta} + 1 + b_{r,p,t,\eta} + a_{r,p,t,\eta} = \binom{r+t}{r},\ 0 \le b_{r,p,t,\eta} \le r-1 \tag{5.1}$$

For all $t > \eta$ from (5.1) we get

$$\begin{aligned} &p'(t) + a_{r,p,t-1,\eta} + t(a_{r,p,t,\eta} - a_{r,p,t-1,\eta}) + b_{r,p,t,\eta} - b_{r,p,t-1,\eta} \\ &+ a_{r,p,t,\eta} - a_{r,p,t-1,\eta} = \binom{r+t-1}{r-1} \end{aligned} \tag{5.2}$$

with either $a_{r,p,t-1,\eta} > 0$ and $a_{r,p,t,\eta} - a_{r,p,t-1,\eta} = -1$ or $a_{r,p,t,\eta} = a_{r,p,t-1,\eta} = 0$.

Call E_t, $t > t_1$, the following assertion:

E_t, $t > t_1$: Let $W \subset \mathbb{P}^r$ be a general union of $b_{r,p,t}$ lines and a general rational curve of degree $a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1}$. Then $h^1(\mathcal{I}_{Y\cup W}(t)) = 0$.

Claim 1: For all $t > t_1$, E_t is true.

Proof of Claim 1: We first need to check that E_t is meaningful, i.e. that $a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1} \ge 0$. This is true if $t_1 \gg 0$(depending only on r, not from p), because $b_{r,p,t_1} < t_1$ and $a_{r,0,t} \sim t^{r-1}/r!$ if $t \gg 0$ and hence, $a_{r,0,t_1+1} - a_{r,0,t_1} \sim t^{r-2}/r \cdot (r-2)!$ if $t_1 \gg 0$. Now we check E_{t_1+1}. Let $U \subset \mathbb{P}^r$ be a general union of t_1 lines. By Lemma 5.1 we have $h^1(\mathcal{I}_{Y\cup U}(t_1)) = 0$. Let $F \subset H$ be a general rational curve of degree $a_{r,0,t_1+1} - a_{r,0,t_1} - b_{r,p,t_1+1}$. The inductive assumption on Theorem 1.2 (or a weaker statement like Lemma 4.8 for a rational curve) gives $h^1(H, \mathcal{I}_{Y\cap H\cup F}(t_1+1)) = 0$. Since $ta_{r,0,t_1} \geq 2t_1$, (5.2) gives $p'(t_1+1) + t_1 + t(a_{r,0,t_1+1} - a_{r,0,t_1} - b_{r,p,t_1}) + 1 \leq \binom{r+t_1}{r-1}$. Since $E \cap H$ is general in H and $b_{r,p,t_1+1} \leq t_1$, we get $h^1(H, \mathcal{I}_{Y\cap H\cup F}(t_1+1)) = 0$. Hence a Castelnuovo's sequence gives $h^1(\mathcal{I}_{Y\cup U\cup F}(t_1+1)) = 0$.

Now assume $t \geq t_1+2$ and that E_{t-1} is true. Lemmas 5.2 (if $r=4$) and 5.3 (if $r>4$) with $e = b_{r,p,t}$ show that $h^1(H, \mathcal{I}_{Y\cap H\cup F}(t)) = 0$ for a general union $F \subset H$ of $b_{r,p,t}$ lines and a rational curve F_1 of degree $a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1}$. Let $G \subset \mathbb{P}^r$ be a general union of $b_{r,p,t}$ lines and a general rational curve G_1 of degree $a_{r,0,t} - a_{r,0,t_1} - b_{r,p,t_1}$. Set $G_2 := G \setminus G_1$. By Remark 2.3 we may assume that $G \cap H$ is a general union of $a_{r,0,t} - a_{r,0,t_1}$ points of H. Increasing if necessary t_1 we may assume that $a_{r,0,t} - a_{r,0,t_1} - b_{r,0,t_1} \geq rt$ (indeed $a_{r,0,t} \cong t^{r-1}/r!$ for $t \gg 0$). Since $a_{r,0,t} - a_{r,0,t_1} - b_{r,0,t_1} \geq rt$ and E_1 is general, it is easy to check that E_1 passes through at least t general points of H (alternatively use Remark 2.3 in the projective space H and, calling $N_{E_1,H}$, the normal bundle of E_1 in H, that $h^1(E_1, N_{E_1,H}(-G)) = 0$ for a general $G \subset E_1$ with $\sharp(G) = t$). Hence without losing the generality of F we may assume that $G_2 \cap H \subset F_1$, that $F_2 \cap G = \emptyset$ and that $G_1 \cap F_1$ is a single point. Hence $G \cup F$ is a union of an element of $Z'(r, a_{r,0,t} - a_{r,0,t} - b_{r,p,t})$ and $b_{r,p,t}$ lines ([6], [7]). Since $h^0(\mathcal{O}_{Y\cap H\cup F\cup(G\cap(H\setminus F))}(t)) = p'(t) + t(a_{r,0,t} - a_{r,0,t-1}) + 1 + b_{r,0,t} + (a_{r,0,t-1} - a_{r,0,t_1} - b_{r,0,t-1} - 1) = p'(t) + t(a_{r,0,t} - a_{r,0,t_1}) + b_{r,}$

Call B_t, $t \geq \eta$, the following assertion:

B_t, $t \geq \eta$: Let $W \subset H$ be a general union of $b_{r,p,t,\eta}$ general lines and the general rational curve of degree $a_{r,p,t,\eta} - b_{r,p,t,\eta}$. Then $h^1(\mathcal{I}_{Y\cup W}(t)) = 0$.

Claim 2: For all $t \geq \eta$, B_t is true.

Proof of Claim 2: B_η is true, because E_η is true. Assume $t > \eta$ and that B_{t-1} is true. Take a general union $E \subset \mathbb{P}^r$ of $b_{r,p,t-1,\eta}$ general lines and the general rational curve E_1 of degree $a_{r,p,t-1,\eta} - b_{r,p,t-1,\eta}$. Since B_{t-1} is true, we have $h^1(\mathcal{I}_{Y\cup E}(t-1)) = 0$ i.e. $h^0(\mathcal{I}_{Y\cup E}(t-1)) = a_{t-1,\eta}$. Assume for the moment $a_{t-1,\eta} > 0$. Since we may take as H a general hyperplane, we have $h^1(\mathcal{I}_{Y\cup E\cup\{P\}}(t-1)) = 0$ for a general $P \in H$.

Let $A \subset H$ be a general smooth rational curve of degree $a_{r,p,t,\eta} - a_{r,p,t-1,\eta} - 1 - b_{r,p,t,\eta}$, L a general 2-secant line of A and $B \subset H$ be a general union of $b_{r,p,t,\eta}$ lines. By Lemma 5.2 (case $r=4$) and Lemma 5.3 (case $r>4$), we have $h^1(H, \mathcal{I}_{Y\cap H\cup A\cup B\cup D}(t)) = 0$. Since $a_{r,p,t,\eta} - a_{r,p,t-1,\eta} - 1 - b_{r,p,t,\eta} \geq r(t-1)$ we may assume that A contains exactly one point of $E_1 \cap H$, that $(E \setminus E_1) \cap H \subset A$, and that one of the points, P, of $D \cap A$, is a general point of H. Let $V \subset \mathbb{P}^r$ be any 3-dimensional linear space containing the plane $V' \subset H$ spanned by D and the tangent line to A at P. Let $Z \subset V$ be the 2-point of V with P as its reduction, i.e., the closed subscheme of V with $(\mathcal{I}_{P,V})^2$ as its ideal sheaf. Set $W := E \cup A \cup D \cup Z \cup B$. Since $W \cap Y = \emptyset$ and W is a flat limit of a family of disjoint unions of $b_{r,p,t,\eta}$ lines and a rational curve of degree $a_{r,p,t,\eta}$ ([6], [7]), to prove B_t it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup W}(t)) = 0$. Since $\mathrm{Res}_H(Y\cup W) = Y\cup E\cup\{P\}$, it is sufficient to prove that $h^i(H,\mathcal{I}_{Y\cap H\cup E\cap H\cup A\cup B\cup D}(t)) = 0$, $i = 0,1$. This is true, because $h^1(H,\mathcal{I}_{Y\cap H\cup A\cup B\cup D}(t)) = 0$, $E\cap H$ is general in H and $h^0(\mathcal{O}_{Y\cap H\cup A\cup B\cup D}(t)) = p'(t)+a_{r,p,t-1,\eta}+t(a_{r,p,t,\eta}-a_{r,p,t-1,\eta})+b_{r,t,p,\eta}-2-b_{r,p,t,\eta} = \binom{r+t-1}{r-1}$ (by (5.2), because $a_{t,\eta} = a_{t,\eta}-1$).

Now assume $a_{t-1,\eta} = 0$, i.e., assume $t \ge a_{\eta,\eta}+2$. We make the construction just given, taking a general rational curve $A' \subset H$ of degree $a_{r,p,t,\eta}-a_{r,p,t-1,\eta}-b_{r,p,t,\eta}$ instead of $A\cup D\cup Z$.

(ii) To conclude the proof of Theorem 1.2 it is sufficient to check the h^0-part of Remark 3.4, i.e. it is sufficient to prove that $h^0(\mathcal{I}_{Y\cup W}(t)) = 0$ for a general rational curve of degree $a_{r,p,t}+1$. Let $W\subset H$ be a general union of $b_{r,p,t-1,\eta}$ general lines and the general rational curve of degree $a_{r,p,t-1,\eta}$. Since $t-1 \ge \eta + a_{\eta,\eta}$, we have $b_{r,p,t-1,\eta} = b_{r,p,t-1}$ and $a_{r,p,t-1,\eta} = a_{r,p,t-1}$. Hence E_{t-1} gives $h^i(\mathcal{I}_W(t-1)) = 0$, $i = 0,1$. Let $E\subset H$ be a general rational curve of degree $a_{r,p,t}-a_{r,p,t-1}+1$ with the only restriction that it contains exactly one point of each connected component of W. Since $a_{r,p,t}-a_{r,p,t-1}+1 \ge rt$, E may be seen as a general rational curve of H with degree $a_{r,p,t}-a_{r,p,t-1}+1$ (as seen in the proof of $B_{t-1} \implies B_t$). Hence for large t and the inductive assumption either $h^0(H,\mathcal{I}_{Y\cap H\cup E}(t)) = 0$ or $h^1(H,\mathcal{I}_{Y\cap H\cup E}(t)) = 0$. Since $W\cap H$ is general in H, we get $h^0(H,\mathcal{I}_{Y\cap H\cup W\cap H\cup E}(t)) = 0$. A Castelnuovo's sequence gives $h^0(\mathcal{I}_{Y\cup W\cup E}(t)) = 0$.

5.2. **Proofs of Theorems 1.1 and 1.3.** We recall the following easy lemma ([3, Lemma 6]).

Lemma 5.4. *Fix integers $n\ge 3$, $t>0$, $d\ge 0$, $e\ge 0$ such that $(t+1)(d+2e) \le \binom{n+t}{n}$. Let $X\subset \mathbb{P}^n$ be a general union of d lines and e reducible conics. Then $h^1(\mathcal{I}_X(t)) = 0$.*

Proof of Theorem 1.1: We use Lemma 3.1 and Remark 3.4. We first check the inductive step, i.e. we assume that the statement of Theorem 1.1 is true in $\mathbb{P}^{r-1}$ for the admissible polynomial p' and we prove Theorem 1.1 in $\mathbb{P}^r$ for the Hilbert polynomial p. We use that $c_{r,p,t}\sim c_{r,0,t}\sim a_{r,p,t}\sim t^{r-1}/r!$ for $t\gg 0$ and hence $c_{r,p,t}-c_{r,p,t-1}\sim \frac{t^{r-2}}{r\cdot(r-2)!} \gg 3t$ for all $r\ge 4$. For all integers $n\ge 3$ and $t\ge 0$ and all polynomials f define the integers $x_{n,f,t}$ and $y_{n,f,t}$ by the following relations

$$f(t)+(t+1)x_{n,f,t}+(2t+1)y_{n,f,t} = \binom{r+t}{r},\ 0\le y_{n,f,t}\le t \tag{5.3}$$

For all $t>0$ from (5.3) we get

$$f(t)-f(t-1)+x_{n,f,t-1}+(t+1)(x_{n,f,t}-x_{n,f,t-1})+$$
$$2y_{n,f,t}-2y_{n,f,t-1} = \binom{r+t-1}{r-1}$$

Notice that $x_{n,f,t}\le c_{n,f,t}\le x_{n,f,t}+2y_{n,f,t}\le x_{n,f,t}+2t$. Fix an integer t_1 larger that the Gotzmann's number of p.

Instead of E_t we use the following statement E'_t:

E'_t: Let $W \subset \mathbb{P}^r$ be a general union of $y_{r,p,t}$ reducible conics with singular point contained in H and $x_{r,0,t} - x_{r,0,t_1}$ lines. Then $h^1(\mathcal{I}_{Y\cup W}(t)) = 0$.

We prove E'_t for all $t > t_1$ using Lemma 5.4 instead of Lemmas 5.2 and 5.3 (see the proof of Claim 1 below for more details). Then we fix an integer $\eta > t_1$ as in the proof of Theorem 1.2 and set $a'_{\eta,\eta} := \binom{r+\eta}{r} - (\eta+1)(x_{r,0,\eta} - x_{r,0,t_1}) - (2\eta+1)y_{r,p,\eta}$. For all integers $t \geq \eta$ set $a'_{t,\eta} := \max\{0, a_{\eta,\eta} + \eta - t\}$. For all integer $t \geq \eta$ define the integers $x_{r,p,t,\eta}$ and $y_{r,p,t,\eta}$ by the relations

$$p(t) + (t+1)x_{r,p,t,\eta} + (2t+1)y_{r,p,t,\eta} + a'_{t,\eta} = \binom{r+t}{r}, \ 0 \leq y_{r,p,t,\eta} \leq t$$

Notice that $x_{r,p,\eta,\eta} = x_{r,0,\eta} - x_{r,0,t_1}$ and $y_{r,p,\eta,\eta} = y_{r,p,t}$. For all $t > \eta$ we have

$$\begin{aligned} p(t) - p(t-1) = x_{r,p,t-1,\eta} + (t+1)(x_{r,p,t,\eta} - x_{r,p,t-1,\eta}) + \\ 2y_{r,p,t,\eta} - 2y_{r,p,t-1,\eta} + a'_{t,\eta} - a'_{t-1,\eta} = \binom{r+t-1}{r-1} \end{aligned}$$

Since $a_{t,\eta} = 0$ for all $t \geq \eta + a_{\eta,\eta}$ we have $x_{r,p,t,\eta} = x_{r,p,t}$ and $y_{r,p,t,\eta} = y_{r,p,t}$ for all $t \geq \eta + a_{\eta,\eta}$. For all $t \geq \eta$ call B'_t the following assertion:

B'_t, $t \geq \eta$: Let $W \subset \mathbb{P}^r$ be a general union of $x_{r,p,t,\eta}$ lines and $y_{r,p,t,\eta}$ reducible conics whose singular point is contained in H. Then $h^1(\mathcal{I}_{Y\cup W}(t)) = 0$ (or, equivalently, $h^0(\mathcal{I}_{Y\cup W}(t)) = a_{t,\eta}$).

Claim 1: B'_t is true for all $t \geq \eta$.

Proof of Claim 1: B'_η is true, because it is equivalent to E'_η. Assume $t \geq \eta+2$ and that B'_{t-1} is true. Take W satisfying B'_{t-1}. For a general W no irreducible component of W is contained in H and hence for each conic $T \subset W$ the scheme $T \cap H$ is a tangent vector of H. Hence for a general W the scheme $W \cap H$ is a general union of $y_{r,p,t-1,\eta}$ tangent vectors of H and $x_{r,p,t-1,\eta}$ general points of H. First assume $y_{r,p,t,\eta} \geq y_{r,p,t-1,\eta}$. Let $E \subset H$ be a general union of $x_{r,p,t,\eta} + 2y_{r,p,t,\eta} - x_{r,p,t-1,\eta} - 2y_{r,p,t-1,\eta}$ lines with the only restriction that $y_{r,p,t,\eta} - y_{r,p,t-1,\eta}$ of these lines contain a point of a different line of W; this is possible, because for large η we have $x_{r,p,t,\eta} - x_{r,p,t-1,\eta} \sim x^{r-2}/r \cdot (r-2)! > 2t$. We need to check that $h^1(H, \mathcal{I}_{Y\cap H\cup(W\cap H)\cup E}(t)) = 0$. Hence $W \cap H$ is a general union of $y_{r,p,t-1,\eta}$ tangent vectors of H and $x_{r,p,t-1,\eta}$ general points of H. In characteristic zero a general tangent vector gives independent conditions to any non-trivial linear system ([11], [9, Lemma 1.5]); in arbitrary characteristic (as in [15] or [3]) we may use that the number of tangent vectors is very small. A Castelnuovo's sequence proves that $h^1(\mathcal{I}_{Y\cup W\cup E}(t)) = 0$. By the semicontinuity theorem B'_t is true. Now assume $y_{r,p,t,\eta} < y_{r,p,t-1,\eta}$. Fix $S \subseteq \mathrm{Sing}(W)$ with $\sharp(S) = y_{r,p,t-1,\eta} - y_{r,p,t,\eta}$. For each $P \in S$ call U_P the connected component of W containing P. For each $P \in S$ fix any 3-dimensional linear subspace of $\mathbb{P}^r$ containing U_P, but not contained in H. Let $Z_P \subset V_P$ be the closed subscheme of V_P with $(\mathcal{I}_{P,V_P})^2$ as its ideal sheaf. Notice that $C_P := U_P \cup Z_P$ is a sundial. Let $E \subset H$ be a general union of $x_{r,p,t,\eta} + 2y_{r,p,t,\eta} - x_{r,p,t-1,\eta} + 2y_{r,p,t-1,\eta}$. Set $G := W \cup E \cup_{P\in S} Z_P$. Since each C_P is a sundial, G is a flat limit of a family of disjoint unions of $x_{r,p,t,\eta}$ lines and $y_{r,p,t,\eta}$ reducible conics whose singular point is contained in H. To get $h^1(\mathcal{I}_{Y\cup G}(t)) = 0$ we need to control the union of $Y \cap H$, $W \cap H \setminus S$ and $Z_P \cap H$, $P \in S$. To control $W \cap H \setminus S$ we only need to control the postulation of a general union of $y_{r,p,t,\eta}$

tangent vectors on H. The scheme $Z_P \cap H$ is a degree 3 scheme supported by P and spanning the plane $V_P \cap H$ (it is called a triple point in [15]), since the sum of the numbers of tangent vectors and triple points is small, this is easy (e.g., [15, Assertion $H''_{n,N}$ in §3]); this sum is so small that we could take 2-points of H instead of tangent vectors and triple points in the sense of [15] and still get an h^1-vanishing using the Alexander-Hirschowitz theorem.

From B'_t we easily get Theorem 1.1.

To start the induction we need to do the case $r = 4$ only using Lemma 4.8 with p' as the Hilbert polynomial. Take ψ as in the statement of Lemma 4.8 with respect to the admissible polynomial p'. We have $c_{4,p,t} - c_{4,p,t-1} \sim c_{4,0,t} - c_{4,p,t-1} \sim t^2/8 \gg (c_{3,0,\psi} + 4)(t+1)$ Hence we may use Lemma 4.8 instead of the case $r = 3$ of Theorem 1.1 (which we do not claim, although we believe that it is true). $\square$

Proof of Theorem 1.3 in $\mathbb{P}^r$, $r \geq 4$: We use Lemma 3.3 and Remark 3.4. Fix p, $r \geq 4$, Y and an integer $g > 0$. Let y_0 be an integer such that Theorem 1.2 is true for all integers $t \geq y_0$. We also assume $y_0 > 2g + r$ and $a_{r,p,g,t} \geq 2gr + r^2$. For all $y \geq y_0$ set $g(y) = \min\{g, y - y_0\}$. For all integer $t \geq y_0$ define the integers a_{r,p,g,t,y_0} and b_{r,p,g,t,y_0} by the relations

$$p(t) + ta_{r,p,g,t,y_0} + 1 - g(t) + b_{r,b,g,t,y_0} = \binom{r+t}{r}, \ 0 \leq b_{r,p,g,t,y_0} \leq t-1$$

Since $y_0 > g \geq g(t)$ if $b_{r,p,t} \leq t - g(y) - 1$, then $a_{r,p,g,t} = a_{r,p,t}$ and $b_{r,p,g,t} = b_{r,p,t} + g(t)$, while $a_{r,p,g,t} = a_{r,p,t} + 1$ and $b_{r,p,g,t} = b_{r,p,t} + g(t) - t$ if $b_{r,p,t} \geq t - g(t)$. Let $\Psi(t)$ denote the following assertion:

$\Psi(t)$, $t \geq y_0$: Let $W \subset \mathbb{P}^r$ be a general union of a general curve of genus $g(t)$ and degree $a_{r,p,g,t} - b_{r,p,g,t}$ and $b_{r,p,g,t}$ lines. Then $h^1(\mathcal{I}_{Y\cup W}(t)) = 0$ (equivalently, $h^0(\mathcal{I}_{Y\cup W}(t)) = 0$.

(a) Fix an integer $t \geq y_0$. Let $W \subset \mathbb{P}^r$ be a general curve of genus $g(t)$ and degree $a_{r,p,g,t}$. Let N_W be the normal bundle of W in $\mathbb{P}^r$. Since $a_{r,p,g,t} \geq 2gr + r^2$ and W is a general element of $Z(r, a_{r,p,g,t}, g(t)$, it is easy to check that $h^1(W, N_W(-1)) = 0$ ([8, part (2) of Théorème 5]). Hence for general W the set $W \cap H$ is a general subset of H with cardinality $a_{r,p,g,t}$ (Remark 2.3).

(b) $\Psi(y_0)$ is true, because $g(y_0) = 0$, y_0 is large (say $y_0 \geq \eta$ with η as in the proof of Theorem 1.2) and hence it is true the assertion B_{y_0} introduced during the proof of Theorem 1.2 is true.

Claim 1: $\Psi(t-1) \Longrightarrow \Psi(t)$ for all $t > y_0$

Proof of Claim 1: First assume $y_0 \leq t \leq y_0 + g$. Let $W \subset \mathbb{P}^r$ be a general union of a general curve W_1 of genus $g(t-1)$ and degree $a_{r,p,g,t-1} - b_{r,p,g,t-1}$ and $b_{r,p,g,t-1}$ lines. Since $\Psi(t-1)$ is true, we have $h^i(\mathcal{I}_{Y\cup W}(t-1)) = 0$, $i = 0, 1$. We have $a_{r,p,g,t} - a_{r,p,g,t-1} \geq (r+2)t$ for large y_0. By Lemmas 5.2 and 5.3 and the semicontinuity theorem for cohomology we have $h^1(H, \mathcal{I}_{Y\cap H\cup A'\cup B}(t)) = 0$, where A' is a general elliptic curve of degree $a_{r,p,g,t} - a_{r,p,g,t-1} - b_{r,p,g,t}$ and B is a general union of $b_{r,p,g,t}$ lines (A' is a flat limit of a family $A \cup D$ with A rational and D 2-secant line of A). By Remark 2.3 and [8, part (2) of Théorème 5 for the genus 1]) A' contains $b_{r,p,g,t-1} + 1$ general points of H. Hence we may assume that A' contains exactly one point of $W_1 \cap H$ and all points of $(W \setminus W_1) \cap H$. Since $W \cap (H \setminus A')$ is general in H by part (a), we get $h^i(H, \mathcal{I}_{Y\cap H\cup (W\cap H)\cup A'\cup B}(t)) = 0$,

$i = 0, 1$, by (5.2). A Castelnuovo's sequence gives $h^i(\mathcal{I}_{Y \cup W \cup A' \cup B}(t)) = 0$, $i = 0, 1$. Since $W \cup A' \cup B$ is a flat limit of a family of disjoint unions of $b_{r,p,g,t}$ lines and a smooth curve of genus $g(t)$ and degree $a_{r,p,g,t} - b_{r,p,g,t}$, $\Psi(t)$ is true in this case. Now assume $t > y_0$. In this range we only add $b_{r,p,g,t}$ lines and a rational curve of degree $a_{r,p,g,t} - a_{r,p,g,t-1} - b_{r,p,g,t}$ containing one point of $W \cap H$.

To conclude the proof of Theorem 1.2 it is sufficient to use $\Psi(t)$ instead of B_t as in the proof of Theorem 1.2. □

Acknowledgement. The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

1. T. Aladpoosh and E. Ballico, *Postulation of a general union of an m-point and a general smooth rational curve*, Rend. Semin. Mat. Univ. Politec. Torino (to appear).
2. E. Ballico, *Postulation of general unions of lines and decorated lines*, preprint.
3. E. Ballico, *Postulation of a union $X \subset \mathbb{P}^r$, $r \ge 4$, of a given zero-dimensional scheme and several general lines*, ISRN Geometry Volume 2013, Article ID 391721, 6 pages http://dx.doi.org/10.1155/2013/391721
4. E. Ballico, *Postulation of disjoint unions of lines and a multiple point*, Mediterr. J. Math. (to appear).
5. E. Ballico and Ph. Ellia, *The maximal rank conjecture for nonspecial curves in $\mathbf{P}^3$*, Invent. Math. **79** (1985), no. 3, 541–55.
6. E. Ballico and Ph. Ellia, *On the hypersurfaces containing a general projective curve*, Compositio Math. **60** (1986), no. 1, 85–95.
7. E. Ballico and Ph. Ellia, *The maximal rank conjecture for non-special curves in $\mathbb{P}^n$*, Math. Z. **196** (1987), 355–367.
8. E. Ballico and Ph. Ellia, *Bonnes petites composantes des schmas de Hilbert de courbes lisses de $\mathbf{P^n}$*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), no. 4, 187–190.
9. A. Bernardi, M. V. Catalisano, A. Gimigliano, and M. Idà, *Secant varieties to osculating varieties of Veronese embeddings of $\mathbb{P}^n$*, J. Algebra **321** (2009), no. 3, 982–1004.
10. E. Carlini, M. V. Catalisano, and A. V. Geramita, *3-dimensional sundials*, Cent. Eur. J. Math. **9** (2011), no. 5, 949–971.
11. C. Ciliberto and R. Miranda, *Interpolations on curvilinear schemes*, J. Algebra **203** (1998), no. 2, 677–678.
12. F. Cioffi, P. Lella, M. G. Marinari and M. Roggero, *Segments of the Hilbert schemes of points*, Discrete Math. **311**, no. 20 (2011), 2238–2252, doi 10.1016/j.disc.2011.07.011 .
13. G. Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. **158** (1978), 61–70.
14. R. Hartshorne, Algebraic Geometry, Springer-Verlag, Berlin–Heidelberg–New York, 1977.
15. R. Hartshorne and A. Hirschowitz, *Droites en position générale dans $\mathbb{P}^n$*, Algebraic Geometry, Proceedings, La Rábida 1981, 169–188, Lect. Notes in Math. 961, Springer, Berlin, 1982.
16. A. Hirschowitz, *Sur la postulation générique des courbes rationnelles*, Acta Math. **146** (1981), 209–230.
17. A. Iarrobino and V. Kanev, Power sums, Gorenstein algebras, and determinantal loci, Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, Berlin, 1999, Appendix C by Iarrobino and Steven L. Kleiman.
18. D. Mumford, Lectures on curves on an algebraic surface. With an appendix by G. Bergamm, Annals of Mathematics Studies, No. 59 Princeton University Press, Princeton, N.J. 1966.
19. D. Perrin, *Courbes passant par m points g'enéraux de $\mathbb{P}^3$*, Bull. Soc. Math. France, Mémoire **28/29** (1987).

20. Z. Ran, *Normal bundles of rational curves in projective spaces*, Asian J. Math. **11** (2007), no. 4, 567–608.
21. J. Rathmann, *The uniform position principle for curves in characteristic p*, Math. Ann. **276** (1987), no. 4, 565–579.
22. G. Sacchiero, *Normal bundles of rational curves in projective space*, Ann. Univ. Ferrara Sez. VII (N. S.) **26** (1980), 33–40.
23. E. Sernesi, *On the existence of certain families of curves*, Invent. Math. **75** (1984), no. 1, 25–57.

[1] Dept. of Mathematics, University of Trento, 38123 Povo (TN), Italy
E-mail address: ballico@science.unitn.it