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ANALYSIS OF A FRACTIONAL-ORDER MATHEMATICAL MODEL FOR OCCUPATIONAL TRANSITION FROM FARMING TO OTHER SECTORS

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Abstract. In this article, we developed a fractional-order mathematical model to study the dynamics of farmers transitioning to other professions, incorporating memory effects through Caputo-type fractional derivatives. We ensured the model's biological viability by proving non-negativity and boundedness of solutions, and demonstrated existence and uniqueness using fixed-point theory. The model has non-negative equilibrium points, with local and global stability conditions analyzed. The basic reproduction number $\mathfrak{R}_\mathfrak{o}$ was determined, showing that when $\mathfrak{R}_\mathfrak{o} < 1$, the equilibrium point E_1 is locally asymptotically stable. The stability of the endemic equilibrium E_* was assessed using the Routh-Hurwitz criterion. The global stability of the equilibrium points is also analyzed using the Lyapunov function. Sensitivity analysis identified optimal intervention strategies, and numerical simulations confirmed the theoretical findings.

1. Introduction

The agricultural sector has long been a cornerstone of economies around the world, particularly in rural areas where farming is often the primary occupation. However, in recent decades, significant shifts have been observed as individuals increasingly transition from farming to other sectors. This occupational transition is driven by various factors, including economic diversification, technological advancements, urbanization, and environmental changes, all of which have significantly altered the economic landscape. From 1991 to 2019, the employability rate in agriculture decreased from 63% to 43% [6].

Young people's migration to urban areas is one of the key reasons to leave farming. There are numerous opportunities for young people to pursue careers outside of farming. Only a small number of young people living in rural areas are interested in working on a farm. This is because young people seek better working conditions, and agriculture is often perceived as a physically demanding occupation. The rise of industrialization has also contributed to the migration

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of people away from rural areas worldwide. People go to cities in search of more appealing professions and careers because they view agricultural labour as "dirty" [17, 30, 31]. A large number of young people are unwilling to continue working in agriculture because of the uncertainty that comes with climate change, which demotivates the younger generation [7, 20].

Mathematical modeling is a tool that transforms real-world problems into mathematical representations, often utilizing systems of differential equations, and solves them using mathematical methods. It offers valuable insights into complex systems. As technology advances, mathematical modeling will become increasingly essential in developing effective solutions.

Nowadays, fractional differential equations (FDEs) are widely applied in many scientific and engineering fields to provide novel and more effective results. Because fractional derivatives possess a non-local property, they are essential for analyzing the memory effects in complex physical scenarios and deriving more accurate solutions. Fractional-order systems provide higher accuracy and align more closely with real-world data compared to conventional integer-order systems [5, 24]. Over the past few decades, various fractional operators with distinct kernels have been developed, including the Caputo operator with a singular kernel [27], the Caputo-Fabrizio operator [3], and the Atangana-Baleanu operator [1] with a non-singular kernel, as well as the fractal-fractional operator [2]. Because of their various practical applications, these derivatives have been extensively utilized to model real-world challenges in a variety of scientific, economic, biological, and engineering fields [13, 28, 35, 14, 22]. Among these operators, the Caputo derivative is widely recognized and frequently employed to simulate the memory of the system [23, 9].

Building on the insights gained from previous research and studies on fractional operators, we apply the Caputo fractional derivative to the susceptible-infectious-recovered-susceptible (SIRS) model to explore the dynamics of farmers transitioning from agriculture to other professions. These transitions are influenced by memory, which encompasses the farmers' knowledge and experiences. Positive farming experiences promote continued agricultural practices, while challenges like crop failure or market instability may drive shifts to alternative sectors. Farmers with diverse experiences and transferable skills (management, marketing, technical) adapt more easily to new industries. Government policies can impact this memory effect by either supporting skill development in non-agricultural sectors or reinforcing agricultural practices, thus influencing farmers' willingness to transition. Recognizing the influence of past experiences, we have employed a fractional-order approach in our study to accurately capture the effects of memory, allowing for a nuanced understanding of how historical factors impact current and future agricultural outcomes. This paper aims to understand the reasons behind farmers in India transitioning from agriculture to other sectors by developing a mathematical model that simulates real-world scenarios. It seeks to identify key factors driving these changes and provide a framework for predicting future transitions and their impacts on rural communities and the broader economy. This approach offers both theoretical insights and practical tools for designing effective policies and interventions to address this critical issue. To the best of our knowledge, no mathematical model has been made for addressing this critical social issue. Therefore, this paper provides valuable insights for researchers and policymakers by offering a novel mathematical perspective for analyzing the dynamics of farmer transitions, allowing for more precise modeling of these complex processes. Additionally, it helps in designing targeted policies that can better address the socioeconomic challenges faced by rural communities and promote sustainable development in the agricultural sector.

The structure of this paper is as follows: Section 2 deals with the preliminaries of fractional calculus. In Section 3, we explain the formulation of the model. Section 4 illustrates the positivity and boundedness of solutions of the proposed model. In section 5, we prove the existence and uniqueness of solutions. The model's equilibrium points are found in Section 6. Sections 7 and 8, respectively, examine the equilibrium points' local and global stability. Section 9 illustrates the numerical sensitivity analysis of the model. A numerical simulation is carried out in section 10. Section 11 concludes the work.

2. Preliminaries

Definition 2.1. [26] For any function $f \in \mathcal{C}^n([0,\infty),\mathbb{R})$, the Caputo derivative of order θ can be defined as

$$
{}_0^C D_t^\theta f(t) = \begin{cases} \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} f^{(n)}(s)\, ds, & \text{for } n-1<\theta<n, n\in\mathbb{N} \\ \frac{d^n}{dt^n} f(t), & \text{for } \theta = n. \end{cases}
$$

where $\Gamma(.)$ is the Gamma function, $t \geq 0$.

Lemma 2.2. [11] *Suppose $\delta, \epsilon \in \mathbb{C}, f(t), g(t)$ are differentiable functions and $n-1<\theta\leq n, n\in\mathbb{N}$. Then*

$$
{}_0^C D_t^\theta(\delta f(t) + \epsilon g(t)) = \delta\, {}_0^C D_t^\theta f(t) + \epsilon\, {}_0^C D_t^\theta g(t)
$$

Theorem 2.3. [25, 11] *Let $f(t)$ be a function that is n times continuously differentiable, and assume that ${}_0^C D_t^\theta f(t)$ is piece-wise continuous on the interval $[0,\infty)$ where $n-1<\theta\leq n, n\in\mathbb{N}$, then its Laplace transform is*

$$
\mathcal{L}\{{}_0^C D_t^\theta f(t), s\} = s^\theta F(s) - \sum_{i=0}^{n-1} s^{\theta-i-1} f^{(i)}(0),
$$

Where $F(s) = \mathcal{L}\{f(t)\}$.

Lemma 2.4. [15, 18] *The Laplace Transformation of the function $t^{m-1}E_{l,m}(\pm\lambda t^l)$ is given by*

$$
\mathcal{L}\{t^{m-1}E_{l,m}(\pm\lambda t^l)\} = \frac{s^{l-m}}{s^l \mp \lambda}, \qquad s>0,
$$

where $E_{l,m}$ denote the Mittag-Leffler function.

Theorem 2.5. [26] *Consider the following fractional-order dynamical model:*

$$
{}_{t_0}^C D_t^\theta G(t) = \Phi(G(t)), G(t=t_0) = (g_{t_0}^1, g_{t_0}^2, ..., g_{t_0}^n), g_{t_0}^i > 0, i = 1,2,...,n
$$

with $0<\theta\leq 1$, $G(t) = (g^1(t), g^2(t), ..., g^n(t))$ and $\Phi(G) : [t_0,\infty) \to \mathbb{R}^{n\times n}$. The following equation may be used to find the equilibrium points of the model

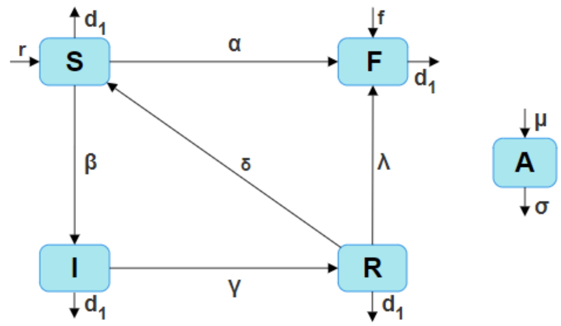


FIGURE 1. The flow diagram of the model

shown above: $\Phi(G) = 0$. *If all of the eigenvalues* λ *of the Jacobian matrix* $J(G) = \frac{\partial(\Phi_1,\Phi_2,\ldots,\Phi_n)}{\partial(g^1,g^2,\ldots,g^n)}$ *computed at the equilibrium points satisfy* $|arg(\lambda_i)| > \frac{\theta\pi}{2}$, *then each equilibrium point will be locally asymptotically stable. If not, it will be unstable.*

3. Model Formulation

In the considered model, we divided the total population into four sub-groups representing infected status. Occasional or susceptible farmers (denoted by S) are those who are small-scale farmers but susceptible to changing their occupation from farming to another. Permanent farmers (denoted by F) are those who have committed to farming as their long-term primary occupation. Infected farmers (denoted by I) are those who changed their occupation from farming to another profession. The recovered class (denoted by R) are those farmers who come back to farming from other professions, and A denotes the cumulative density of aids (e.g., government policies and funds, awareness programs, etc.) to motivate the people towards farming. The flow diagram of the proposed model is shown in Figure 1. Thus, the governing nonlinear system of fractional order differential equations of the proposed model is

$$\begin{aligned}
{}_0^C D_t^\theta S(t) &= rS(1-\frac{S}{K}) - \alpha AS - \beta SI + \delta R - d_1 S,\\
{}_0^C D_t^\theta F(t) &= f + \alpha AS + \lambda RF - d_1 F,\\
{}_0^C D_t^\theta I(t) &= \beta SI - \gamma I - d_1 I,\\
{}_0^C D_t^\theta R(t) &= \gamma I - \delta R - \lambda RF - d_1 R,\\
{}_0^C D_t^\theta A(t) &= \mu\beta SI - \sigma A,
\end{aligned} \tag{3.1}$$

Where θ represents the fractional order, $0 < \theta \leq 1$ and initial conditions:

$$S(0) \geq 0,\ F(0) \geq 0,\ I(0) \geq 0,\ R(0) \geq 0,\ A(0) \geq 0$$

All the parameters are assumed to be positive. r denotes the birth rate of the human population. K be the carrying capacity. α be the rate of transmission from susceptible farmers (S) to permanent farmers (F) due to being influenced by multiple motivating factors, including economic incentives, access to resources, education, technological advancements, government policies, and support programs. β denotes the infection rate of farmers, i.e., the rate at which farmers change their occupation from farming to another profession. λ denotes the rate at which the recovered farmers become permanent farmers. The constant f denotes the people who never give up farming; they are basically large-scale farmers from generation to generation. The parameter d_1 represents the natural death rate of the population. γ be the recovery rate of farmers, i.e., the rate at which infected farmers come back to the farming profession. δ represents the rate at which the recovered farmers become susceptible. μ denotes the growth rate of cumulative density of aids (e.g., government policies and funds, awareness programs, etc.), which is assumed to be proportional to the number of people who are affected by other professions. σ represents the dissemination rate of aid provided to the susceptible farmers.

Lemma 3.1. *(Lemma 5, [29]) Let $0 < \theta \leq 1$, $\lambda < 0$ and $|arg(\lambda)| > \frac{\pi\theta}{2}$. Then $E_{\theta,\theta}(\lambda t^{\theta})$ and $E_{\theta,\theta+1}(\lambda t^{\theta})$ tend monotonically to zero as $t \to \infty$.*

Lemma 3.2. *(corollary 6, [29]) Let $0 < \theta \leq 1$ and $\lambda < 0$. Then $t^{\theta} E_{\theta,\theta+1}(\lambda t^{\theta}) = -\frac{1}{\lambda}$ as $t \to \infty$.*

4. Positivity and boundedness of solutions

Lemma 4.1. [25] *Assume that $g(x) \in C[h_1, h_2]$ and ${}_{t_0}^{C}D_t^{\theta} g(x) \in C[h_1, h_2]$ for $0 < \theta \leq 1$. If ${}_{t_0}^{C}D_t^{\theta} g(x) \geq 0$ for all $x \in (h_1, h_2)$, then $g(x)$ is non decreasing for each $x \in [h_1, h_2]$. If ${}_{t_0}^{C}D_t^{\theta} g(x) \leq 0$ for all $x \in (h_1, h_2)$, then $g(x)$ is non-increasing for each $x \in [h_1, h_2]$.*

Theorem 4.2. *All solutions of the model that start from $(S(t_0), F(t_0), I(t_0), R(t_0), A(t_0))$ remain non-negative.*

Proof. Suppose that $Y(t_0) = (S(t_0), F(t_0), I(t_0), R(t_0), A(t_0)) \in \Gamma^+ = \{(S, F, I, R, A) \in \Gamma^+ : S, F, I, R, A \in \mathbb{R}^+\}$ as the initial point of solution of the model (3.1).

Let us consider that there exists a constant τ, $t_0 \leq t < \tau$ such that

$$\begin{aligned} &I(t) > 0 \;\; t_0 \leq t < \tau \\ &I(\tau) = 0, \\ &I(\tau^+) < 0 \end{aligned}$$

From the third equation of the system, we have

$${}_{0}^{C}D_t^{\theta} I(t)|_{I(t)=0} = 0$$

It follows from Lemma that $I(t)$ is non-decreasing, i.e. $I(\tau^+) \geq 0$ which contradicts our assumption that $I(\tau^+) < 0$. Therefore, we get $I(t) \geq 0, \forall t \in [t_0, \infty)$. Similarly, we can show that $S(t) \geq 0, F(t) \geq 0, R(t) \geq 0$ and $A(t) \geq 0, \forall t \in [t_0, \infty)$. $\square$

Theorem 4.3. *The region given by* $\Omega \in \mathbb{R}^5_+$ *such that* $\Omega = \{(S, F, I, R, A) \in \mathbb{R}^5_+ : 0 < S + F + I + R \leq \frac{rK+4f}{4d_1}, 0 < A \leq \mu\beta\frac{(rK+4f)^2}{16d_1^2\sigma}\}$ *is positively invariant for the system (3.1) with the initial conditions in* $\mathbb{R}^5_+$.

Proof. Let $N(t) = S(t) + F(t) + I(t) + R(t)$. Then using the Caputo derivative on both sides we get

$$ {}_0^C D_t^\theta N(t) = {}_0^C D_t^\theta S(t) + {}_0^C D_t^\theta F(t) + {}_0^C D_t^\theta I(t) + {}_0^C D_t^\theta R(t), $$

Then from the system (3.1), we get

$$ {}_0^C D_t^\theta N(t) = rS\left(1 - \frac{S}{K}\right) + f - d_1 N $$
$$ {}_0^C D_t^\theta N(t) + d_1 N(t) \leq \frac{rK}{4} + f, $$

where maximum of the quadratic function $rS\left(1 - \frac{S}{K}\right)$ in S is $\frac{rK}{4}$.
Applying the Laplace transform to the above equation, we have

$$ \mathcal{L}\left[{}_0^C D_t^\theta N(t) + d_1 N(t)\right] \leq \mathcal{L}[\frac{rK}{4} + f], $$
$$ s^\theta \mathcal{N}(s) - s^{\theta-1}N(0) + d_1\mathcal{N}(s) \leq \frac{1}{s}\left(\frac{rK+4f}{4}\right), \text{ where } \mathcal{N}(s) = \mathcal{L}[N(t)], $$
$$ \mathcal{N}(s) \leq N(0)\frac{s^{\theta-1}}{s^\theta + d_1} + \frac{rK+4f}{4s(s^\theta + d_1)}, $$

Using the Laplace inverse transform, we have

$$ N(t) \leq N(0)E_{\theta,1}(-d_1 t^\theta) + \frac{rK+4f}{4}t^\theta E_{\theta,\theta+1}(-d_1 t^\theta), $$

Using the Lemma (3.1) and (3.2), we get

$$ N(t) \leq 0 + \frac{rK+4f}{4}(-\frac{1}{(-d_1)}), $$
$$ \text{i.e. } S(t) + F(t) + I(t) + R(t) \leq \frac{rK+4f}{4d_1}, $$

Again, from last equation of the system we get

$$ {}_0^C D_t^\theta A(t) = \mu\beta SI - \sigma A, $$
$$ \leq \mu\beta\frac{rK+4f}{4d_1}\frac{rK+4f}{4d_1} - \sigma A, \because \ 0 \leq S, I \leq \frac{rK+4f}{4d_1}, $$
$$ {}_0^C D_t^\theta A(t) + \sigma A \leq \mu\beta\frac{(rK+4f)^2}{16d_1^2}, $$

Using Laplace Tramsform, we get

$$\mathcal{L}\left[{}_{0}^{C}D_{t}^{\theta}A(t)+\sigma A\right]\leq\mathcal{L}\left[\mu\beta\frac{(rK+4f)^2}{16d_1^2}\right],$$
$$s^{\theta}\mathcal{A}(s)-s^{\theta-1}A(0)+\sigma\mathcal{A}(s)\leq\mu\beta\frac{(rK+4f)^2}{16d_1^2}\frac{1}{s},\ \text{where}\ \mathcal{A}(s)=\mathcal{L}[A(t)],$$
$$\mathcal{A}(s)\leq A(0)\frac{s^{\theta-1}}{s^{\theta}+\sigma}+\frac{\mu\beta(rK+4f)^2}{16d_1^2 s(s^{\theta}+\sigma)},$$

Taking Laplace inverse transform on both sides, we get

$$A(t)\leq A(0)E_{\theta,1}(-\sigma t^{\theta})+\mu\beta\frac{(rK+4f)^2}{16d_1^2}t^{\theta}E_{\theta,\theta+1}(-\sigma t^{\theta}),$$

Applying lemma (3.1) and (3.2), we get

$$\begin{aligned}A(t)&\leq 0+\mu\beta\frac{(rK+4f)^2}{16d_1^2}\left(-\frac{1}{(-\sigma)}\right),\\&\leq\mu\beta\frac{(rK+4f)^2}{16d_1^2\sigma}.\end{aligned}$$

Therefore, for all solutions of model (3.1) with starting conditions in Ω, $\forall t>0$ remains in Ω. Hence, the region Ω is positively invariant. ☐

5. Existence and Uniqueness

Lemma 5.1. [16] *Consider the fractional-order system defined as follows:*

$$ {}_{t_0}^{C}D_{t}^{\theta}x(t)=f(t,x),t_0>0\ and\ x(t_0)=x_0 \tag{5.1}$$

where $\theta\in(0,1]$, $f:[t_0,\infty)\times\mathcal{R}\to\mathbb{R}^n,\mathcal{R}\subset\mathbb{R}^n$. If the Lipschitz condition with respect to x is satisfied by the function $f(t,x)$, then the system described in (5.1) possesses a unique solution on the interval $[t_0,\infty)\times\mathcal{R}$.

To discuss the existence and uniqueness of the system (3.1), consider the region $\Pi\times(0,T)$, $T<\infty$, where $\Pi=\{(S,F,I,R,A)\in\mathbb{R}^5:max(|S|,|F|,|I|,|R|,|A|)\leq\mathcal{M}\}$. The system (3.1) can be rewritten as:

$$\begin{aligned}{}_{0}^{C}D_{t}^{\theta}S(t)&=X_1(t,S),\\{}_{0}^{C}D_{t}^{\theta}F(t)&=X_2(t,F),\\{}_{0}^{C}D_{t}^{\theta}I(t)&=X_3(t,I),\\{}_{0}^{C}D_{t}^{\theta}R(t)&=X_4(t,R),\\{}_{0}^{C}D_{t}^{\theta}A(t)&=X_5(t,A),\end{aligned}\tag{5.2}$$

Where

$$\begin{aligned}
X_1(t,S) &= rS(1-\frac{S}{K}) - \alpha AS - \beta SI + \delta R - d_1 S,\\
X_2(t,F) &= f + \alpha AS + \lambda RF - d_1 F,\\
X_3(t,I) &= \beta SI - \gamma I - d_1 I,\\
X_4(t,R) &= \gamma I - \delta R - \lambda RF - d_1 R,\\
X_5(t,A) &= \mu\beta SI - \sigma A,
\end{aligned}$$

Theorem 5.2. *The Lipschitz condition is satisfied by the kernels X_i, $i = 1, 2, ..., 5$.*

Proof. Let us start with Kernel X_1. For any two functions S and $\bar{S}$, we obtain

$$\begin{aligned}
&\|X_1(t,S) - X_1(t,\bar{S})\|\\
&= \Big\| rS(1-\frac{S}{K}) - \alpha AS - \beta SI + \delta R - d_1 S\\
&\quad - \Big(r\bar{S}(1-\frac{\bar{S}}{K}) - \alpha A\bar{S} - \beta\bar{S}I + \delta R - d_1\bar{S}\Big)\Big\|\\
&= \Big\| r(S-\bar{S}) - \frac{r}{K}(S^2-\bar{S}^2) - \alpha A(S-\bar{S}) - \beta I(S-\bar{S}) - d_1(S-\bar{S})\Big\|\\
&\le r\|S-\bar{S}\| + \frac{r}{K}\|S^2-\bar{S}^2\| + \alpha\|A\|\|S-\bar{S}\| + \beta\|I\|\|S-\bar{S}\| + d_1\|S-\bar{S}\|\\
&\le \Big(r + \frac{r}{K}\|S+\bar{S}\| + \alpha\|A\| + \beta\|I\| + d_1\Big)\|S-\bar{S}\|
\end{aligned}$$

Taking $\Lambda_1 = r + \frac{r}{K}2\mathcal{M} + \alpha\mathcal{M} + \beta\mathcal{M} + d_1$, where $max(|S|, |F|, |I|, |R|, |A| \le \mathcal{M})$. So we get

$$\|X_1(t,S) - X_1(t,\bar{S})\| \le \Lambda_1\|S(t) - \bar{S}(t)\|,$$

Similarly, we can obtain

$$\begin{aligned}
\|X_2(t,F) - X_2(t,\bar{F})\| &\le \Lambda_2\|F(t) - \bar{F}(t)\|,\\
\|X_3(t,I) - X_3(t,\bar{I})\| &\le \Lambda_3\|I(t) - \bar{I}(t)\|,\\
\|X_4(t,R) - X_4(t,\bar{R})\| &\le \Lambda_4\|R(t) - \bar{R}(t)\|,\\
\|X_5(t,A) - X_5(t,\bar{A})\| &\le \Lambda_5\|A(t) - \bar{A}(t)\|,
\end{aligned}$$

Where

$$\Lambda_2 = \lambda\mathcal{M} + d_1,\ \Lambda_3 = \beta\mathcal{M} + \gamma + d_1,\ \Lambda_4 = \delta + \lambda\mathcal{M} + d_1,\ \Lambda_5 = \sigma,$$

Thus, the Kernels X_i, $i = 1, 2, ..., 5$ satisfies Lipschitz criteria [16]. Further, if $0 \le \Lambda_i \le 1$, then X_i, $i = 1, 2, ..., 5$ is a contraction mapping. Hence, the system (3.1) has a unique solution. □

6. Equilibrium Points

To find the equilibrium points of the system (3.1), we have the following system of equations

$$\begin{aligned}
rS(1-\frac{S}{K})-\alpha AS-\beta SI+\delta R-d_1S&=0,\\
f+\alpha AS+\lambda RF-d_1F&=0,\\
\beta SI-\gamma I-d_1I&=0,\\
\gamma I-\delta R-\lambda RF-d_1R&=0,\\
\mu\beta SI-\sigma A&=0,
\end{aligned}$$

On solving, we get three non-negative equilibrium points, namely $E_0=(0,\frac{f}{d_1},0,0,0)$, $E_1=(K(1-\frac{d_1}{r}),\frac{f}{d_1},0,0,0)$ and the co-existing equilibrium point $E_*=(S^*,F^*,I^*,R^*,A^*)$, where

$$\begin{aligned}
S^*&=\frac{\gamma+d_1}{\beta},\\
F^*&=\frac{f+m_3I^*}{d_1-\lambda(m_1I^*+m_2)},\ \text{ provided }\ d_1\neq\lambda(m_1I^*+m_2),\\
R^*&=m_1I^*+m_2,\\
A^*&=\frac{\mu(\gamma+d_1)}{\sigma}I^*,
\end{aligned}$$

and I^* can be obtained from the quadratic equation

$$h_1I^2+h_2I+h_3=0, \tag{6.1}$$

where

$$\begin{aligned}
h_1&=\gamma\lambda m_1+\lambda m_1m_3+\lambda m_1^2d_1-\lambda\delta m_1^2,\\
h_2&=\gamma\lambda m_2-\gamma d_1+m_1\delta(d_1-\lambda m_2)+m_1\lambda f-d_1^2m_1+2\lambda d_1m_1m_2\\
&\quad+\lambda m_2m_3-\delta\lambda m_1m_2,\\
h_3&=m_2\{(\delta-d_1)(d_1-\lambda m_2)+\lambda f\},
\end{aligned}$$

with

$$\begin{aligned}
m_1&=\frac{\alpha\mu(\gamma+d_1)^2+\sigma\beta(\gamma+d_1)}{\delta\sigma\beta},\ m_2=\frac{r(\gamma+d_1-\beta K)(\gamma+d_1)}{\delta\beta^2K},\\
m_3&=\frac{\alpha\mu(\gamma+d_1)}{\sigma\beta},
\end{aligned}$$

Thus, the roots of the quadratic equation (6.1) are $I=\frac{-h_2\pm\sqrt{h_2^2-4h_1h_3}}{2h_1}$. Now, for real values of I, $h_2^2-4h_1h_3\geq 0$. Let $D=\sqrt{h_2^2-4h_1h_3}$. If $D>h_2$ and $h_1>0$ or $D<h_2$ and $h_1<0$, then $I_1=\frac{D-h_2}{2h_1}$ will be positive root. Again, $I_2=\frac{-D-h_2}{2h_1}$ will be positive root if $h_2>0,\ h_1<0$.

6.1. **The system's basic reproduction number.** The basic reproduction number, denoted as $\mathfrak{R}_0$, is defined as the estimated number of secondary infections that a single infected individual is expected to generate in a fully susceptible population during their infectious period. It can be calculated using the next-generation matrix approach [8].

Assume there are $x = (x_1, x_2, ..., x_n)^T$ individuals in each compartment, where the first $m < n$ compartments contain infected individuals [32]. In the absence of disease, we consider that the disease-free equilibrium (DFE) x_0 exists and is stable. The linearized equations for $x_1, ..., x_m$ around the DFE can be expressed as decoupled equations of the form: $\frac{dx_i}{dt} = F_i(x) - V_i(x)$ for $i = 1, 2, ..., m$. Here, the rate of new infections entering compartment i is denoted by $F_i(x)$, whereas the rate of other transitions from compartment i to other infected compartments is represented by $V_i(x)$. $F_i = 0$ if $i \in [m+1, n]$. Now define $\mathcal{F} = \frac{\partial F_i(x_0)}{\partial x_i}$ and $\mathcal{V} = \frac{\partial V_i(x_0)}{\partial x_i}$ for $1 \leq i, j \leq m$, where $\mathcal{F}$ is entry wise non-negative and $\mathcal{V}$ is a non-singular matrix. Thus $\mathcal{F}\mathcal{V}^{-1}$ is the next generation matrix and $\mathfrak{R}_\mathrm{o}$ is the maximum eigenvalue of the matrix $\mathcal{F}\mathcal{V}^{-1}$.

Now, for our considered model (3.1),

$$F = \begin{bmatrix} \beta SI \\ 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} (\gamma + d_1)I \\ (\delta + \lambda F + d_1)R - \gamma I \end{bmatrix},$$

The Jacobian matrices of F and V at the equilibrium point $E_1 = (K(1 - \frac{d_1}{r}), \frac{f}{d_1}, 0, 0, 0)$ are given by

$$\mathcal{F} = \begin{bmatrix} \beta K(1 - \frac{d_1}{r}) & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathcal{V} = \begin{bmatrix} \gamma + d_1 & 0 \\ -\gamma & \delta + \frac{\lambda f}{d_1} + d_1 \end{bmatrix},$$

$$\therefore W = \mathcal{F}\mathcal{V}^{-1} = \frac{1}{(\gamma + d_1)(\delta + \frac{\lambda f}{d_1} + d_1)} \begin{bmatrix} \beta K(1 - \frac{d_1}{r})(\delta + \frac{\lambda f}{d_1} + d_1) & 0 \\ 0 & 0 \end{bmatrix},$$

Hence, the basic reproduction number is the maximum eigenvalue of W, i.e.

$$\mathfrak{R}_\mathrm{o} = \rho(W) = \frac{\beta K(r - d_1)}{r(\gamma + d_1)}, \text{ provided } r > d_1.$$

7. Local Stability

This section analyzes the local stability of three equilibrium points, determined using the Jacobian matrix

$$J = \begin{pmatrix} r(1 - \frac{2S}{K}) - \alpha A - \beta I - d_1 & 0 & -\beta S & \delta & -\alpha S \\ \alpha A & \lambda R - d_1 & 0 & \lambda F & \alpha S \\ \beta I & 0 & \beta S - \gamma - d_1 & 0 & 0 \\ 0 & -\lambda R & \gamma & -\delta - \lambda F - d_1 & 0 \\ \mu\beta I & 0 & \mu\beta S & 0 & -\sigma \end{pmatrix}$$

Theorem 7.1. *The equilibrium point* $E_0 = (0, \frac{f}{d_1}, 0, 0, 0)$ *is always unstable.*

Proof. The Jacobian matrix of the system (3.1) at E_0 is

$$J(E_0) = \begin{pmatrix} r - d_1 & 0 & 0 & \delta & 0 \\ 0 & -d_1 & 0 & \frac{\lambda f}{d_1} & 0 \\ 0 & 0 & -\gamma - d_1 & 0 & 0 \\ 0 & 0 & \gamma & -\delta - \frac{\lambda f}{d_1} - d_1 & 0 \\ 0 & 0 & 0 & 0 & -\sigma \end{pmatrix}$$

The eigenvalues of the above jacobian matrix are

$$\lambda_1 = -d_1,\ \lambda_2 = -d_1 - \gamma,\ \lambda_3 = r - d_1,\ \lambda_4 = -\sigma,\ \lambda_5 = -\delta - \frac{\lambda f}{d_1} - d_1,$$

In general, the birth rate of human beings is greater than the natural death rate, i.e. $r > d_1$. So, $\lambda_3 = r - d_1 > 0$. Thus according to the theorem (2.5), the equilibrium point E_0 is unstable. $\square$

Theorem 7.2. *The equilibrium point* $E_1 = (K(1-\frac{d_1}{r}), \frac{f}{d_1}, 0, 0, 0)$ *is locally asymptotically stable if* $\mathfrak{R}_\circ < 1$.

Proof. The Jacobian matrix of the system (3.1) at E_1 is

$$J(E_1) = \begin{pmatrix} -(r-d_1) & 0 & -\beta K(1-\frac{d_1}{r}) & \delta & -\alpha K(1-\frac{d_1}{r}) \\ 0 & -d_1 & 0 & \frac{\lambda f}{d_1} & \alpha K(1-\frac{d_1}{r}) \\ 0 & 0 & \beta K(1-\frac{d_1}{r}) - \gamma - d_1 & 0 & 0 \\ 0 & 0 & \gamma & -\delta - \frac{\lambda f}{d_1} - d_1 & 0 \\ 0 & 0 & \mu\beta K(1-\frac{d_1}{r}) & 0 & -\sigma \end{pmatrix}$$

The eigenvalues of the above jacobian matrix are

$$\lambda_1 = -(r-d_1),\ \lambda_2 = -d_1,\ \lambda_3 = \beta K(1-\frac{d_1}{r}) - \gamma - d_1,\ \lambda_4 = -(\delta + \frac{\lambda f}{d_1} + d_1),$$
$$\lambda_5 = -\sigma,$$

Here $\lambda_1, \lambda_2, \lambda_4$ and λ_5 are negative, but $\lambda_3 = \beta K(1-\frac{d_1}{r}) - \gamma - d_1 = (\gamma + d_1)(\frac{\beta K(r-d_1)}{r(\gamma+d_1)} - 1) = (\gamma+d_1)(\mathfrak{R}_\circ - 1)$ will be negative if $\mathfrak{R}_\circ < 1$. So, $|arg(\lambda_i)| > \frac{\theta\pi}{2}$ for all $i = 1, 2, ..., 5$. Hence by the theorem (2.5), E_1 is locally asymptotically stable if $\mathfrak{R}_\circ < 1$. Further, if $\mathfrak{R}_\circ > 1$ then one eigenvalue will be positive. Therefore, it is unstable. $\square$

Theorem 7.3. *The co-existing equilibrium point* $E_* = (S^*, F^*, I^*, R^*, A^*)$ *is locally asymptotically stable if the conditions specified in (7.2) are satisfied.*

Proof. The Jacobian matrix of the system (3.1) at the equilibrium E_* is

$$J(E_*) = \begin{pmatrix} m_{11} & 0 & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & 0 & m_{24} & m_{25} \\ m_{31} & 0 & m_{33} & 0 & 0 \\ 0 & m_{42} & m_{43} & m_{44} & 0 \\ m_{51} & 0 & m_{53} & 0 & m_{55} \end{pmatrix}$$

Where

$$m_{11} = r(1 - \frac{2S^*}{K}) - \alpha A^* - \beta I^* - d_1,\ m_{13} = -\beta S^*,\ m_{14} = \delta,\ m_{15} = -\alpha S^*,$$
$$m_{21} = \alpha A^*,\ m_{22} = \lambda R^* - d_1,\ m_{24} = \lambda F^*,\ m_{25} = \alpha S^*,$$
$$m_{31} = \beta I^*,\ m_{33} = \beta S^* - \gamma - d_1,$$
$$m_{42} = -\lambda R^*,\ m_{43} = \gamma,\ m_{44} = -\delta - \lambda F^* - d_1,$$
$$m_{51} = \mu\beta I^*,\ m_{53} = \mu\beta S^*,\ m_{55} = -\sigma,$$

So, the characteristic equation of the Jacobian matrix $J(E_*)$ is given by

$$x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0, \tag{7.1}$$

Here a_1, a_2, a_3, a_4 and a_5 are the coefficients that can be derived from the matrix $J(E_*)$. Thus, the equilibrium point E_* is locally asymptotically stable if the following Routh-Hurwitz criteria are satisfied:

$$\begin{aligned}
&\Delta_1 = a_1 > 0,\\
&\Delta_2 = a_1a_2 - a_3 > 0,\\
&\Delta_3 = a_1a_2a_3 - a_3^2 - a_1^2a_4 + a_1a_5 > 0,\\
&\Delta_4 = a_1a_2a_3a_4 - a_3^2a_4 - a_1^2a_4^2 - a_1a_2^2a_5 + a_2a_3a_5 + 2a_1a_4a_5 - a_5^2 > 0,\\
&\Delta_5 = a_5(a_1a_2a_3a_4 - a_3^2a_4 - a_1^2a_4^2 - a_1a_2^2a_5 + a_2a_3a_5 + 2a_1a_4a_5 - a_5^2) > 0.
\end{aligned} \tag{7.2}$$

□

8. Global Stability

In this section, we will study the global stability of the equilibrium points E_1 and E_* which are determined earlier.

Theorem 8.1. *The equilibrium point E_1 is globally asymptotically stable if $\mathfrak{R}_o < 1$.*

Proof. Let us consider the Lyapunav function as $L_1(t) = \nabla_1 I + \nabla_2 R$. Applying the Caputo fractional derivative on both sides, we get

$$\begin{aligned}
{}_0^C D_t^\theta\, L_1(t) &= \nabla_1\, {}_0^C D_t^\theta\, I(t) + \nabla_2\, {}_0^C D_t^\theta\, R(t),\\
&= \nabla_1(\beta SI - \gamma I - d_1 I) + \nabla_2(\gamma I - \delta R - \lambda RF - d_1 R),\\
&= [\nabla_1\{\beta K(1 - \frac{d_1}{r}) - \gamma - d_1\} + \nabla_2\gamma]I - \{\nabla_2(\delta + \frac{\lambda f}{d_1} + d_1)\}R,\\
&= [\nabla_1(\gamma + d_1)(\mathfrak{R}_o - 1) + \nabla_2\gamma]I - \{\nabla_2(\delta + \frac{\lambda f}{d_1} + d_1)\}R,\\
&= [\nabla_2\gamma - \nabla_1(\gamma + d_1)(1 - \mathfrak{R}_o)]I - \{\nabla_2(\delta + \frac{\lambda f}{d_1} + d_1)\}R,
\end{aligned}$$

Taking $\nabla_1 = \gamma$ and $\nabla_2 = (\gamma + d_1)(1 - \mathfrak{R}_o)$, we obtain

$$\begin{aligned}
{}_0^C D_t^\theta\, L_1(t) &= -(\gamma + d_1)(1 - \mathfrak{R}_o)(\delta + \frac{\lambda f}{d_1} + d_1)R,\\
&< 0 \text{ if } \mathfrak{R}_o < 1.
\end{aligned}$$

As all the parameters of the system are assumed to be positive, so ${}_0^C D_t^\theta\, L_1(t) < 0$ if $\mathfrak{R}_o < 1$ and ${}_0^C D_t^\theta\, L_1(t) = 0$ if $I = R = 0$. Hence, E_1 is globally asymptotically stable if $\mathfrak{R}_o < 1$. □

Lemma 8.2. *(Lemma 3.1, [33]). Let $x(t) \in \mathbb{R}^+$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$*

$${}_{t_0}^C D_t^\theta \left[x(t) - x^* - x^*\, ln\frac{x(t)}{x^*}\right] \leq \left(1 - \frac{x^*}{x(t)}\right) {}_{t_0}^C D_t^\theta x(t),\ x^* \in \mathbb{R}^+,\ \forall\theta \in (0,1).$$

Theorem 8.3. *The co-existing equilibrium point $E_* = (S^*, F^*, I^*, R^*, A^*)$ is globally asymptotically stable if the following conditions hold:*

$$1 < \frac{F^*}{F}; 1 < \frac{A}{A^*}; 1 < \frac{R}{R^*} < \frac{S}{S^*}; \frac{I}{I^*} < \frac{R}{R^*}; \frac{SI}{S^*I^*} < \frac{A}{A^*}; \lambda R^* < d_1.$$

Proof. Consider the Lyapunov function as

$$\begin{aligned} L_2(t) = &\left[S(t) - S^* - S^* \, ln\left(\frac{S(t)}{S^*}\right)\right] + \left[F(t) - F^* - F^* \, ln\left(\frac{F(t)}{F^*}\right)\right] \\ &+ \left[I(t) - I^* - I^* \, ln\left(\frac{I(t)}{I^*}\right)\right] + \left[R(t) - R^* - R^* \, ln\left(\frac{R(t)}{R^*}\right)\right] \\ &+ \left[A(t) - A^* - A^* \, ln\left(\frac{A(t)}{A^*}\right)\right], \end{aligned}$$

Using the lemma 8.2, we get

$$\begin{aligned} {}_0^C D_t^\theta \, L_2(t) \leq &\left(1 - \frac{S^*}{S(t)}\right) {}_0^C D_t^\theta \, S(t) + \left(1 - \frac{F^*}{F(t)}\right) {}_0^C D_t^\theta \, F(t) \\ &+ \left(1 - \frac{I^*}{I(t)}\right) {}_0^C D_t^\theta \, I(t) + \left(1 - \frac{R^*}{R(t)}\right) {}_0^C D_t^\theta \, R(t) \\ &+ \left(1 - \frac{A^*}{A(t)}\right) {}_0^C D_t^\theta \, A(t), \\ = &\left(1 - \frac{S^*}{S}\right) \left[rS(1 - \frac{S}{K}) - \alpha AS - \beta SI + \delta R - d_1 S\right] \\ &+ \left(1 - \frac{F^*}{F}\right) [f + \alpha AS + \lambda RF - d_1 F] \\ &+ \left(1 - \frac{I^*}{I}\right) [\beta SI - \gamma I - d_1 I] + \left(1 - \frac{R^*}{R}\right) [\gamma I - \delta R - \lambda RF - d_1 R] \\ &+ \left(1 - \frac{A^*}{A}\right) [\mu\beta SI - \sigma A], \\ = &(S - S^*) \left[r - \frac{rS}{K} - \alpha A - \beta I + \delta \frac{R}{S} - d_1\right] \\ &+ \left(1 - \frac{F^*}{F}\right) [f + \alpha AS + \lambda RF - d_1 F] + \left(1 - \frac{I^*}{I}\right) [\beta SI - \gamma I - d_1 I] \\ &+ \left(1 - \frac{R^*}{R}\right) [\gamma I - \delta R - \lambda RF - d_1 R] + \left(1 - \frac{A^*}{A}\right) [\mu\beta SI - \sigma A], \end{aligned}$$

At the equilibrium point E_*,

$$
\begin{aligned}
r &= \frac{rS^*}{K} + \alpha A^* + \beta I^* + d_1 - \delta\frac{R^*}{S^*},\\
f &= d_1 f^* - \alpha A^* S^* - \lambda R^* F^*,\\
d_1 + \gamma &= \beta S^*,\\
\delta + d_1 &= \frac{\gamma I^* - \lambda R^* F^*}{R^*},\\
\sigma &= \frac{\mu\beta S^* I^*}{A^*},
\end{aligned}
$$

Putting these values in above equation, we get

$$
\begin{aligned}
{}_0^C D_t^\theta\, L_2(t) \le{}& (S - S^*)\left[r - \frac{rS}{K} - \alpha A - \beta I + \delta\frac{R}{S} - d_1\right] + \left(1 - \frac{F^*}{F}\right)\\
& [f + \alpha AS + \lambda RF - d_1 F]\\
& + \left(1 - \frac{I^*}{I}\right)[\beta SI - \gamma I - d_1 I] + \left(1 - \frac{R^*}{R}\right)[\gamma I - \delta R - \lambda RF - d_1 R]\\
& + \left(1 - \frac{A^*}{A}\right)[\mu\beta SI - \sigma A],\\
={}& (S - S^*)\left[-\frac{r}{K}(S - S^*) - \alpha(A - A^*) - \beta(I - I^*) + \delta\left(\frac{R}{S} - \frac{R^*}{S^*}\right)\right]\\
& + \left(1 - \frac{F^*}{F}\right)[-d_1(F - F^*) + \alpha(AS - A^*S^*) + \lambda(RF - R^*F^*)]\\
& + (I - I^*)[\beta S - \beta S^*] + \frac{(R - R^*)}{RR^*}\left[\gamma\left(I - \frac{RI^*}{R^*}\right) - \lambda R(F - F^*)\right]\\
& + \left(1 - \frac{A^*}{A}\right)\left(\mu\beta SI - \frac{\mu\beta S^* I^*}{A^*}A\right),\\
={}& -\frac{r}{K}(S - S^*)^2 - \alpha(S - S^*)(A - A^*) + \delta(S - S^*)\left(\frac{R}{S} - \frac{R^*}{S^*}\right)\\
& -\frac{d_1}{F}(F - F^*)^2 + \alpha\frac{(F - F^*)}{F}(AS - A^*S^*) + \lambda\frac{(F - F^*)}{F}(RF - R^*F^*)\\
& + \frac{\gamma}{RR^*}(R - R^*)(R^*I - RI^*) + \lambda(R - R^*)(F^* - F)\\
& + \frac{\mu\beta}{AA^*}(A - A^*)(SIA^* - S^*I^*A),
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{r}{K}(S-S^*)^2 - \alpha(S-S^*)(A-A^*) + \delta(S-S^*)\left(\frac{R}{S}-\frac{R^*}{S^*}\right)\\
&\quad -\frac{d_1}{F}(F-F^*)^2 + \alpha\frac{(F-F^*)}{F}[S(A-A^*)+A^*(S-S^*)]\\
&\quad +\lambda\frac{(F-F^*)}{F}[F(R-R^*)+R^*(F-F^*)] + \frac{\gamma}{RR^*}(R-R^*)(R^*I-RI^*)\\
&\quad -\lambda(R-R^*)(F-F^*) + \frac{\mu\beta}{AA^*}(A-A^*)(SIA^*-S^*I^*A),\\
&= -\frac{r}{K}(S-S^*)^2 - \alpha(S-S^*)(A-A^*) + \delta(S-S^*)\left(\frac{R}{S}-\frac{R^*}{S^*}\right)\\
&\quad -(d_1-\lambda R^*)\frac{(F-F^*)^2}{F} + \frac{\alpha S}{F}(F-F^*)(A-A^*) + \frac{\alpha A^*}{F}(F-F^*)\\
&\quad (S-S^*) + \frac{\gamma}{RR^*}(R-R^*)(R^*I-RI^*) + \frac{\mu\beta}{AA^*}(A-A^*)(SIA^*-S^*I^*A),
\end{aligned}
$$

Thus, ${}_0^C D_t^\theta L_2(t) < 0$ if the following conditions hold:

$$
1<\frac{F^*}{F};1<\frac{A}{A^*};1<\frac{R}{R^*}<\frac{S}{S^*};\frac{I}{I^*}<\frac{R}{R^*};\frac{SI}{S^*I^*}<\frac{A}{A^*};\lambda R^*<d_1; \tag{8.1}
$$

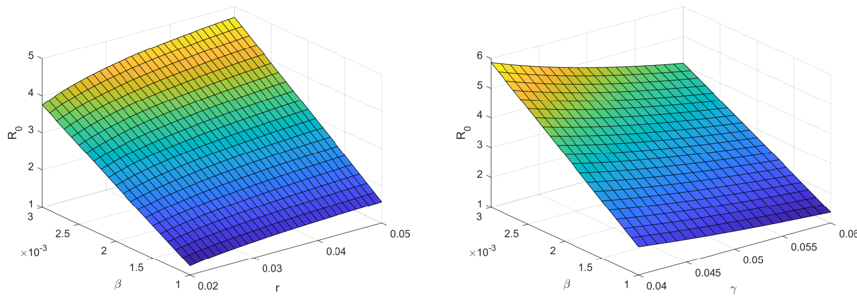
and ${}_0^C D_t^\theta L_2(t) = 0$ if $S=S^*,\ F=F^*,\ I=I^*,\ R=R^*,\ A=A^*$. Hence, the endemic equilibrium point E_* is globally asymptotically stable if the conditions in (8.1) hold. □

Bifurcation Analysis

The equilibrium point E_1 exists for all values of $\mathfrak{R}_o$ and it is asymptotically stable when $\mathfrak{R}_o < 1$. As the value of $\mathfrak{R}_o$ crosses 1, then its stability changes from being asymptotically stable to unstable. The other equilibrium point E_* exists when $\mathfrak{R}_o > 1$ [i.e. some of its coordinates are negative when $\mathfrak{R}_o < 1$]. E_1 loses stability when $\mathfrak{R}_o$ becomes larger than 1, but E_* becomes asymptotically stable. This phenomenon is known as transcritical bifurcation.

9. Numerical sensitivity analysis

In this section, we analyse the sensitivity of the model (3.1) in the sense of [4] and [19] to know the parameters that have a high impact on the reproduction number $\mathfrak{R}_o$ for effective response to the problem under study. When a variable u is differentiable with respect to a parameter b, then its normalized forward sensitivity index is defined as $\Pi_b^u = \frac{\partial u}{\partial b}\cdot\frac{b}{u}$.



(a) (b)

FIGURE 2. The plot of sensitivity analysis for basic reproductive number $\mathfrak{R}_o$ with different parameters

With respect to the model parameters, the sensitivity indices of the basic reproduction number $\mathfrak{R}_o$ are computed as follows:

$$\Pi_r^{\mathfrak{R}_o} = \frac{d_1}{r - d_1} = 0.1364, \text{ (at the parameter values for our numerical study)},$$
$$\Pi_K^{\mathfrak{R}_o} = 1,$$
$$\Pi_\beta^{\mathfrak{R}_o} = 1,$$
$$\Pi_{d_1}^{\mathfrak{R}_o} = -\frac{r+\gamma}{(\gamma+d_1)(r-d_1)} = -40.5844,$$
$$\Pi_\gamma^{\mathfrak{R}_o} = -\frac{\gamma}{(\gamma+d_1)} = -0.8929,$$

If the sign of Π_b^u is positive, then it indicates a direct variation relationship, that is, a rise or fall in the associated parameter results in an equivalent increase or reduction in the reproduction number. On the other hand, the negative sign implies inverse proportionality relationship, that is, a reduction or increase in a certain model parameter results in an equivalent rise or fall in the reproduction number. The sensitivity of the parameters β and r with regard to $\mathfrak{R}_o$ is shown in Figure 2(a). It illustrates that when β and r increases,then $\mathfrak{R}_o$ is also increases. Figure 2(b) shows the sensitivity of the parameter γ in relation to $\mathfrak{R}_o$. It depicts that when γ increases, then the value of $\mathfrak{R}_o$ decreases.

Therefore, it is feasible to get insight into the best intervention options to control and manage the expansion of farmers profession change through the use of sensitivity analysis.

10. Numerical simulations and discussion

In this section, we perform comprehensive numerical simulations using MATLAB to evaluate and validate the analytical conclusions of our model system.

Stability Analysis.

In this case, we investigate the stability of our proposed model. First, We employed the parameter values $r = 0.05$, $K = 100$, $\alpha = 0.0002$, $\beta = 0.0011$,

$\delta = 0.003$, $d_1 = 0.02$, $f = 2$, $\lambda = 0.003$, $\gamma = 0.05$, $\mu = 0.09$ and $\sigma = 0.002$ for numerical simulations. The time series solutions of all individuals corresponding to $\theta = 0.6, 0.7, 0.8, 0.9, 1$ when $\mathfrak{R}_\circ = 0.94 < 1$ are shown in Fig. 3(a) to 3(e). We have found from the following figures that the system (3.1) is locally asymptotically stable at $E_1 \approx (60, 100, 0, 0, 0)$, which validates our theoretical findings. Figure 3(f) shows the global stability of the model system (3.1) at E_1 with different initial conditions taking $\theta = 0.85$, which confirms our theoretical results.

Next, if we employ the parameter value $\beta = 0.002$ and others are same as above, then $\mathfrak{R}_\circ = 3.14 > 1$. In this case, the time series solutions of all individuals corresponding to $\theta = 0.6, 0.7, 0.8, 0.9, 1$ are shown in Fig. 4(a) to 4(e). We have observed from the figures that the system (3.1) is locally asymptotically stable at $E_* \approx (28.05, 456.68, 12.14, 4.03, 31.63)$ when $\mathfrak{R}_\circ > 1$, which validates our theoretical findings. Figure 4(f) shows the global stability of the model system (3.1) at E_* with different initial conditions taking $\theta = 0.85$. Again, from figure 3 and 4, we have seen that all the population moves more rapidly towards the equilibrium point as the fractional order decreases.

Impact of β for θ =0.85.

The impact of β on each individual is depicted in Fig. 5(a) to 5(e). From figure, we observed that the susceptible and permanent farmers decreases when β changes from 0.001 to 0.008 for $\theta = 0.85$. It is evident from Fig. 5(c) that the number of infected people increases at first, but after a certain time it declines as β increases from 0.001 to 0.008 for $\theta = 0.85$.

Impact of μ for θ =0.85.

The impact of μ (i.e., the growth rate of density of aids) on each class is depicted in Fig. 6(a) to 6(e). It is observed from Fig. 6(c) that the number of infected farmers decreases when μ changes from 0.01 to 0.3 for $\theta = 0.85$. On the other hand, Fig. 6(b) shows that the number of permanent farmers increases.

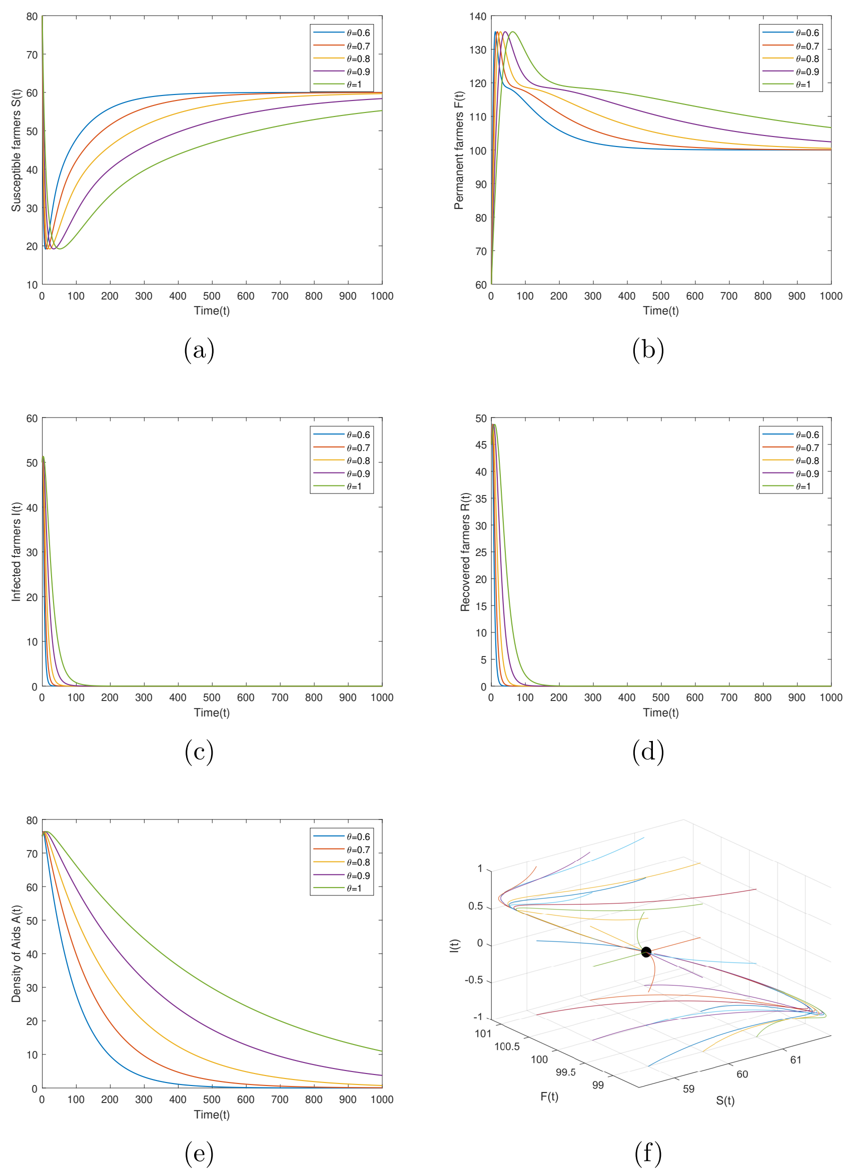


FIGURE 3. Time series solution of (a) Susceptible farmers, (b) Permanent farmers, (c) Infected farmers, (d) Recovered farmers, and (e) Density of Aids for values of $\theta = 0.6, 0.7, 0.8, 0.9, 1$ when $\mathfrak{R}_o < 1$, (f) Phase diagram for $\theta = 0.85$ when $\mathfrak{R}_o < 1$.

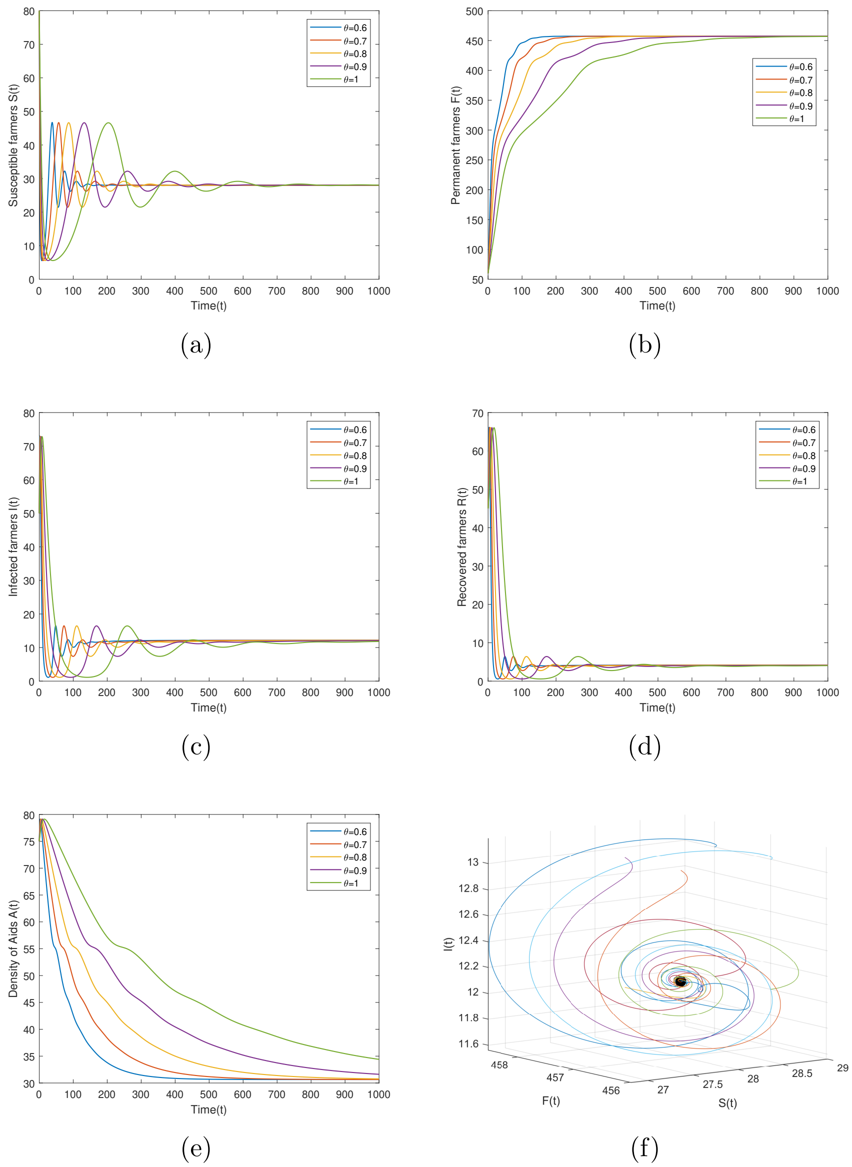


FIGURE 4. Time series solution of (a) Susceptible farmers, (b) Permanent farmers, (c) Infected farmers, (d) Recovered farmers, and (e) Density of Aids for values of $\theta = 0.6, 0.7, 0.8, 0.9, 1$ when $\mathfrak{R}_{\mathfrak{o}} > 1$, (f) Phase diagram for $\theta = 0.85$ when $\mathfrak{R}_{\mathfrak{o}} > 1$.

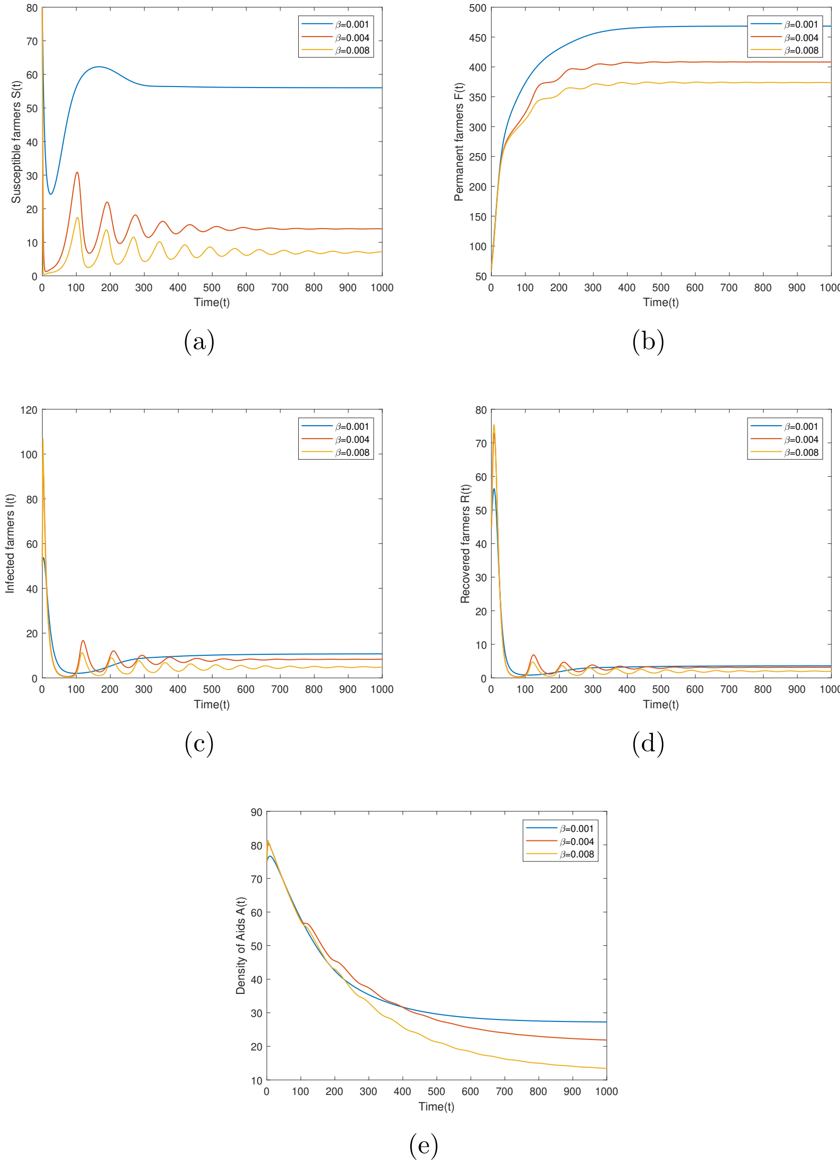
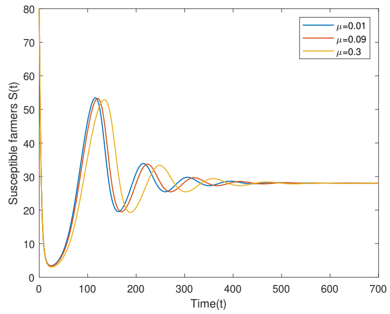
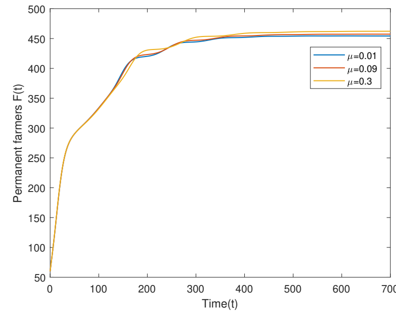


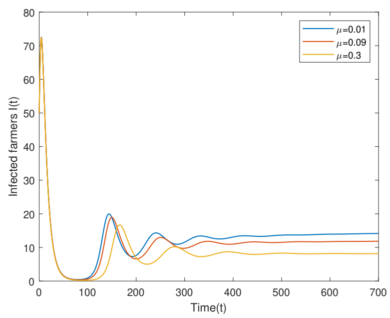
FIGURE 5. Effect of parameter β on (a) Susceptible farmers (b) Permanent farmers, (c) Infected farmers, (d) Recovered farmers, and (e) Density of Aids for $\theta = 0.85$



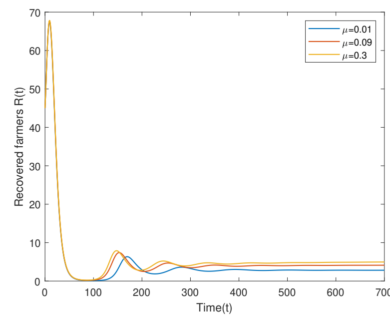
(a)



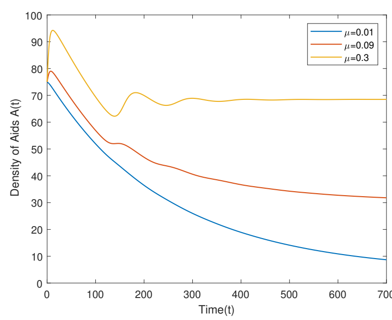
(b)



(c)



(d)



(e)

FIGURE 6. The effect of parameter μ on (a) Susceptible farmers, (b) Permanent farmers, (c) Infected farmers, (d) Recovered farmers, and (e) Density of Aids for $\theta = 0.85$

11. Conclusions

In this work, we developed and analyzed a fractional-order model to describe the dynamics of farmers transitioning from agriculture to other occupations, utilizing the Caputo derivative. The numerical simulations reveal that as the fractional order decreases, all populations converge more rapidly towards the equilibrium point, emphasizing the significance of fractional calculus in accurately modeling and managing dynamic systems. By adjusting the fractional order, we can control the speed at which a system reaches its desired state, offering a powerful tool for researchers and practitioners across various disciplines. Our study concludes that the availability and effectiveness of support mechanisms, such as government policies, funding, and awareness programs, are crucial in controlling the transition of farmers to other sectors.

In conclusion, examining the transition of farmers from agriculture to other sectors is vital for promoting a resilient and sustainable agricultural industry. By understanding the factors that influence farmers' decisions to commit to farming permanently, stakeholders can design better support mechanisms and policies. This understanding not only aids in stabilizing the agricultural workforce but also ensures the long-term viability of farming as a profession, ultimately contributing to broader economic and social well-being.

The realism of the model in this study can be improved by incorporating additional parameters and state variables. Future research endeavors can utilize other fractional operators to further elaborate the model in order to get deeper insights into the dynamics of the farmer's occupation change.

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