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TRANSFER OF PRÜFER-LIKE CONDITIONS

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ABSTRACT. This paper deals with five extensions of the Prüfer domain concept to commutative rings with zero divisors. We investigate the stability of these Prüfer-like conditions under localization and homomorphic image. Our results generate new and original examples of Prüfer-like rings.

1. Introduction and preliminaries

Throughout this paper all rings are commutative with identity element and all modules are unital. In his article [24], Prüfer introduced a new class of integral domains; namely those domains in which all finitely generated ideals are invertible. Through the years, Prüfer domains acquired a great many equivalent characterizations, each of which was extended to rings with zero-divisors in different ways. More precisely, we consider the following Prüfer-like properties on a commutative ring R [1, 3, 4]: 1) R is semihereditary, i.e., every finitely generated ideal of R is projective. 2) The weak global dimension of R is at most one. 3) R is arithmetical, i.e., every finitely generated ideal of R is locally principal. 4) R is Gaussian, i.e., $c(fg) = c(f)c(g)$ for any polynomials $f, g \in R[X]$, where $c(f)$ is the content of f, that is, the ideal of R generated by the coefficients of f. 5) R is Prüfer, i.e., every finitely generated regular ideal is invertible (equivalently, every two generated regular ideal is invertible).

In [11], it is proved that each one of the above conditions implies the following next one: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). Also examples are given to show that, in general, the implications cannot be reversed. Moreover, an investigation is carried out to see which conditions may be added to any of these properties in order to reverse the implications. Recall that in the domain context, the above five classes of Prüfer-like rings collapse to the notion of Prüfer domain. From Bazzoni and Glaz [4, Theorem 3.12], we note that a Prüfer ring R satisfies anyone of the five conditions if and only if its total ring of quotients $Tot(R)$ satisfies the same condition. For more details on these notions, we refer the reader to [1, 3, 4, 7, 10, 11, 13, 17, 21, 27].

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This paper investigates the stability of the above five Prüfer-like conditions under localization and homomorphic image. In Section 2, we prove that if R is a Prüfer ring then so is $S^{-1}R$ for any multiplicative subset $S \subseteq R \setminus Z(R)$, where $Z(R)$ is the set of zero-divisors of R (Theorem 2.1). We further show that the Prüfer property is not, in general, stable under localization (Example 2.3). However, Theorem 2.5 asserts that if R is Gaussian (resp., arithmetical, $w.dim(R) \leq 1$, or semihereditary) then so is $S^{-1}R$ for any multiplicative subset S of R.

In Section 3, Theorem 3.1 states that the homomorphic image of an arithmetical (resp., Gaussian) ring is arithmetical (resp., Gaussian). We also show that the remaining three Prüfer-like conditions are not stable under homomorphic image (Examples 3.2 and 3.3). The section closes with a result (Theorem 3.4) which investigates the transfer of the five Prüfer-like conditions to a particular case of homomorphic image; namely, direct product of rings.

2. Localization of Prüfer conditions

In this section we present a detailed treatment of the localization of the pre-mentioned Prüfer-like conditions. We prove that, unlike Prüfer rings, the classes of Gaussian rings, arithmetical rings, rings with $w.dim(R) \leq 1$ and semihereditary rings are stable under localization. We start with the following theorem which states a condition under which the class of Prüfer rings is stable under localization.

Theorem 2.1. *Let R be a Prüfer ring and S a multiplicative subset of R which is contained in $R \setminus Z(R)$. Then $S^{-1}R$ is a Prüfer ring.*

The proof will use the following Lemma.

Lemma 2.2. *Let R be a Prüfer ring and S a multiplicative subset of R which is contained in $R \setminus Z(R)$. Then $Tot(R) = Tot(S^{-1}R)$.*

Proof. We claim that $R \subseteq S^{-1}R$. Indeed, let $0 \neq a \in R$ such that $\frac{a}{1} = \frac{0}{1}$. Then, there exists $t \in S$ such that $ta = 0$ and hence $a = 0$ since $t \in S \subseteq R \setminus Z(R)$. This means that $R \subseteq S^{-1}R$. Hence, every element $\frac{a}{b}$ of $Tot(R)$, where $a \in R$ and $b \in R \setminus Z(R)$, can be written as $\frac{a}{b} = \frac{a/1}{b/1}$ with $\frac{b}{1} \in S^{-1}R \setminus Z(S^{-1}R)$. Thus, $Tot(R) \subseteq Tot(S^{-1}R)$.
Conversely, let $x = \frac{a/s}{a'/s'} \in Tot(S^{-1}R)$, where $\frac{a}{s} \in S^{-1}R$ and $\frac{a'}{s'} \in S^{-1}R \setminus Z(S^{-1}R)$. We claim that $a's \in R \setminus Z(R)$. Indeed, let $b \in R$ such that $ba's = 0$. Then, $(\frac{bs}{1})(\frac{a'}{s'}) = \frac{0}{1}$ $(\in S^{-1}R)$ and so $\frac{bs}{1} = \frac{0}{1}$ since $\frac{a'}{s'} \in S^{-1}R \setminus Z(S^{-1}R)$. Thus, there exists $t \in S$ such that $tsb = 0$ and so $b = 0$ since $ts \in S \subseteq R \setminus Z(R)$. Hence, x can be written as $x = \frac{a/1}{a'/1}\frac{1/s}{1/s'} = \frac{(a/1)(1/s)(ss'/1}{(a'/1)(1/s')(ss'/1)} = \frac{(as')/1}{(a's/1)}$ since $\frac{ss'}{1} \in S^{-1}R \setminus Z(S^{-1}R)$. Therefore, $x = \frac{as'}{a's}$ since $a's \in R \setminus Z(R)$ which means that $x \in Tot(R)$. □

Proof of Theorem 2.1. One of the many characterizations of Prüfer rings is that each overrings is integrally closed. It is also clear by Lemma 2.2 that $R \subseteq S^{-1}R \subseteq Tot(R)(= Tot(S^{-1}R))$ for each subset $S \subseteq R \setminus Z(R)$. Thus R Prüfer implies $S^{-1}R$ is Prüfer. □

The next example shows that the condition $S \subseteq R \setminus Z(R)$ cannot be dropped in Theorem 2.1. For this, we appeal to the notion of trivial ring extension. We recall that for a ring A and an A-module E, the trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a_1, e_1)(a_2, e_2) = (a_1a_2, a_1e_2 + a_2e_1)$. Considerable work, part of it summarized in Glaz's book [9] and Huckaba's book [16], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory, see for instance [9, 16, 18, 23, 24].

Example 2.3. Let $A = K[[X_1, X_2, X_3]] = K + M$ be a power series ring over a field K and $M := (X_1, X_2, X_3)$. Let E be an A-module such that $ME = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E. Let S be the multiplicative subset of R given by $S := \{(X_1, 0)^n \ / \ n \in \mathbb{N}\}$ and S_0 the multiplicative subset of A given by $S_0 := \{X_1^n \ / \ n \in \mathbb{N}\}$. Then:

(1) R is a Prüfer ring.
(2) $S_0^{-1}A$ is a domain which is not Prüfer.
(3) $S^{-1}R$ and $S_0^{-1}A$ are isomorphic rings. In particular, $S^{-1}R$ is not Prüfer.

Proof. 1) One may easily verify that R is local with maximal ideal $M \propto E$ and each element of R is either a unit or a zero divisor. Thus, $R = Tot(R)$ is Prüfer.

2) We clearly have

$$S_0^{-1}A = S_0^{-1}K[[X_1, X_2, X_3]] = (S_0^{-1}K[[X_1]])[[X_2, X_3]] \subseteq qf(K[[X_1]])[[X_2, X_3]].$$

Since $S_0^{-1}K[[X_1]]$ is Noetherian, by [9, Theorem 8.1.1] $S_0^{-1}A$ is a domain with

$$w.dim(S_0^{-1}A) = w.dim(S_0^{-1}K[[X_1]])[[X_2, X_3]] = w.dim(S_0^{-1}K[[X_1]]) + 2 \geq 2.$$

In particular, $S_0^{-1}A$ is not a Prüfer domain.

3) Since $X_1E \subseteq ME = 0$ and $X_1 \in S_0^{-1}$, then $S_0^{-1}E = 0$. Thus, $S^{-1}(0 \propto E) = 0$ and so $S^{-1}R = \{\frac{(a,0)}{(s,0)} \ / \ a \in A \text{ and } s \in S_0\}$. Now, we easily check that:

$$\begin{aligned} f : S_0^{-1}A &\longrightarrow S^{-1}R \\ \frac{a}{s} &\longmapsto \frac{(a,0)}{(s,0)} \end{aligned}$$

is a ring isomorphism. In particular, R is not a Prüfer domain by 2). □

Corollary 2.4. *Let R be a ring and S a multiplicative subset of R which is contained in $R \setminus Z(R)$. Then:*

(1) *If R is Gaussian, then so is $S^{-1}R$.*
(2) *If R is arithmetical, then so is $S^{-1}R$.*
(3) *If $w.dim(R) \leq 1$, then $w.dim(S^{-1}R) \leq 1$.*
(4) *If R is semihereditary, then so is $S^{-1}R$.*

Proof. If R satisfies one of the five Prüfer-like conditions, then so is $Tot(R)(= Tot(S^{-1}R))$ by [4, Theorem 3.12]. Also, R is, in all cases, a Prüfer ring and so $S^{-1}R$ is a Prüfer ring by Theorem 2.1. Therefore, $S^{-1}R$ satisfies the same Prüfer-like condition by [4, Theorem 3.12]. □

The localization of a Prüfer ring is not always a Prüfer ring by Example 2.3. For the other Prüfer-like conditions, we have:

Theorem 2.5. *Let R be a ring and S a multiplicative subset of R. Then:*

(1) *If R is Gaussian, then so is $S^{-1}R$.*
(2) *If R is arithmetical, then so is $S^{-1}R$.*
(3) *If $w.dim(R) \leq 1$, then $w.dim(S^{-1}R) \leq 1$.*
(4) *If R is semihereditary, then so is $S^{-1}R$.*

Proof. 1) Assume that R is a Gaussian ring. Our aim is to prove that for all polynomials $S^{-1}f$ and $S^{-1}g$, we have $c_{S^{-1}R}(S^{-1}fS^{-1}g) = c_{S^{-1}R}(S^{-1}f)c_{S^{-1}R}(S^{-1}g)$. Without loss of generality, we may assume that $f = \sum_{i=0}^{n} a_i X_i$ and $g = \sum_{i=0}^{m} b_i X_i \in R[X]$. But, $c_R(fg) = c_R(f)c_R(g)$ since R is Gaussian. Hence:

$$\begin{array}{rcl} c_{S^{-1}R}(S^{-1}f)c_{S^{-1}R}(S^{-1}g) & = & S^{-1}(c_R(f))S^{-1}(c_R(g)) \\ & = & S^{-1}[c_R(f)c_R(g)] \\ & = & S^{-1}(c_R(fg)) \ (\textit{since } R \textit{ is Gaussian}) \\ & = & c_{S^{-1}R}(S^{-1}fS^{-1}g), \text{as desired.} \end{array}$$

2) Let J be a finitely generated ideal of $S^{-1}R$ and M be a maximal ideal of $S^{-1}R$. There exist a finitely generated ideal I of R and a prime ideal m of R such that $J = S^{-1}I$ and $M = S^{-1}m$. Next note that $S^{-1}R_M$ is naturally isomorphic to R_m. As R is arithmetical, IR_P is principal for each prime P. It follows that $J_M \cong IR_m$ is locally principal.

3) and 4) are clear, completing the proof of Theorem 2.5. □

Now, we are able to construct a non-Gaussian Prüfer ring.

Example 2.6. Let R and S be as in Example 2.3. Then:

(1) R is a Prüfer ring.
(2) R is not a Gaussian ring.

Proof. 1) R is a Prüfer ring by Example 2.3(1).

2) We claim that R is not Gaussian. Deny, $S^{-1}R$ is a Gaussian ring by Theorem 2.5(1). Then $S^{-1}R$ is a Prüfer domain by Example 2.3, which contradicts Example 2.3(3). Hence, R is not a Gaussian ring. □

3. Homomorphic image of Prüfer conditions

This section studies the homomorphic image of Prüfer-like rings. We show that the homomorphic image of a Gaussian (resp., arithmetical) ring is Gaussian (resp., arithmetical). On the other hand, we show that the other three classes of Prüfer-like rings are not stable under homomorphic image (Examples 3.2 and 3.3).

Theorem 3.1. (1) *The homomorphic image of a Gaussian ring is Gaussian.*
(2) *The homomorphic image of an arithmetical ring is arithmetical.*

Proof. 1) By [12]. 2) By [5]. □

There have many examples presented of Prüfer rings such that some homomorphic image is not Prüfer. One of the earliest sources for R Prüfer not implying R/I Prüfer is the paper [6]. There are several such examples in the Example section of Huckaba's book [16].
Now, we give a new example showing that the homomorphic image of a Prüfer ring is not, in general, a Prüfer ring. This is also an example of a non-Gaussian Prüfer ring.

Example 3.2. Let A be a non-valuation local domain, M its maximal ideal, and E an A-module with $ME = 0$. Let $R := A \propto E$ be the trivial ring extension of A by E. Consider the following ring homomorphism:

$$h : R \longrightarrow A$$

$$(a, e) \longmapsto a.$$

Then:

(1) R is a Prüfer ring (since it is a total ring).
(2) R is not a Gaussian ring by Theorem 3.1(1) since $h(R) = A$ is not a Gaussian domain.

Now, we construct an arithmetical ring R with $w.dim(R) = \infty$ which shows that the homomorphic image of a semihereditary ring (resp., a ring with $w.dim \leq 1$) is not necessarily semihereditary (resp., of weak dimension less than or equal to one).

Example 3.3. Let $A = K[[X]]$ be the power series ring over a field K, $I = (X^n)$ the ideal of A generated by X^n, where $n \geq 2$. Set $R := A/I$. Then:

(1) A is a discrete valuation domain.
(2) R is an arithmetical Noetherian ring.
(3) $w.dim(R) = \infty$. In particular, R is not semihereditary.

Proof. 1) Trivial.
2) R is an arithmetical ring by Theorem 3.1(2) since A is a Prüfer domain. Also, R is Noetherian since the homomorphic image of Noetherian ring is Noetherian.
3) Let x^i be the image of X^i in $R = A/I$. We denote by (x^i) the principal ideal of R generated by x^i. It is easy to check that the following sequences:

$$0 \longrightarrow (x^{n-1}) \longrightarrow R \xrightarrow{u} (x) \longrightarrow 0$$

$$0 \longrightarrow (x) \longrightarrow R \xrightarrow{v} (x^{n-1}) \longrightarrow 0$$

where $u(r) = rx$ and $v(r) = rx^{n-1}$ for each $r \in R$, are exact. But, the principal ideal (x) of R is not a projective ideal since $xx^{n-1} = 0$ and R is local. Therefore, by the above two exact sequences of R, we have $pd_R((x))(= fd_R((x)) = \infty$ (since R is Noetherian) which means that $w.dim(R) = \infty$. In particular, R is not semihereditary. □

Finally, we study a particular case of homomorphic images, that is, the direct product of Prüfer-like rings.

Theorem 3.4. *Let $(R_i)_{1\leq i\leq n}$ be a family of rings. Then:*

(1) $\prod_{i=1}^{n} R_i$ *is Prüfer if and only if so is* R_i *for each* $i=1,\ldots,n$.

(2) $\prod_{i=1}^{n} R_i$ *is Gaussian if and only if so is* R_i *for each* $i=1,\ldots,n$.

(3) $\prod_{i=1}^{n} R_i$ *is arithmetical if and only if so is* R_i *for each* $i=1,\ldots,n$.

(4) $w.dim(\prod_{i=1}^{n} R_i) = \sup\{w.dim(R_i) \mid i=1,\ldots,n\}$.

(5) $\prod_{i=1}^{n} R_i$ *is semihereditary if and only if so is* R_i *for each* $i=1,\ldots,n$.

Proof. The proofs of the first three assertions are done by induction on n and it suffices to check it for $n=2$.

1) By [8].

2) If $R_1\times R_2$ is a Gaussian ring, then, for each $i=1,2$, R_i is a Gaussian ring as homomorphic image of a Gaussian ring by Theorem 3.1(1).

Conversely, assume that R_1 and R_2 are Gaussian rings. Let $f=\sum_{i=0}^{n}(a_i,b_i)X_i$ and $g=\sum_{i=0}^{m}(c_i,d_i)X_i$ be two polynomials in $(R_1\times R_2)[X]$ and let $f_1=\sum_{i=0}^{n}a_iX_i\in R_1[X]$, $f_2=\sum_{i=0}^{n}b_iX_i\in R_2[X]$, $g_1=\sum_{i=0}^{m}c_iX_i\in R_1[X]$, and $g_2=\sum_{i=0}^{m}d_iX_i\in R_2[X]$. We will prove that $c_{R_1\times R_2}(fg)=c_{R_1\times R_2}(f)c_{R_1\times R_2}(g)$. First we note that $c_{R_1\times R_2}(f)=(c_{R_1}(f_1),c_{R_2}(f_2))$. Hence, it is easy to see that $c_{R_1\times R_2}(fg)=(c_{R_1}(f_1g_1),c_{R_2}(f_2g_2))=(c_{R_1}(f_1)c_{R_1}(g_1),c_{R_2}(f_2)c_{R_2}(g_2))$ (since R_1 and R_2 are Gaussian rings) $=(c_{R_1}(f_1),c_{R_2}(f_2))(c_{R_1}(g_1),c_{R_2}(g_2))=c_{R_1\times R_2}(f)c_{R_1\times R_2}(g)$.

3) If $R_1\times R_2$ is an arithmetical ring, then, for each $i=1,2$, R_i is an arithmetical ring as homomorphic image of an arithmetical ring by Theorem 3.1(2).

Conversely, assume that R_1 and R_2 are arithmetical rings. Let J be a finitely generated ideal of the ring $R_1\times R_2$, and M a maximal ideal of $R_1\times R_2$. Write $J=I_1\times I_2$, where I_i is a finitely generated ideal of R_i, and M is either $m_1\times R_2$ or $R_1\times m_2$, where $m_i\in Max(R_i)$ for $i=1,2$. We may assume that $M=m_1\times R_2$ (the case $M=R_1\times m_2$ is similar). We will prove that $J_M=(I_1\times I_2)_{m_1\times R_2}$ is principal. For this, consider an element $\frac{(a_1,a_2)}{(s_1,s_2)}\in J_M$, where $a_i\in I_i$ for $i=1,2$ and $(s_1,s_2)\in(R_1\times R_2)-(m_1\times R_2)$. So, we have $\frac{(a_1,a_2)}{(s_1,s_2)}=\frac{(a_1,a_2)(1,0)}{(s_1,s_2)(1,0)}=\frac{(a_1,0)}{(s_1,0)}$ since $(1,0)\in(R_1\times R_2)-(m_1\times R_2)$. Assume that $I_{1_{m_1}}=aR_{1_{m_1}}$, where $a\in I_1$ since R_1 is supposed to be arithmetical. Therefore, $\frac{(a_1,a_2)}{(s_1,s_2)}=\frac{(a_1,0)}{(s_1,0)}\in(a,0)(R_1\times R_2)_{m_1\times R_2}$. On the other hand, $(a,0)\in J$ imply that $(a,0)(R_1\times R_2)_{m_1\times R_2}\subseteq J_M=J(R_1\times R_2)_{m_1\times R_2}$. Finally, $J_M=(a,0)(R_1\times R_2)_M$ is principal and so $R_1\times R_2$ is an arithmetical ring.

4) and 5) are clear since a ring R is semihereditary if and only if it is coherent and $w.dim(R)\leq 1$, a finite direct product of rings is coherent if and only if each

component is coherent, and $w.dim(\prod_{i=1}^{n} R_i) = sup\{w.dim(R_i) \mid i = 1, \ldots, n\}$. □

We close this paper by constructing a non-total non-Gaussian Prüfer ring.

Example 3.5. Let R be as in Example 2.3, T a Prüfer domain which is not a field, and $L := R \times T$ the direct product of R by T. Then:

(1) L is a non-total ring since $(1, a)(\in L)$ is neither unit nor zero-divisor in L for each non invertible element $a \in T$.
(2) L is a Prüfer ring by Theorem 3.4(1) since R and T are Prüfer rings.
(3) L is not a Gaussian ring by Theorem 3.4(2) since R is not a Gaussian ring.

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