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GRAPH DOMINATION IN ROOTED PRODUCTS: EQUITABLE AND OUTDEGREE EQUITABLE APPROACHES

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Abstract. In this paper, equitability related domination parameters such as equitable domination number and outdegree equitable domination number are investigated in the context of rooted product of graphs. Obtained a lower bound for the equitable domination number of rooted product of graphs. Obtained upper as well as lower bound for the outdegree equitable domination number of rooted product of graphs. Some particular rooted product of graphs are examined in detail.

1. Introduction and preliminaries

The notion of Domination has a long history, even though the formalization of the theory came much later. Early references can be found from nineteenth century through the attempts to solve the chess board puzzles. Ore [11] and Berge [2] are credited with the theoretical introduction of Domination in the field of Graph Theory. Domination Theory finds applications in the areas of Network Design, Facility Location and Optimization problems. Research works of Cockayne [5], Lasker [1], Haynes [7], et al explained methods to explore various aspects and applications of Domination.

This paper focuses solely on finite, undirected, simple and connected graphs. We adopt the conventions related to definitions and notations from [4]. For a vertex v in a graph G, $N(v)$ represents the neighborhood of v which is the collection of all vertices adjacent to v. The number of edges incident to a vertex v in a graph G is termed as the vertex's degree, $d_G(v)$.

A dominating set of a graph G is a subset D of vertices of G such that for each vertex u in $V - D$ there exists some vertex v in D such that $N(v)$ contains u. The **domination number** $\gamma(G)$ is the size of the smallest dominating set of G. A dominating set of G with size $\gamma(G)$ is termed as a γ- set of G.

An **equitable dominating set** of a graph $G = (V, E)$, is a subset of vertices $D \subseteq V$ such that each vertex $v \in V - D$, is adjacent to at least one vertex $u \in D$ and the degree of u and v differ by at most 1. The **equitable domination**

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number $\gamma^e(G)$ is the size of the smallest equitable dominating set of G. A γ^e-set of G is an equitable dominating set of G with exactly $\gamma^e(G)$ vertices.

For a vertex u in a dominating set D of a graph G, the number of edges with one end vertex u and the other end vertex in $V - D$ is called the outdegree of u with respect to D, $d^o_{D,G}(u)$. A dominating set D is called the **outdegree equitable dominating set** if the absolute value of the differences of outdegrees of any two vertices of D is at most one. The **outdegree equitable domination number** of G $\gamma_{oe}(G)$ is the size of the smallest outdegree equitable dominating set of G. An outdegree equitable dominating set of G with exactly $\gamma_{oe}(G)$ vertices is termed as a γ_{oe}- set of G.

Example

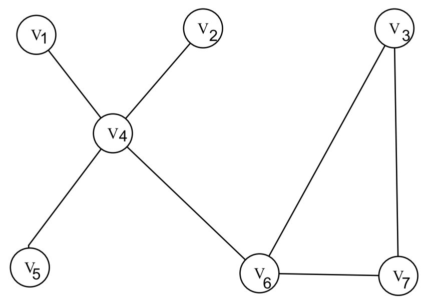


FIGURE 1. G

In Figure 1,

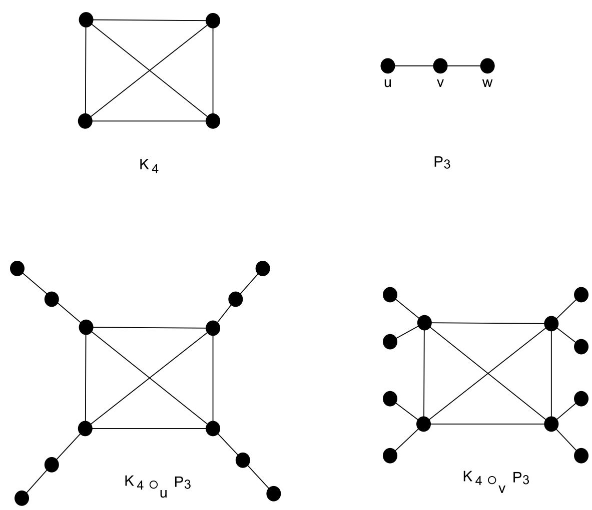
$\gamma(G) = 2$, { v$_4$, v$_3$} is a γ - set of G
$\gamma^e(G) = 4$, {v$_1$, v$_2$, v$_5$, v$_6$} is a γ^e- set of G
$\gamma_{oe}(G) = 2$, {v$_4$, v$_6$} is a γ_{oe}- set of G

{ v$_4$, v$_7$} is a dominating set of G but it is neither an equitable dominating set nor an outdegree equitable dominating set of G.

V. Swaminathan and K. M. Dharmalingham pioneered the concept of equitable domination [13] whereas outdegree equitable domination was introduced by K. M. Dharmalingham, [6]. Further studies can be seen in the works, [3], [9], [10], [12], [14], [15].

Given a graph G with vertices $u_1, u_2, ..., u_n$ and a graph H with root vertex v(H is a graph with the vertex v, designated as the root vertex), the rooted product Go_vH is defined as the graph obtained by taking one copy of G and n copies of H and identifying the vertex u_i of G with the vertex v in the i^{th} copy of H for all i, $1 \leq i \leq n$, [8]. In [8], various domination parameters such as roman domination number, independent domination number, super domination number, connected and convex domination number are studied. This work aims to investigate the equitable domination number and the outdegree equitable domination number of rooted product of graphs.

Example



This work looks into the above mentioned domination parameters of rooted product of graphs.

2. Main results

Theorem 2.1. *Given a graph G of order n and a graph H with root v where $|V(H)| \geq 2$, then $\gamma^e(Go_vH) \geq n$.*

Proof. Suppose $H = (V, E)$. Let $H_i = (V_i, E_i)$ denote the i^{th} copy of H where $i = 1, 2..., n$ and let v_i denote the copy of v in H_i. For convenience, we denote the vertex in Go_vH obtained by identifying the vertex v_i with the vertex of G as v_i itself. Let D represents a γ^e - set of Go_vH. Let $D_i = D \cap V_i$. Then either $v_i \in D_i$ or $v_i \notin D_i$. If $v_i \notin D_i$, then as $|V(H)| \geq 2$, in order to dominate vertices of H_i, D should contain at least one vertex from H_i, other than v_i. Thus in both cases $|D_i| \geq 1$. Now $D = \bigcup_{i=1}^{n} D_i$ and $D_i \cap D_j = \phi$, for all $i, j = 1, 2, ...n$ and $i \neq j$. Therefore

$$\gamma^e(Go_vH) = |D| = \sum_{i=1}^{n} |D_i| \geq \sum_{i=1}^{n} 1 = n$$

□

Theorem 2.2. *Given a graph G of order n and a graph H with root v, where v is an equitable isolate of H, then $\gamma^e(Go_vH) = \gamma^e(G) + n(\gamma^e(H) - 1)$.*

Proof. Let $u_1, u_2, ..., u_n$ denote the vertices of G. Suppose $H = (V, E)$. Let $H_i = (V_i, E_i)$ denote the i^{th} copy of H where $i = 1, 2..., n$ and let v_i denote the copy of v in H_i. Suppose $deg_H(v) = l$. Let D represents a γ^e - set of H. Let D_i denote the copy of D in H_i, $i = 1, 2, ..., n$. Since v_i is an equitable isolate of H_i, $v_i \in D_i$ for all $i = 1, 2, ..., n$. Let x_i denote the vertex obtained by identifying the vertices u_i and v_i in Go_vH. Let $T = \bigcup_{i=1}^{n}(D_i - v_i)$ and suppose S be a γ^e - set of G. Let $S' = \{x_i : u_i \in S$ for some $i = 1, 2, ...n\}$. Then S and T are subsets of $V(Go_vH)$ and clearly $S' \cap T = \phi$

Claim: $S' \cup T$ is an equitable dominating set of Go_vH.

Take a vertex x from $V(Go_vH) - (S' \cup T)$. Being a vertex in Go_vH, either $x = x_i$ or $x \in V_i - v_i$ for some $i = 1, 2, ..., n$. Assume $x = x_i$ for some $i = 1, 2, ..., n$. Now $deg_{Go_vH}(x_i) = deg_G(u_i) + l$, for all $i = 1, 2, ..., n$. As $x_i \in V(Go_vH) - (S' \cup T)$, $x_i \notin S'$ which implies that $u_i \notin S$ and as S is a dominating set of G, there exists some vertex u_j, $j \neq i$ in S such that u_i is adjacent to u_j and $|deg_G(u_i) - deg_G(u_j)| \leq 1$. Since $u_j \in S$, we get $x_j \in S'$ and x_i is adjacent to x_j in Go_vH with

$$|deg_{Go_vH}(x_i) - deg_{Go_vH}(x_j)| = |deg_G(u_i) + l - (deg_G(u_j) + l)| \leq 1$$

Suppose $x \in V_i - v_i$. Then as D_i is an equitable dominating set of H_i, there exists a vertex y in D_i such that x is adjacent to y and $|deg_{H_i}(x) - deg_{H_i}(y)| \leq 1$. Also $deg_{Go_vH}(x) = deg_{H_i}(x)$ and $deg_{Go_vH}(y) = deg_{H_i}(y)$. As x is adjacent to y in H_i, x and y are adjacent in Go_vH. Now

$$|deg_{Go_vH}(x) - deg_{Go_vH}(y)| = |deg_{H_i}(x) - deg_{H_i}(y)| \leq 1$$

Thus our claim is established. Also

$$\gamma^e(Go_vH) \leq |S' \cup T| = |S'| + |T| = \gamma^e(G) + n(\gamma^e(H) - 1)$$

Therefore

$$\gamma^e(Go_vH) \leq \gamma^e(G) + n(\gamma^e(H) - 1)$$

If possible suppose $\gamma^e(Go_vH) < \gamma^e(G) + n(\gamma^e(H) - 1)$. Let F be a γ^e - set of Go_vH. Then by our assumption,

$$|F| < \gamma^e(G) + n(\gamma^e(H) - 1) \tag{2.1}$$

Let $F_i = F \cap V_i, i = 1, 2..., n$. As v_i is an equitable isolate of H_i, x_i cannot equitably dominate any vertices of H_i. Therefore in order to dominate vertices of $H_i - x_i$ equitably, F_i should contain at least $\gamma^e(H_i) - 1$ vertices. This is true for all $i = 1, 2, ..., n$. Hence F contains at least $n(\gamma^e(H_i) - 1)$ in order to equitably dominate the vertices of $\bigcup_{i=1}^{n}(H_i - x_i)$. Now let F' be the set of vertices obtained from F by excluding the above mentioned vertices. Then F' equitably dominates the remaining vertices $x_1, x_2, ...x_n$ of Go_vH. Let $F' = \{x_j : j \in J\}$, where $J \subseteq \{1, 2, 3, ..., n\}$. Now suppose $E' = \{u_j : x_j \in F'\}$.

Claim : E' yields an equitable dominating set of G

Let $u_k \in U - E'$. Then $x_k \notin F'$. As F' equitably dominates the vertices $x_1, x_2, ...x_n$ there exists some $j \in J$, such that x_j is adjacent to x_k and

$$|deg_{Go_vH}(x_j) - deg_{Go_vH}(x_k)| \leq 1 \tag{2.2}$$

Now for each $i = 1, 2, ..., n$, $deg_{Go_vH}(x_i) = deg_G(u_i) + l$. Since x_j is adjacent to x_k in Go_vH, u_j is adjacent to u_k in G and from Eq.(2.2) we get

$$|deg_G(u_j) - deg_G(u_k)| = |deg_{Go_vH}(x_j) - l - (deg_{Go_vH}(x_k) - l)| \leq 1$$

Our claim is thus validated. Now from Eq. (2.1)

$$|E'| = |F'| < |F| - n(\gamma^e(H) - 1) < \gamma^e(G)$$

which gives a contradiction. Thus

$$\gamma^e(Go_vH) = \gamma^e(G) + n(\gamma^e(H) - 1)$$

□

Corollary 2.3. *Let G be a graph of order n. Then*

1. $\gamma^e(Go_vK_{p,q}) = \gamma^e(G)+n(p+q-1)$, *where v is any vertex in $K_{p,q}$ and $|p-q| \geq 2$.*

2. $\gamma^e(Go_vW_m) = \gamma^e(G) + \lceil \frac{m-1}{3} \rceil$, *where $m \geq 6$ and v is the center of the wheel graph W_m.*

Theorem 2.4. *Given a graph G of order n and a graph H of order at least 2 with root v then $n \leq \gamma_{oe}(Go_vH) \leq n\gamma_{oe}(H)$.*

Proof. Let $u_1, u_2..., u_n$ denote the vertices of G. Suppose $H = (V, E)$ and let $H_i = (V_i, E_i)$ denote the i^{th} copy of H where $i = 1, 2..., n$ and let v_i denote the copy of v in H_i. Let x_i represents the vertex obtained by identifying the vertices u_i and v_i in Go_vH. Suppose $X = \{x_1, x_2, ..., x_n\}$. Let D be a γ_{oe} - set of H. Now denote the copy of D in H_i as D_i . Now for all $s, t \in D$

$$|d^o_{D,H}(s) - d^o_{D,H}(t)| \leq 1$$

Therefore for each $i = 1, 2, ..., n$ and for all $s, t \in D_i$

$$|d^o_{D_i,H_i}(s) - d^o_{D_i,H_i}(t)| \leq 1 \tag{2.3}$$

Also for $s \in D_i, t \in D_j$ where $i \neq j$

$$|d^o_{D_i,H_i}(s) - d^o_{D_j,H_j}(t)| = |d^o_{D_i,H_i}(s) - d^o_{D_i,H_i}(t')|$$

where t' denote the copy of t in D_i and hence from Eq. (2.3) we get

$$|d^o_{D_i,H_i}(s) - d^o_{D_j,H_j}(t)| \leq 1 \tag{2.4}$$

Case 1: $v \notin D$

Then $v_i \notin D_i$, for all $i = 1, 2, ..., n$. Let $D' = \bigcup_{i=1}^n D_i$. Clearly D' is a dominating set of Go_vH.

Claim : D' yields an outdegree equitable dominating set of Go_vH.

Let $s, t \in D'$. Suppose $s, t \in D_i$, for some $i = 1, 2, ..., n$, then from Eq.(2.3) we get

$$|d^o_{D',Go_vH}(s) - d^o_{D',Go_vH}(t)| = |d^o_{D_i,H_i}(s) - d^o_{D_i,H_i}(t)| \leq 1$$

Suppose $s \in D_i$ and $t \in D_j$ where $i, j \in \{1, 2, ..., n\}$ and $i \neq j$.
Then $d^o_{D',Go_vH}(s) = d^o_{D_i,H_i}(s)$; $d^o_{D',Go_vH}(t) = d^o_{D_j,H_j}(t)$ and hence from Eq.(2.4) we get

$$|d^o_{D',Go_vH}(s) - d^o_{D',Go_vH}(t)| = |d^o_{D_i,H_i}(s) - d^o_{D_j,H_j}(t)| \leq 1$$

Our claim is thus validated. Therefore

$$\gamma_{oe}(Go_vH) \leq n\gamma_{oe}(H)$$

Case 2: $v \in D$

Then $v_i \in D_i$ for all $i = 1, 2, ..., n$. Let $deg_H(v) = l$ and let $T = \bigcup_{i=1}^{n}(D_i - v_i)$ and let $D' = T \cup X$. Then clearly D' is a dominating set of Go_vH.

Claim : D' is an outdegree equitable dominating set of Go_vH.

Suppose $s, t \in D'$. If $s, t \in T$, then by case 1 we get that

$$|d^o_{D',Go_vH}(s) - d^o_{D',Go_vH}(t)| \leq 1$$

Remaining cases are

(a) $s, t \in X$

Suppose $s = x_p$ and $t = x_q$, for some $p, q \in 1, 2, ..., n$, then as $X \subseteq D'$, we get $d^o_{D',Go_vH}(s) = d^o_{D',Go_vH}(x_p) = d^o_{D_p,H_p}(v_p)$ and $d^o_{D',Go_vH}(t) = d^o_{D',Go_vH}(x_q) = d^o_{D_q,H_q}(v_q)$. Hence from Eq.(2.4) we get

$$|d^o_{D',Go_vH}(s) - d^o_{D',Go_vH}(t)| \leq 1$$

(b) $s \in T, t \in X$

As $T = \bigcup_{i=1}^{n}(D_i - v_i)$, $s \in T$ implies $s \in D_i$ for some $i = 1, 2, ..., n$. Now $d^o_{D',Go_vH}(s) = d^o_{D_i,H_i}(s)$. Now $t \in X$ implies $t = x_j$ for some $j = 1, 2, ..., n$ and $d^o_{D',Go_vH}(t) = d^o_{D',Go_vH}(x_j) = d^o_{D_j,H_j}(v_j)$. Hence from Eq.(2.4) we get

$$|d^o_{D',Go_vH}(s) - d^o_{D',Go_vH}(t)| \leq 1$$

Consequently, our claim has been proved. Therefore

$$\gamma_{oe}(Go_vH) \leq |D'| = n(\gamma_{oe}(H) - 1) + n = n\gamma_{oe}(H) \tag{2.5}$$

Now it remains to prove that $n \leq \gamma_{oe}(Go_vH)$. Let F be a γ_{oe} set of Go_vH.
Possibilities are

(a) $F \subseteq X$

(b) $F \subseteq \bigcup_{i=1}^{n}(H_i - v_i)$

(c) $F \cap X \neq \phi$ and $F \cap \bigcup_{i=1}^{n}(H_i - v_i) \neq \phi$

If $F \subseteq X$, then $F = X$ because if $F \neq X$, then there exists at least one x_j, $j = 1, 2, ...n$ such that $x_j \notin F$. Therefore F does not dominate the vertices in $H_i - v_i$ wich is a contradiction to the fact that F ia a dominating set of Go_vH. Thus $\gamma_{oe}(Go_vH) = n$.

If $F \subseteq \bigcup_{i=1}^{n}(H_i - v_i)$, then for each $i = 1, 2, ...n$, F should contain at least one vertex of $H_i - v_i$ in order to dominate the vertices of $H_i - v_i$. Thus $\gamma_{oe}(Go_vH) \geq n$.

Suppose $F \cap X \neq \phi$ and $F \cap \bigcup_{i=1}^{n}(H_i - v_i) \neq \phi$. If $x_j \in F$, for all $j = 1, 2, ...n$, then clearly $\gamma_{oe}(Go_vH) \geq n$. Suppose there exists a vertex x_j of X for some $j = 1, 2...n$ such that $x_j \notin F$. Then F should contain at least one vertex of $H_j - v_j$ in order to dominate the vertices of $H_j - v_j$. Thus $\gamma_{oe}(Go_vH) \geq n$. Therefore in all the cases

$$\gamma_{oe}(Go_vH) \geq n \tag{2.6}$$

Thus from Eq. (2.5)and Eq. (2.6) we get $n \leq \gamma_{oe}(Go_vH) \leq n\gamma_{oe}(H)$.

□

Theorem 2.5. *Given a graph G of order n and a graph H with root v and $|V(H)| \geq 2$, then $\gamma_{oe}(Go_vH) = n$ if and only if $\gamma_{oe}(H) = 1$.*

Proof. Suppose $H = (V, E)$. Let $H_i = (V_i, E_i)$ denote the i^{th} copy of H where $i = 1, 2..., n$ and let v_i denote the copy of v in H_i. Suppose $\gamma_{oe}(H) = 1$. Then from Theorem 2.3 we get that $\gamma_{oe}(Go_vH) = n$. Conversely suppose that $\gamma_{oe}(H) = n$. Let D be a γ_{oe} - set of Go_vH. Then by our assumption $|D| = n$. Let $D_i = D \cap V_i$. Then $D_i \cap D_j = \phi$ for all i, j where $i \neq j$ and $\bigcup_{i=1}^{n} D_i = D$. Hence

$$|D| = \sum_{i=1}^{n} |D_i|$$

For convenience denote the vertex of Go_vH obtained by identifying the vertex of G with the root v_i of the copy H_i as v_i itself. For each $i = 1, 2, ..., n$, either $v_i \in D_i$ or $v_i \notin D_i$. Let $S = \{i \in N : 1 \leq i \leq n, v_i \in D_i\}$ and let $|S| = p$.

If $p = n$, then $v_i \in D_i$ for all $i = 1, 2, ..., n$ which implies that $D = \{v_1, v_2, ..., v_n\}$. Therefore for all $i = 1, 2, ..., n$; v_i dominates H_i and hence $\gamma_{oe}(H_i) = 1$ which gives that $\gamma_{oe}(H) = 1$.

Suppose $p = 0$. Then for all $i = 1, 2, ..., n$; $v_i \notin D_i$. Hence $D_i \subset H_i - v_i$, for all $i = 1, 2, ..., n$. As D is a γ_{oe} - set of Go_vH and $|D| = n$, we can assure that for all $i = 1, 2, ..., n$; D_i contains exactly one vertex from$H_i - v_i$ which dominates the vertices in the i^{th} copy of H attached to the graph G. Hence $\gamma_{oe}(H_i) = 1$ which gives that $\gamma_{oe}(H) = 1$.

Suppose $1 \leq p \leq n - 1$. If possible suppose $\gamma_{oe}(H) > 1$. Our assumption $p \leq n - 1$ implies that there exists a vertex v_i, for some $i = 1, 2, ...n$ of Go_vH such that $v_i \notin D_i$. Then as $\gamma_{oe}(H) > 1$, D_i should contain at least two vertices in order to dominate i^{th} copy of H attached to the graph G. Therefore

$$|D| = \sum_{i=1}^{n} |D_i| \geq p + 2(n-p) = 2n - p \geq 2n - (n-1) = n+1$$

which is a contradiction to our assumption that $|D| = n$. Hence
$\gamma_{oe}(H) = 1$. □

Theorem 2.6. *Given a graph H with root v and $\gamma_{oe}(H - v) \geq \gamma_{oe}(H)$, then for any graph G with n verices, $\gamma_{oe}(Go_vH) = n\gamma_{oe}(H)$.*

Proof. Suppose $H = (V, E)$ and let $H_i = (V_i, E_i)$ denote the i^{th} copy of H where $i = 1, 2..., n$ and let v_i denote the copy of v in H_i. Let D be a γ_{oe} - set of Go_vH. For convenience denote the vertex of Go_vH obtained by identifying the vertex of G with the root v_i of the copy H_i as v_i itself. Let $D_i = D \cap V_i$, $i = 1, 2, ..., n$. Then $D = \bigcup_{i=1}^{n} D_i$ and $D_i \cap D_j = \phi$ for all i, j where $i \neq j$. Hence

$$|D| = \sum_{i=1}^{n} |D_i|$$

Suppose $\gamma_{oe}(H - v) \geq \gamma_{oe}(H)$. Suppose $v_i \in D_i$ for some i, then D_i serves as an outdegree equitable dominating set of H_i and hence $|D_i| \geq \gamma_{oe}(H)$. Suppose $v_i \notin D_i$, $i = 1, 2, ..., n$. Then D_i is an outdegree equitable dominating set of $H_i - v_i$. Therefore $|D_i| \geq \gamma_{oe}(H_i - v_i) \geq \gamma_{oe}(H_i) = \gamma_{oe}(H)$.

Thus for all $i = 1, 2, ..., n$, $|D_i| \geq \gamma_{oe}(H)$, which implies that

$$|D| \geq n\gamma_{oe}(H)$$

Hence from Theorem 2.3, we get that $\gamma_{oe}(Go_vH) = n\gamma_{oe}(H)$. □

Corollary 2.7. *Let G be a graph of order m. Then*

$$\gamma_{oe}(Go_vP_n) = \begin{cases} \gamma_{oe}(G) & \textit{if} \quad n = 1 \\ m & \textit{if} \quad n = 2, 3 \textit{ and } v \textit{ is any vertex of } P_n \\ m\lceil \frac{n}{3} \rceil & \textit{if} \quad n \geq 4, n \not\equiv 1(mod3) \textit{ and } v \textit{ is any vertex of } P_n \\ m\lceil \frac{n}{3} \rceil & \textit{if} \quad n \geq 4, n \equiv 1(mod3) \textit{ and } v \textit{ is any vertex of } P_n \\ & \qquad \textit{of degree two other than the unique center of } P_n \end{cases}$$

Proof. Suppose $n = 1$, then $Go_vP_n \cong G$. Hence $\gamma_{oe}(Go_vP_n) = \gamma_{oe}(G)$.

Suppose $n = 2, 3$ then $\gamma_{oe}(P_n) = 1$ and hence by Theorem 2.4, we get $\gamma_{oe}(Go_vP_n) = m$.

Suppose $n \geq 4, n \not\equiv 1(mod3)$ and v is any vertex of P_n, then
$\gamma_{oe}(P_n - v) \geq \gamma_{oe}(P_n)$. Also $\gamma_{oe}(p_n) = \lceil \frac{n}{3} \rceil$. Hence by Theorem 2.5 we get that $\gamma_{oe}(Go_vP_n) = m\lceil \frac{n}{3} \rceil$.

Suppose $n \geq 4, n \equiv 1(mod3)$ and v is any vertex of P_n, then
$\gamma_{oe}(P_n - v) \geq \gamma_{oe}(P_n)$. Hence by Theorem 2.6 we get that $\gamma_{oe}(Go_vP_n) = m\lceil \frac{n}{3} \rceil$. □

Theorem 2.8. *Let G be a graph of order m and P_n be the path on n vertices with root v, where v is a pendant vertex of P_n. Suppose $n \geq 4$ and $n \equiv 1(mod3)$, then*

$$m(\frac{n-1}{3}) + \gamma_{oe}(G) \leq \gamma_{oe}(Go_vP_n) \leq m\lceil\frac{n}{3}\rceil$$

Proof. Let $P_n = v_1, v_2, ..., v_n$ be the path with root v_1. Suppose $P_n^i = v_1^i, v_2^i, ..., v_n^i$ denote the i^{th} copy of $P_n, i = 1, 2, ..., m$ and let v_1^i denote the copy of v_1 in P_n^i. Let $Q^i = v_2^i, v_3^i, ..., v_n^i$. Let D be a γ_{oe} - set of P_n. Suppose D_i denote the copy of D in P_n^i. Then clearly $\bigcup_{i=1}^m D_i$ gives an outdegree equitable dominating set of Go_vP_n. Therefore

$$\gamma_{oe}(Go_vP_n) \leq \sum_{i=1}^{m} |D_i| = m\lceil\frac{n}{3}\rceil$$

Now if possible suppose $\gamma_{oe}(Go_vP_n) < \gamma_{oe}(G) + m(\frac{n-1}{3})$. Let E be a γ_{oe} - set of Go_vP_n. Let $E_i = E \cap V(P_n^i), i = 1, 2, ..., m$. Then each E_i should contain at least $\frac{n-1}{3}$ vertices in order to dominate Q^i which is a path of length $n-1$. Let F_i be the set obtained by excluding these vertices from E_i. Therefore $|E_i| \geq \frac{n-1}{3}$, for all $i = 1, 2, ..., m$.

$$|\bigcup_{i=1}^{m} E_i| = \sum_{i=1}^{m} E_i \geq m(\frac{n-1}{3}).$$

Now $\bigcup_{i=1}^n F_i$ outdegree equitably dominates the vertices of G identified with the root of copies of P_n. But

$$|\bigcup_{i=1}^{m} F_i| \leq E - |\bigcup_{i=1}^{n} E_i|$$

$$|\bigcup_{i=1}^{m} F_i| < \gamma_{oe}(G) + m(\frac{n-1}{3}) - m(\frac{n-1}{3})$$

$$|\bigcup_{i=1}^{m} F_i| < \gamma_{oe}(G)$$

which is a contradiction. Hence $\gamma_{oe}(Go_vP_n) \geq \gamma_{oe}(G)+m(\frac{n-1}{3})$. Thus we conclude that

$$m(\frac{n-1}{3}) + \gamma_{oe}(G) \leq \gamma_{oe}(Go_vP_n) \leq m\lceil\frac{n}{3}\rceil$$

□

Theorem 2.9. *Let G be a graph of order m and P_n be the path on n vertices with root v, where v is a pendant vertex of P_n. Suppose $n \geq 4$ and $n \equiv 1(mod3)$. Then $\gamma_{oe}(Go_vP_n) = m(\frac{n-1}{3}) + \gamma_{oe}(G)$ if and only if there exists a γ_{oe} set of G with outdegrees of vertices as 1 or 2.*

Proof. Suppose there exists a γ_{oe} set of G say D with outdegrees of vertices as 1 or 2. Let $P_n = v_1, v_2, ..., v_n$ be the path with root $v = v_1$. Suppose $P_n^i = v_1^i, v_2^i, ..., v_n^i$ denote the i^{th} copy of $P_n, i = 1, 2, ..., m$ and let v_1^i denote the copy of v_1 in P_n^i. Let $Q^i = v_2^i, v_3^i, ..., v_n^i$. Let $D_i = \{v_3^i, v_6^i, ..., v_{n-1}^i\}$. Then $|D_i| = \frac{n-1}{3} = \lceil\frac{n-1}{3}\rceil$ and D_i is a dominating set of Q^i. Also D_i is an outdegree equitable dominating set of Q^i as outdegrees of all vertices in D_i is 2. As $\gamma_{oe}(P_{n-1}) = \lceil \frac{n-1}{3} \rceil$, D_i is a $\gamma_{oe}-$set of Q_i. Now $\bigcup_{i=1}^{m} D_i$ dominates all vertices of Go_vP_n except those vertices obtained by identifying the vertices of G with the root of the copies of P_n. Let $T = D \cup \bigcup_{i=1}^{m} D_i$. Then T is a dominating set of Go_vP_n. Now for each $u \in T$,

$$d^o_{T,Go_vP_n}(u) = \begin{cases} 2 \quad or \quad 3 & \text{if u} \in \text{ D} \\ 2 & \text{if u} \in \bigcup_{i=1}^{m} D_i \end{cases}$$

Therefore $|d^o_{T,Go_vP_n}(u) - d^o_{T,Go_vP_n}(x)| \leq 1$, for all $x, u \in T$, which concludes T as an outdegree equitable dominating set of Go_vP_n. Hence

$$\gamma_{oe}(Go_vP_n) \leq |T| = \gamma_{oe}(G) + m(\frac{n-1}{3})$$

Therefore from Theorem 2.6 we conclude that

$$\gamma_{oe}(Go_vP_n) = m(\frac{n-1}{3}) + \gamma_{oe}(G)$$

Conversely suppose that $\gamma_{oe}(Go_vP_n) = m(\frac{n-1}{3}) + \gamma_{oe}(G)$. Suppose there exists no γ_{oe} set of G with outdegrees of vertices as 1 or 2. This implies that every γ_{oe} set of G contains vertices with outdegrees greater than or equal to 3. Let S be a γ_{oe} set of G. Then outdegrees of vertices in S is greater than or equal to 3. Let $R^i = v^i_3, v^i_4, ..., v^i_n$, which is a path of length $n-2$. Let E be a γ_{oe} set of Go_vP_n. Then as $\gamma_{oe}(P_n) = \lceil \frac{n}{3} \rceil$, we get that E contains at least $m(\lceil \frac{n-2}{3} \rceil)$ vertices from $\bigcup_{i=1}^{m} V(R^i)$ with maximum outdegree 2 in order to dominate the vertices of $\bigcup_{i=1}^{m} V(R^i)$ in Go_vP_n.

Let L denote the set of above mentioned vertices in E. Let $F = E - L$, then F dominates the vertices of G in Go_vP_n. By our assumption $|E| = m(\frac{n-1}{3}) + \gamma_{oe}(G)$ and $E = F \cup L$. As $F \cap L = \phi$, $|E| = |F| + |L|$. We have already obtained that $|L| \geq m\lceil \frac{n-2}{3} \rceil$. Thus from $|E| = |F| + |L|$, we can say that $|L| = m\lceil \frac{n-2}{3} \rceil = m(\frac{n-1}{3})$, [as $n \equiv 1(mod 3)$], for if $|L| > m\lceil \frac{n-2}{3} \rceil$, then $|F| < \gamma_{oe}(G)$ which is a contradiction to the minimality of γ_{oe} set of G. Now $|L| = m\lceil \frac{n-2}{3} \rceil = m(\frac{n-1}{3})$ implies that $|F| = \gamma_{oe}(G)$.

Now Let S' denote those vertices of Go_vP_n obtained by identifying the root of the copies of P_n with the vertices of S. As outdegrees of vertices in S is greater than or equal to 3, we get that outdegrees of vertices in S' is greater than or equal to 4. Therefore $S' \neq F$. This then implies that $|F| > \gamma_{oe}(G)$ giving a contradiction to the assumption that $|E| = \gamma_{oe}(Go_vP_n) = m(\frac{n-1}{3}) + \gamma_{oe}(G)$. Hence there exists a γ_{oe} set of G with outdegrees of vertices as 1 or 2.

□

Theorem 2.10. *Let G be a graph of order m and C_n be the cycle on n vertices, $n \geq 4$. Let v be any vertex of C_n. Suppose $n \not\equiv 1(mod 3)$, then $\gamma_{oe}(Go_vC_n) = m\lceil \frac{n}{3} \rceil$.*

Proof. We know that $\gamma_{oe}(C_n) = \gamma_{oe}(P_n) = \lceil \frac{n}{3} \rceil$. Now $C_n - v \cong P_{n-1}$. Therefore $\gamma_{oe}(C_n - v) = \gamma_{oe}(P_{n-1}) = \lceil \frac{n-1}{3} \rceil$. Given that $n \not\equiv 1(mod 3)$. Hence $\lceil \frac{n}{3} \rceil = \lceil \frac{n-1}{3} \rceil$ which implies that $\gamma_{oe}(C_n - v) = \gamma_{oe}(C_n)$. Hence from Theorem 2.5 we can conclude that $\gamma_{oe}(Go_vC_n) = m\lceil \frac{n}{3} \rceil$. □

References

1. Allan, R. B. & Laskar, R. C. (1978). On Domination and Independent Domination Numbers of a Graph, Discrete Math. 23. https://doi.org/10.1016/0012-365X(78)90105-X
2. Berge, C. (1962). Theory of graphs and its applications, Methuen, London.
3. Caay, Mark & Durog, Marjun (2021).On some independent equitable domination of graphs, Gulf Journal of Mathematics, 11(1), 57 - 64. https://doi.org/10.56947/gjom.v11i1.668
4. Clark, J. & Holton, D. A. (1991). A First Look At Graph Theory, World Scientific Publishing, Singapore.
5. Cockayne, E. J. & Hedetniemi, S.T. (1977). Towards a Theory of Domination in Graphs, Networks 7, 247-261. https://doi.org/10.1002/net.3230070305
6. Dharmalingham, K. M. (2010). Bounds for Outdegree Equitable Domination Number in Graphs, Bulletin of the International Mathematical Virtual Institute, Volume 3, 149-154.
7. Haynes, T.W. , Hedetniemi, S.T. & Slater, P.J. (1998). Fundamentals of Domination in Graphs, Marcel Dekker, Inc. New York.
8. Kuziak, Dorota, Lemańska, Magdalena & Yero, Ismael G. (2016). Domination-Related Parameters in Rooted Product Graphs, The Bulletin of the Malaysian Mathematical Society, 39(1), 199-217. https://doi.org/10.1007/S40840-015-0182-5
9. Mahesh, M. S. & Namasivayam, P. (2017). Triple connected two out degree equitable domination number, Journal of Discrete Mathematical Sciences and Cryptography, 20(3), 653-661. https://doi.org/10.1080/09720529.2016.1262320
10. Nayaka S.R., Anwar Alwardi & Puttaswamy. (2018). Transversal Domination in Graphs, Gulf Journal of Mathematics, 6(2), 41-49. https://doi.org/10.56947/gjom.v6i2.132
11. Ore, O. (1962).Theory of graphs, Amer. Math. Soc. Colloq. Publ. Providence, RI.
12. Sahal, A.& Mathad, V. (2013). Two - Outdegree Equitable Domination Number in Graphs, Transactions on Combinatorics, 2(3), 13-19.
13. Swaminathan V., Markandan Dharmalingam K. (2011). Degree Equitable Domination on Graphs, Kragujevac Journal of Mathematics, 35(1), 191-197.
14. T. R. T & Menon, M. K.(2023). Equitable and Outdegree Equitable Domination Number of Graphs, TWMS Journal Of Applied And Engineering Mathematics, 13(4), 1631-1640.
15. T. R. T & Menon, M. K.(2024). Outdegree equitable domination number of certain graph operators, Proyecciones, 43(5), 1037-1053. https://doi.org/10.22199/issn.0717-6279-6275

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