

$\mathcal{I}$ - CONVERGENCE OF SEQUENCES OF SUBSPACES IN AN INNER PRODUCT SPACE

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ABSTRACT. In this paper we introduce the notion of $\mathcal{I}$ -convergence of sequences of k -dimensional subspaces of an inner product space, where $\mathcal{I}$ is an ideal of subsets of $\mathbb{N}$, the set of all natural numbers and $k \in \mathbb{N}$. We also study some basic properties of this notion.

1. INTRODUCTION AND BACKGROUND

The notion of convergence of sequences of points was extended to the notion of convergence of sequences of sets by many authors [18, 19, 20]. Manuharawati et al. [12], [13] introduced the concepts of convergences of sequences of 1-dimensional and 2-dimensional subspaces of a normed linear space and in [14] they have introduced the convergence notion of sequences of k -dimensional ($k \in \mathbb{N}$) subspaces of an inner product space $\mathcal{X}$ of dimension k or higher (may be infinite).

On the other hand the notion of convergence of sequences of real numbers was extended to the notion of statistical convergence by Fast [4] (and also independently by Schoenberg [17]) with the concept of natural density. A lot of work have been done in this direction, after the work of Salat [16] and Fridy [5]. For more primary works in this field one can see [6, 9] etc.

The notion of statistical convergence of sequences of real numbers further extended to the notion of $\mathcal{I}$ -convergence by Kostyrko et al. [10], with the notion of an ideal $\mathcal{I}$ of subsets of the set of all natural numbers $\mathbb{N}$. For more works in this direction one can see [1, 2, 3, 11].

Recently in 2021, using the notion of natural density, F. Nuray [15] has introduced and studied the notion of statistical convergence of a sequence of k -dimensional ($k \in \mathbb{N}$) subspaces of an inner product space. It seems therefore reasonable to introduce and study the notion of $\mathcal{I}$ -convergence of a sequence of k -dimensional subspaces of an inner product space. In this paper we do the same and investigate some basic properties of this notion. Our results extend the results of Nuray [15] and Manuharawati et al. [14].

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2. BASIC DEFINITIONS AND NOTATIONS

We first recall some basic definitions and notation related to inner product space from the literature. Throughout the paper $\mathcal{X}$ stands for an inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ of dimension k ($k \in \mathbb{N}$) or higher (may be infinite) over the field $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$), U_n ($n \in \mathbb{N}$), V and W denote k -dimensional subspaces of $\mathcal{X}$. Also throughout the paper, it is supposed that $U_n = \text{span}\{u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)}\}$ ($n \in \mathbb{N}$), $V = \text{span}\{v_1, v_2, \dots, v_k\}$ and $W = \text{span}\{w_1, w_2, \dots, w_k\}$, where $\{u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)}\}$ is an orthonormal basis of U_n ($n \in \mathbb{N}$) and $\{v_1, v_2, \dots, v_k\}$, $\{w_1, w_2, \dots, w_k\}$ are respective orthonormal bases of V and W . Also for any $x \in \mathcal{X}$, $\|x\| = \sqrt{\langle x, x \rangle}$.

The orthogonal projection of a vector $u \in \mathcal{X}$ onto the subspace V is denoted by $P_V(u)$ and is defined by

$$P_V(u) = \sum_{j=1}^k \langle u, v_j \rangle v_j.$$

The distance between a vector $u \in \mathcal{X}$ and a subspace V of $\mathcal{X}$ is denoted by $\tilde{d}(u, V)$ and is defined by

$$\begin{aligned} \tilde{d}(u, V) &= \inf\{\|u - v\| : v \in V\} \\ &= \|u - P_V(u)\| \end{aligned}$$

where $P_V(u)$ is the orthogonal projection of the vector u upon V .

Distance or gap between two subspaces U and V of $\mathcal{X}$ (see [8, 14]) is denoted by $\tilde{d}(U, V)$ and is defined by

$$\begin{aligned} \tilde{d}(U, V) &= \sup\{\inf\{\|u - v\| : v \in V\} : u \in U, \|u\| = 1\} \\ &= \sup_{u \in U, \|u\|=1} \|u - P_V(u)\|. \end{aligned}$$

Using the notion of distance between two subspaces, in [14] Manuwarawati et al. introduced the notion of convergence of a sequence of k -dimensional subspaces of an inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$.

Definition 2.1. [14]: Let $\{U_n\}_{n \in \mathbb{N}}$ and V be k -dimensional subspaces of $\mathcal{X}$. Then the sequence $\{U_n\}_{n \in \mathbb{N}}$ is said to converge to the subspaces V , written as $\lim_{n \rightarrow \infty} U_n = V$, if

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{d}(U_n, V) &= 0 \\ \text{i.e. } \lim_{n \rightarrow \infty} \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| &= 0. \end{aligned}$$

On the other hand the usual notion of convergence of real sequences was extended to the notion of statistical convergence [4, 17] using the concept of natural density.

Definition 2.2. [5]: Let $\mathcal{P}$ be a subset of $\mathbb{N}$. The quotient $d_j(\mathcal{P}) = \frac{|\mathcal{P} \cap \{1, 2, \dots, j\}|}{j}$ is called the j^{th} partial density of $\mathcal{P}$, for all $j \in \mathbb{N}$. Now the limit, $d(\mathcal{P}) = \lim_{j \rightarrow \infty} d_j(\mathcal{P})$ (if it exists) is called the natural density or simply density of $\mathcal{P}$.

Definition 2.3. [5]: A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is said to be statistically convergent to $x(\in \mathbb{R})$, if for any $\epsilon > 0$, $d(\mathcal{A}(\epsilon)) = 0$, where $\mathcal{A}(\epsilon) = \{m \in \mathbb{N} : |x_m - x| \geq \epsilon\}$.

Further the notion of statistical convergence was extended to the notion of $\mathcal{I}$ -convergence by Kostyrko et al.[10] using the concept of an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$.

Definition 2.4. [10]: Let $\mathcal{D}$ be a non-empty set. A non-empty class $\mathcal{I}$ of subsets of $\mathcal{D}$ is said to be an ideal in $\mathcal{D}$, provided $\mathcal{I}$ satisfies the conditions (i) $\phi \in \mathcal{I}$, (ii) if $\mathcal{A}, \mathcal{B} \in \mathcal{I}$ then $\mathcal{A} \cup \mathcal{B} \in \mathcal{I}$ and (iii) if $\mathcal{A} \in \mathcal{I}$ and $\mathcal{B} \subset \mathcal{A}$ then $\mathcal{B} \in \mathcal{I}$.

An ideal $\mathcal{I}$ in $\mathcal{D}$ ($\neq \phi$) is called non-trivial if $\mathcal{D} \notin \mathcal{I}$ and $\mathcal{I} \neq \{\phi\}$.

An ideal $\mathcal{I}$ in $\mathcal{D}$ ($\neq \phi$) is called an admissible ideal if $\{z\} \in \mathcal{I}$, $\forall z \in \mathcal{D}$.

Throughout the paper we take $\mathcal{I}$ as a non-trivial admissible ideal in $\mathbb{N}$, unless otherwise mentioned.

Definition 2.5. [10]: A non-empty class $\mathcal{F}$ of subsets of $\mathcal{D}$ ($\neq \phi$) is said to be a filter in $\mathcal{D}$, provided (i) $\phi \notin \mathcal{F}$, (ii) if $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ then $\mathcal{A} \cap \mathcal{B} \in \mathcal{F}$ and (iii) if $\mathcal{A} \in \mathcal{F}$ and $\mathcal{B}$ is a subset of $\mathcal{D}$ such that $\mathcal{B} \supset \mathcal{A}$ then $\mathcal{B} \in \mathcal{F}$.

Let $\mathcal{I}$ be a non-trivial ideal in $\mathcal{D}$. Then $\mathcal{F}(\mathcal{I}) = \{\mathcal{D} - \mathcal{A} : \mathcal{A} \in \mathcal{I}\}$ forms a filter on $\mathcal{D}$, called the filter associated with the ideal $\mathcal{I}$.

Definition 2.6. [10]: A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is said to be $\mathcal{I}$ -convergent to $x \in \mathbb{R}$ if $\forall \epsilon > 0$, the set $\mathcal{A}(\epsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \epsilon\} \in \mathcal{I}$ or in other words for each $\epsilon > 0$, $\exists \mathcal{B}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that $|x_n - x| < \epsilon$, $\forall n \in \mathcal{B}(\epsilon)$. In this case we write $\mathcal{I} - \lim_{n \rightarrow \infty} x_n = x$.

In [15] the concept of statistical convergence of sequences of subspaces of an inner product space was introduced by F. Nuray as follows:

Definition 2.7. [15]: A sequence $\{U_n\}_{n \in \mathbb{N}}$ of k -dimensional subspaces of $\mathcal{X}$, is said to converge statistically to a k -dimensional subspace V of $\mathcal{X}$ if for every $\epsilon > 0$, $d(\mathcal{A}(\epsilon)) = 0$, where

$$\mathcal{A}(\epsilon) = \{n \in \mathbb{N} : \tilde{d}(U_n, V) \geq \epsilon\} = \{n \in \mathbb{N} : \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| \geq \epsilon\}.$$

In this case we write $st - \lim_{n \rightarrow \infty} U_n = V$.

In this paper we extend the above concept of statistical convergence of sequences of subspaces of an inner product space to $\mathcal{I}$ -convergence and study some fundamental properties of this notion. We also establish some equivalent conditions of this convergence notion.

3. MAIN RESULTS

Definition 3.1. : A sequence $\{U_n\}_{n \in \mathbb{N}}$ of k -dimensional subspaces of $\mathcal{X}$ is said to be $\mathcal{I}$ -convergent to a k -dimensional subspace V if $\forall \epsilon > 0$, $\mathcal{A}(\epsilon) = \{n \in \mathbb{N} : \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| \geq \epsilon\} \in \mathcal{I}$.

In other words $\{U_n\}_{n \in \mathbb{N}}$ is said to $\mathcal{I}$ -converges to V if $\forall \epsilon > 0, \exists \mathcal{B}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that

$$\begin{aligned} \tilde{d}(U_n, V) &< \epsilon, & \forall n \in \mathcal{B}(\epsilon) \\ \text{i.e. } \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| &< \epsilon, & \forall n \in \mathcal{B}(\epsilon). \end{aligned}$$

In this case we write $\mathcal{I} - \lim_{n \rightarrow \infty} U_n = V$ and V is called an $\mathcal{I}$ -limit of the sequence $\{U_n\}_{n \in \mathbb{N}}$.

Theorem 3.2. *Let U_n ($n \in \mathbb{N}$) and V be k -dimensional subspaces of an inner product space $\mathcal{X}$. Then $\mathcal{I} - \lim_{n \rightarrow \infty} U_n = V$ if and only if $\mathcal{I} - \lim_{n \rightarrow \infty} \|u_i^{(n)} - P_V(u_i^{(n)})\| = 0, \forall i = 1, 2, \dots, k$.*

Proof. First, let $\mathcal{I} - \lim_{n \rightarrow \infty} U_n = V$ and let $\epsilon > 0$ be given. Then $\exists \mathcal{A}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that

$$d(U_n, V) = \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| < \epsilon, \forall n \in \mathcal{A}(\epsilon). \quad (3.1)$$

Since for each $n \in \mathbb{N}$, $\{u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)}\}$ is an orthonormal basis for U_n , so $\|u_i^{(n)}\| = 1, \forall i = 1, 2, \dots, k$ and hence from (3.1) we have $\forall n \in \mathcal{A}(\epsilon)$,

$$\begin{aligned} \|u_i^{(n)} - P_V(u_i^{(n)})\| &\leq \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| < \epsilon, \forall i = 1, 2, \dots, k \\ \Rightarrow \|u_i^{(n)} - P_V(u_i^{(n)})\| &< \epsilon, \forall i = 1, 2, \dots, k. \end{aligned}$$

Therefore, $\forall i = 1, 2, \dots, k$,

$$\begin{aligned} \mathcal{A}(\epsilon) &\subset \{n \in \mathbb{N} : \|u_i^{(n)} - P_V(u_i^{(n)})\| < \epsilon\} \\ \Rightarrow \{n \in \mathbb{N} : \|u_i^{(n)} - P_V(u_i^{(n)})\| &< \epsilon\} \in \mathcal{F}(\mathcal{I}) \\ \Rightarrow \mathcal{I} - \lim_{n \rightarrow \infty} \|u_i^{(n)} - P_V(u_i^{(n)})\| &= 0. \end{aligned}$$

Conversely, let $\mathcal{I} - \lim_{n \rightarrow \infty} \|u_i^{(n)} - P_V(u_i^{(n)})\| = 0, \forall i = 1, 2, \dots, k$.

Now for $u \in U_n = \text{span}\{u_1^{(n)}, u_2^{(n)}, \dots, u_k^{(n)}\}$ there exists unique scalars $c_1, c_2, \dots, c_k \in \mathbb{F}$ such that $u = c_1 u_1^{(n)} + c_2 u_2^{(n)} + \dots + c_k u_k^{(n)}$. So for $u \in U_n$ with $\|u\| = 1$ we have

$$\begin{aligned} \|u - P_V(u)\| &= \left\| \sum_{i=1}^k c_i u_i^{(n)} - P_V\left(\sum_{i=1}^k c_i u_i^{(n)}\right) \right\| = \left\| \sum_{i=1}^k c_i u_i^{(n)} - \sum_{i=1}^k c_i P_V(u_i^{(n)}) \right\| \\ &= \left\| \sum_{i=1}^k c_i \{u_i^{(n)} - P_V(u_i^{(n)})\} \right\| \leq \sum_{i=1}^k |c_i| \|u_i^{(n)} - P_V(u_i^{(n)})\| \quad (3.2) \end{aligned}$$

As $\|u\| = 1$, therefore $\sum_{i=1}^k |c_i| = c > 0$.

Let $\epsilon > 0$ be given. Then $\forall i = 1, 2, \dots, k$, there exists $B_i(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that

$$\left\| u_i^{(n)} - P_V(u_i^{(n)}) \right\| < \frac{\epsilon}{2c}, \quad \forall n \in B_i(\epsilon).$$

Let $B(\epsilon) = \bigcap_{i=1}^k B_i(\epsilon)$. Then $B(\epsilon) \in \mathcal{F}(\mathcal{I})$. Let $n \in B(\epsilon)$. Then

$$\left\| u_i^{(n)} - P_V(u_i^{(n)}) \right\| < \frac{\epsilon}{2c}, \quad \forall i = 1, 2, \dots, k.$$

Then from (3.2) for $u \in U_n$ with $\|u\| = 1$ we have,

$$\begin{aligned} \|u - P_V(u)\| &\leq \sum_{i=1}^k |c_i| \left\| u_i^{(n)} - P_V(u_i^{(n)}) \right\| \\ &< \frac{\epsilon}{2c} \cdot \sum_{i=1}^k |c_i| = \frac{\epsilon}{2} \end{aligned}$$

$$\Rightarrow \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| \leq \frac{\epsilon}{2} < \epsilon.$$

Therefore, $B(\epsilon) \subset \{n \in \mathbb{N} : \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| < \epsilon\}$. Since $B(\epsilon) \in \mathcal{F}(\mathcal{I})$, so

$\{n \in \mathbb{N} : \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| < \epsilon\} \in \mathcal{F}(\mathcal{I})$. This implies,

$$\mathcal{I} - \lim_{n \rightarrow \infty} \sup_{u \in U_n, \|u\|=1} \|u - P_V(u)\| = 0 \text{ i.e. } \mathcal{I} - \lim_{n \rightarrow \infty} U_n = V. \quad \square$$

Remark 3.3. (i) If $\mathcal{I}$ is an admissible ideal, then the usual convergence of a sequence of k -dimensional subspaces $\{U_n\}_{n \in \mathbb{N}}$ of an inner product space $\mathcal{X}$, implies the $\mathcal{I}$ -convergence of $\{U_n\}_{n \in \mathbb{N}}$ in $\mathcal{X}$.

(ii) If we take $\mathcal{I} = \mathcal{I}_f = \{A \subset \mathbb{N} : A \text{ is a finite subset of } \mathbb{N}\}$, then $\mathcal{I}_f$ -convergence of a sequence $\{U_n\}_{n \in \mathbb{N}}$ of k -dimensional subspaces of an inner product space $\mathcal{X}$ coincides with the usual notion of convergence of sequence $\{U_n\}_{n \in \mathbb{N}}$ of k -dimensional subspaces ([14]).

(iii) Again if we take $\mathcal{I} = \mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then $\mathcal{I}_d$ -convergence of sequence $\{U_n\}_{n \in \mathbb{N}}$ of k -dimensional subspaces of an inner product space $\mathcal{X}$, coincides with the notion of statistical convergence of $\{U_n\}_{n \in \mathbb{N}}$ ([15]).

We now site an example of a sequence of subspaces of an inner product space which is $\mathcal{I}$ -convergent but neither convergent in usual sense nor statistically convergent.

Example 3.4. For each $j \in \mathbb{N}$, let $\mathcal{D}_j = \{2^{j-1}(2s-1) : s \in \mathbb{N}\}$. Then $\mathbb{N} = \bigcup_{j=1}^{\infty} \mathcal{D}_j$, is a decomposition of $\mathbb{N}$, such that $\mathcal{D}_j$'s are infinite subsets of $\mathbb{N}$ and $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ for $i \neq j$.

Let $\mathcal{I} = \{A \subset \mathbb{N} : A \text{ intersects only finitely many } \mathcal{D}'_j s\}$. Then $\mathcal{I}$ is a non-trivial admissible ideal in $\mathbb{N}$. Now we consider the real inner product space $\mathbb{R}^3$ with the standard inner product and let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$. Then $\mathcal{D}_1 = \{2s - 1 : s \in \mathbb{N}\} \in \mathcal{I}$ and $\mathcal{B} = \mathbb{N} - \mathcal{D}_1 \in \mathcal{F}(\mathcal{I})$. We considered the sequence $\{U_n\}_{n \in \mathbb{N}}$ of 1-dimensional subspaces of $\mathbb{R}^3$ defined as follows:

$$U_n = \begin{cases} \text{span}\{u_1^{(n)} = (\frac{1}{n}e_1 + \frac{\sqrt{n^2-1}}{n}e_2)\}, & \text{if } n \in \mathcal{B} \\ \text{span}\{u_1^{(n)} = e_3\}, & \text{if } n \in \mathcal{D}_1. \end{cases}$$

Let $V = \text{span}\{e_2\}$. Let $n \in \mathcal{B}$. Then

$$\begin{aligned} P_V(u_1^{(n)}) &= \langle u_1^{(n)}, e_2 \rangle e_2 = \langle \frac{1}{n}e_1 + \frac{\sqrt{n^2-1}}{n}e_2, e_2 \rangle e_2 \\ &= \left\{ \frac{1}{n} \langle e_1, e_2 \rangle + \frac{\sqrt{n^2-1}}{n} \langle e_2, e_2 \rangle \right\} e_2 = \frac{\sqrt{n^2-1}}{n} e_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u_1^{(n)} - P_V(u_1^{(n)})\| &= \left\| \frac{1}{n}e_1 + \frac{\sqrt{n^2-1}}{n}e_2 - \frac{\sqrt{n^2-1}}{n}e_2 \right\| \\ &= \left\| \frac{1}{n}e_1 \right\| = \frac{1}{n} \|e_1\| = \frac{1}{n}. \end{aligned}$$

Let $\epsilon > 0$ be given. Then there exists $k_o \in \mathbb{N}$ such that

$$\|u_1^{(n)} - P_V(u_1^{(n)})\| = \frac{1}{n} < \epsilon, \forall n > k_o, n \in \mathcal{B}.$$

Again for $n \in \mathcal{D}_1$, $P_V(u_1^{(n)}) = \langle e_3, e_2 \rangle e_2 = 0$ and so

$$\|u_1^{(n)} - P_V(u_1^{(n)})\| = \|e_3\| = 1.$$

Then,

$$\begin{aligned} \{n \in \mathcal{B} : \|u_1^{(n)} - P_V(u_1^{(n)})\| \geq \epsilon\} &\subset \{1, 2, \dots, k_o\} \\ \text{and } \{n \in \mathcal{D}_1 : \|u_1^{(n)} - P_V(u_1^{(n)})\| \geq \epsilon\} &= \begin{cases} \phi, & \text{if } \epsilon > 1 \\ \mathcal{D}_1, & \text{if } 0 < \epsilon \leq 1. \end{cases} \end{aligned}$$

Therefore, $\{n \in \mathbb{N} : \|u_1^{(n)} - P_V(u_1^{(n)})\| \geq \epsilon\} \subset \mathcal{D}_1 \cup \{1, 2, \dots, k_o\}$. Since $\mathcal{I}$ is an admissible ideal of $\mathbb{N}$ and $\mathcal{D}_1 \in \mathcal{I}$, so

$$\begin{aligned} \{n \in \mathbb{N} : \|u_1^{(n)} - P_V(u_1^{(n)})\| \geq \epsilon\} &\in \mathcal{I} \\ \Rightarrow \mathcal{I} - \lim_{n \rightarrow \infty} \|u_1^{(n)} - P_V(u_1^{(n)})\| &= 0. \end{aligned}$$

Then by Theorem 3.2, $\mathcal{I} - \lim_{n \rightarrow \infty} U_n = V$.

Now since $\mathcal{D}_1 \subset \{n \in \mathbb{N} : \|u_1^{(n)} - P_V(u_1^{(n)})\| \geq \frac{1}{2}\}$ and $d(\mathcal{D}_1) = \frac{1}{2}$, so $st - \lim_{n \rightarrow \infty} \|u_1^{(n)} - P_V(u_1^{(n)})\| \neq 0$ and so $st - \lim_{n \rightarrow \infty} U_n \neq V$. Thus the sequence $\{U_n\}_{n \in \mathbb{N}}$ is $\mathcal{I}$ -convergent to V but not statistically convergent to V and hence not convergent in usual sense.

Thus we can say that $\mathcal{I}$ -convergence of sequence of subspaces of an inner product space is a natural generalization of statistical convergence of sequence of subspaces, which is also a generalization of usual notion of convergence of sequences of subspaces.

Theorem 3.5. *Let U_n ($n \in \mathbb{N}$) and V be k -dimensional subspaces of an inner product space $\mathcal{X}$. Then $\mathcal{I} - \lim_{n \rightarrow \infty} \|u_i^{(n)} - P_V(u_i^{(n)})\| = 0$ if and only if $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k |\langle u_i^{(n)}, v_j \rangle|^2 = 1, \forall i = 1, 2, \dots, k$.*

Proof. Fix $i \in \{1, 2, \dots, k\}$. Let $\mathcal{I} - \lim_{n \rightarrow \infty} \|u_i^{(n)} - P_V(u_i^{(n)})\| = 0$. Let $\epsilon > 0$ be given. Then there exists $\mathcal{A}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that

$$\begin{aligned} & \|u_i^{(n)} - P_V(u_i^{(n)})\| < \sqrt{\epsilon}, & \forall n \in \mathcal{A}(\epsilon) \\ \Rightarrow & \|u_i^{(n)} - P_V(u_i^{(n)})\|^2 < \epsilon, & \forall n \in \mathcal{A}(\epsilon) \\ \Rightarrow & \left| \sum_{j=1}^k |\langle u_i^{(n)}, v_j \rangle|^2 - 1 \right| < \epsilon, & \forall n \in \mathcal{A}(\epsilon). \end{aligned}$$

Then,

$$\begin{aligned} & \mathcal{A}(\epsilon) \subset \left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k |\langle u_i^{(n)}, v_j \rangle|^2 - 1 \right| < \epsilon \right\} \\ \Rightarrow & \left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k |\langle u_i^{(n)}, v_j \rangle|^2 - 1 \right| < \epsilon \right\} \in \mathcal{F}(\mathcal{I}) \\ \Rightarrow & \mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k |\langle u_i^{(n)}, v_j \rangle|^2 = 1. \end{aligned}$$

Conversely, let $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k |\langle u_i^{(n)}, v_j \rangle|^2 = 1$. Let $\epsilon > 0$ be given. Then there exists $\mathcal{B}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that

$$\begin{aligned} & \left| \sum_{j=1}^k |\langle u_i^{(n)}, v_j \rangle|^2 - 1 \right| < \epsilon^2, & \forall n \in \mathcal{B}(\epsilon) \\ \Rightarrow & \|u_i^{(n)} - P_V(u_i^{(n)})\|^2 < \epsilon^2, & \forall n \in \mathcal{B}(\epsilon) \\ \Rightarrow & \|u_i^{(n)} - P_V(u_i^{(n)})\| < \epsilon, & \forall n \in \mathcal{B}(\epsilon). \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{B}(\epsilon) &\subset \left\{ n \in \mathbb{N} : \left\| u_i^{(n)} - P_V(u_i^{(n)}) \right\| < \epsilon \right\} \\ \Rightarrow \left\{ n \in \mathbb{N} : \left\| u_i^{(n)} - P_V(u_i^{(n)}) \right\| < \epsilon \right\} &\in \mathcal{F}(\mathcal{I}) \\ \Rightarrow \mathcal{I} - \lim_{n \rightarrow \infty} \left\| u_i^{(n)} - P_V(u_i^{(n)}) \right\| &= 0. \end{aligned}$$

□

Corollary 3.6. Let U_n ($n \in \mathbb{N}$) and V be k -dimensional subspaces of an inner product space $\mathcal{X}$. Then the following statements are equivalent:

- (i) $\mathcal{I} - \lim_{n \rightarrow \infty} U_n = V$
- (ii) $\mathcal{I} - \lim_{n \rightarrow \infty} \left\| u_i^{(n)} - P_V(u_i^{(n)}) \right\| = 0, \forall i = 1, 2, \dots, k$
- (iii) $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1, \forall i = 1, 2, \dots, k.$

Proof. The proof directly follows from Theorem 3.2 and Theorem 3.5. □

Note 3.7. Let U_n ($n \in \mathbb{N}$) and V be k -dimensional subspaces of an inner product space $\mathcal{X}$. For each $i = 1, 2, \dots, k$ we have

$$\left\| P_V(u_i^{(n)}) \right\|^2 = \left\langle P_V(u_i^{(n)}), P_V(u_i^{(n)}) \right\rangle = \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2.$$

Theorem 3.8. Let U_n ($n \in \mathbb{N}$) and V be k -dimensional subspaces of an inner product space $\mathcal{X}$. Then $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1$ if and only if $\mathcal{I} - \lim_{n \rightarrow \infty} \left\| P_V(u_i^{(n)}) \right\| = 1$ for all $i = 1, 2, \dots, k$.

Proof. Fix $i \in \{1, 2, \dots, k\}$. Let $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1$. Let $\epsilon > 0$ be given. Then there exists $\mathcal{A}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that

$$\begin{aligned} &\left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon, \forall n \in \mathcal{A}(\epsilon) \\ \Rightarrow &\left| \left\| P_V(u_i^{(n)}) \right\|^2 - 1 \right| < \epsilon, \forall n \in \mathcal{A}(\epsilon) \\ \Rightarrow &\left| \left\| P_V(u_i^{(n)}) \right\| - 1 \right| < \frac{\epsilon}{1 + \left\| P_V(u_i^{(n)}) \right\|} < \epsilon, \forall n \in \mathcal{A}(\epsilon). \end{aligned}$$

Then, $\left| \left\| P_V(u_i^{(n)}) \right\| - 1 \right| < \epsilon, \forall n \in \mathcal{A}(\epsilon)$. This gives $\mathcal{A}(\epsilon) \subset \left\{ n \in \mathbb{N} : \left| \left\| P_V(u_i^{(n)}) \right\| - 1 \right| < \epsilon \right\}$.

Consequently, $\left\{ n \in \mathbb{N} : \left| \left\| P_V(u_i^{(n)}) \right\| - 1 \right| < \epsilon \right\} \in \mathcal{F}(\mathcal{I})$ and so $\mathcal{I} - \lim_{n \rightarrow \infty} \left\| P_V(u_i^{(n)}) \right\| = 1$.

Conversely, let $\mathcal{I} - \lim_{n \rightarrow \infty} \|P_V(u_i^{(n)})\| = 1$. Let $0 < \epsilon \leq 1$. Then there exists $\mathcal{B}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that

$$\begin{aligned} & \left| \|P_V(u_i^{(n)})\| - 1 \right| < \frac{\epsilon}{3}, \quad \forall n \in \mathcal{B}(\epsilon) \\ & \text{and } \left| \|P_V(u_i^{(n)})\| + 1 \right| < 3, \quad \forall n \in \mathcal{B}(\epsilon). \end{aligned}$$

Then, for each $n \in \mathcal{B}(\epsilon)$

$$\begin{aligned} & \left| \|P_V(u_i^{(n)})\|^2 - 1 \right| = \left(\|P_V(u_i^{(n)})\| + 1 \right) \left| \|P_V(u_i^{(n)})\| - 1 \right| < 3 \frac{\epsilon}{3} = \epsilon \\ \Rightarrow & \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon. \end{aligned}$$

Then, $\mathcal{B}(\epsilon) \subset \left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon \right\}$. Consequently,

$$\left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon \right\} \in \mathcal{F}(\mathcal{I}). \quad (3.3)$$

Now let $\epsilon > 1$. Then,

$$\left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < 1 \right\} \subset \left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon \right\}.$$

Then using (3.3) we have $\left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon \right\} \in \mathcal{F}(\mathcal{I})$. Thus

for any $\epsilon > 0$, $\left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon \right\} \in \mathcal{F}(\mathcal{I})$. Hence, $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1$. □

Corollary 3.9. *Let U_n ($n \in \mathbb{N}$) and V be k -dimensional subspaces of an inner product space $\mathcal{X}$. Then the following statements are equivalent:*

- (i) $\mathcal{I} - \lim_{n \rightarrow \infty} U_n = V$
- (ii) $\mathcal{I} - \lim_{n \rightarrow \infty} \|u_i^{(n)} - P_V(u_i^{(n)})\| = 0, \forall i = 1, 2, \dots, k$
- (iii) $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1, \forall i = 1, 2, \dots, k$
- (iv) $\mathcal{I} - \lim_{n \rightarrow \infty} \|P_V(u_i^{(n)})\| = 1, \forall i = 1, 2, \dots, k$.

Proof. The proof directly follows from Theorem 3.2, Theorem 3.5 and Theorem 3.8. □

Following Gunawan et al. [7], we recall the notion of the standard n -norm of n -vectors in an inner product space $\mathcal{X}$ over $\mathcal{F} (= \mathbb{R} \text{ or } \mathbb{C})$ of dimension k or higher.

The standard n -inner product of n -vectors $x_1, x_2, \dots, x_n$ is defined as

$$G(x_1, x_2, \dots, x_n) = \begin{vmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & \dots & \langle x_1, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \dots & \langle x_2, x_n \rangle \\ \vdots & \vdots & & \vdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \dots & \langle x_n, x_n \rangle \end{vmatrix},$$

here $G(x_1, x_2, \dots, x_n)$ is called the Gramian of the vectors $x_1, x_2, \dots, x_n$.

Clearly the vectors $x_1, x_2, \dots, x_n$ in $\mathcal{X}$ are linearly dependent if and only if the Gramian of the n -vectors i.e. $G(x_1, x_2, \dots, x_n)$ vanishes.

The standard n -norm is defined as

$$\|x_1, x_2, \dots, x_n\| = \sqrt{G(x_1, x_2, \dots, x_n)}.$$

Theorem 3.10. *Let U_n ($n \in \mathbb{N}$) and V be k -dimensional subspaces of an inner product space $\mathcal{X}$. If $\mathcal{I} - \lim_{n \rightarrow \infty} U_n = V$, then $\mathcal{I} - \lim_{n \rightarrow \infty} \|u_i^{(n)}, P_V(u_i^{(n)})\| = 0, \forall i = 1, 2, \dots, k$.*

Proof. Fix $i \in \{1, 2, \dots, k\}$. Then we have,

$$\begin{aligned} \|u_i^{(n)}, P_V(u_i^{(n)})\|^2 &= \begin{vmatrix} \langle u_i^{(n)}, u_i^{(n)} \rangle & \langle u_i^{(n)}, P_V(u_i^{(n)}) \rangle \\ \langle P_V(u_i^{(n)}), u_i^{(n)} \rangle & \langle P_V(u_i^{(n)}), P_V(u_i^{(n)}) \rangle \end{vmatrix} \\ &= \begin{vmatrix} 1 & \sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle \\ \sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle & \sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle^2 \end{vmatrix} \\ &= \left(\sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle^2 \right) \left(1 - \sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle^2 \right) \end{aligned} \quad (3.4)$$

Now, let $\mathcal{I} - \lim_{n \rightarrow \infty} U_n = V$. Then by Corollary 3.6, we have $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle^2 =$

1. Then using Cauchy-Schwarz inequality, we have

$$\sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle^2 \leq \sum_{j=1}^k \|u_i^{(n)}\|^2 \|v_j\|^2 = \sum_{j=1}^k 1 = k.$$

Then,

$$\left(\sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle^2 \right) \left| \sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle^2 - 1 \right| \leq k \left| \sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle^2 - 1 \right|.$$

So from (3.4) we have, $\|u_i^{(n)}, P_V(u_i^{(n)})\|^2 \leq k \left| \sum_{j=1}^k \langle u_i^{(n)}, v_j \rangle^2 - 1 \right|$.

Let $\epsilon > 0$ be given. since $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1$, there exists $\mathcal{A}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that

$$\begin{aligned} & \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \frac{\epsilon^2}{k}, \quad \forall n \in \mathcal{A}(\epsilon) \\ \Rightarrow & \left\| u_i^{(n)}, P_V(u_i^{(n)}) \right\|^2 < \epsilon^2, \quad \forall n \in \mathcal{A}(\epsilon) \\ \Rightarrow & \left\| u_i^{(n)}, P_V(u_i^{(n)}) \right\| < \epsilon, \quad \forall n \in \mathcal{A}(\epsilon). \end{aligned}$$

Then we have,

$$\begin{aligned} & \mathcal{A}(\epsilon) \subset \left\{ n \in \mathbb{N} : \left\| u_i^{(n)}, P_V(u_i^{(n)}) \right\| < \epsilon \right\} \\ \Rightarrow & \left\{ n \in \mathbb{N} : \left\| u_i^{(n)}, P_V(u_i^{(n)}) \right\| < \epsilon \right\} \in \mathcal{F}(\mathcal{I}) \\ \Rightarrow & \mathcal{I} - \lim_{n \rightarrow \infty} \left\| u_i^{(n)}, P_V(u_i^{(n)}) \right\| = 0. \end{aligned}$$

□

The Converse of the above theorem is not true. To show this we consider the following example.

Example 3.11. Let $\mathcal{I}$ be a non-trivial admissible ideal of $\mathbb{N}$. Consider the real vector space $\mathcal{X} = \mathbb{R}^2$ with standard inner product. Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$.

We consider the sequence $\{U_n\}_{n \in \mathbb{N}}$ of one-dimensional subspaces of $\mathcal{X}$, defined as follows:

$$U_n = \text{span}\{u_1^{(n)}\}, \text{ where } u_1^{(n)} = e_1, \quad n \in \mathbb{N} \\ \text{and } V = \text{span}\{e_2\}.$$

Then we have, $P_V(u_1^{(n)}) = \langle u_1^{(n)}, e_2 \rangle e_2 = \langle e_1, e_2 \rangle e_2 = 0$. Then $\left\| u_1^{(n)}, P_V(u_1^{(n)}) \right\| = 0, \forall n \in \mathbb{N}$. Consequently, $\lim_{n \rightarrow \infty} \left\| u_1^{(n)}, P_V(u_1^{(n)}) \right\| = 0$ and hence $\mathcal{I} - \lim_{n \rightarrow \infty} \left\| u_1^{(n)}, P_V(u_1^{(n)}) \right\| = 0$.

Now, $\left\| u_1^{(n)} - P_V(u_1^{(n)}) \right\| = \|e_1 - 0\| = \|e_1\| = 1, \forall n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} \left\| u_1^{(n)} - P_V(u_1^{(n)}) \right\| = 1$ and so $\mathcal{I} - \lim_{n \rightarrow \infty} \left\| u_1^{(n)} - P_V(u_1^{(n)}) \right\| = 1 \neq 0$. Then by Theorem 3.2, we can say that the sequence $\{U_n\}_{n \in \mathbb{N}}$ is not $\mathcal{I}$ -convergent to the subspace V of $\mathcal{X}$ i.e. $\mathcal{I} - \lim_{n \rightarrow \infty} U_n \neq V$.

Theorem 3.12. Let U_n ($n \in \mathbb{N}$) and V be k -dimensional subspaces of an inner product space $\mathcal{X}$. Then $\mathcal{I} - \lim_{n \rightarrow \infty} \left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\| = 0$ if and only if $\mathcal{I} -$

$$\lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1 \text{ for all } i = 1, 2, \dots, k.$$

Proof. Fix $i \in \{1, 2, \dots, k\}$. Then using mathematical induction on k , we have

$$\left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\|^2 = 1 - \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2.$$

First, let $\mathcal{I} - \lim_{n \rightarrow \infty} \left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\| = 0$ and let $\epsilon > 0$ be given. Then there exists $\mathcal{A}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that,

$$\begin{aligned} & \left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\| < \sqrt{\epsilon}, \quad \forall n \in \mathcal{A}(\epsilon) \\ \Rightarrow & \left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\|^2 < \epsilon, \quad \forall n \in \mathcal{A}(\epsilon) \\ \Rightarrow & \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon, \quad \forall n \in \mathcal{A}(\epsilon). \end{aligned}$$

Then we have,

$$\begin{aligned} \mathcal{A}(\epsilon) & \subset \left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon \right\} \\ \Rightarrow & \left\{ n \in \mathbb{N} : \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon \right\} \in \mathcal{F}(\mathcal{I}). \end{aligned}$$

Consequently, $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1$.

Conversely, let $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1$. Let $\epsilon > 0$ be given. Then there exists $\mathcal{B}(\epsilon) \in \mathcal{F}(\mathcal{I})$ such that,

$$\begin{aligned} & \left| \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 - 1 \right| < \epsilon^2, \quad \forall n \in \mathcal{B}(\epsilon) \\ \Rightarrow & \left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\|^2 < \epsilon^2, \quad \forall n \in \mathcal{B}(\epsilon) \\ \Rightarrow & \left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\| < \epsilon, \quad \forall n \in \mathcal{B}(\epsilon). \end{aligned}$$

Then we have,

$$\begin{aligned} \mathcal{B}(\epsilon) & \subset \left\{ n \in \mathbb{N} : \left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\| < \epsilon \right\} \\ \Rightarrow & \left\{ n \in \mathbb{N} : \left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\| < \epsilon \right\} \in \mathcal{F}(\mathcal{I}) \\ \Rightarrow & \mathcal{I} - \lim_{n \rightarrow \infty} \left\| u_i^{(n)}, v_1, v_2, \dots, v_k \right\| = 0. \end{aligned}$$

□

Corollary 3.13. *Let U_n ($n \in \mathbb{N}$) and V be k -dimensional subspaces of an inner product space $\mathcal{X}$. Then the following statements are equivalent:*

- (i) $\mathcal{I} - \lim_{n \rightarrow \infty} U_n = V$
- (ii) $\mathcal{I} - \lim_{n \rightarrow \infty} \|u_i^{(n)} - P_V(u_i^{(n)})\| = 0, \forall i = 1, 2, \dots, k$
- (iii) $\mathcal{I} - \lim_{n \rightarrow \infty} \sum_{j=1}^k \left| \langle u_i^{(n)}, v_j \rangle \right|^2 = 1, \forall i = 1, 2, \dots, k$
- (iv) $\mathcal{I} - \lim_{n \rightarrow \infty} \|P_V(u_i^{(n)})\| = 1, \forall i = 1, 2, \dots, k$
- (v) $\mathcal{I} - \lim_{n \rightarrow \infty} \|u_i^{(n)}, v_1, v_2, \dots, v_k\| = 0, \forall i = 1, 2, \dots, k.$

Proof. The proof directly follows from Theorem 3.2, Theorem 3.5, Theorem 3.8 and Theorem 3.12. $\square$

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