Gulf Journal of Mathematics ISSN: 2309-4966
Vol 17, Issue 1 (2024) 205-224 https://doi.org/10.56947/gjom.v17i1.2081

EXISTENCE AND MULTIPLICITY RESULTS FOR KIRCHHOFF-TYPE SUPERLINEAR PROBLEMS INVOLVING THE FRACTIONAL $p(x)$-LAPLACIAN SATISFYING (C)-CONDITION

MAHMOUD EL AHMADI[1] *, MOHAMED BOUABDALLAH[2] AND ANASS LAMAIZI[3]

Abstract. Our focus in this study revolves around investigating a Kirchhoff problem involving the fractional $p(x)$-Laplacian operator. The purpose is to study the existence and multiplicity of weak solutions for the above problem under appropriate hypotheses on functions f and m. By using the Mountain Pass Theorem with Cerami condition, we show the existence of non-trivial weak solution for the problem without assuming the Ambrosetti-Rabinowitz condition. Furthermore, our second purpose is to determine the precise positive interval of λ for which the above problem admits at least two nontrivial weak solutions. It should be noted that the existence of infinitely many weak solutions is proved by employing the Fountain Theorem.

1. Introduction

In the present article, we study the existence and multiplicity of solutions for the fractional $p(x)$-Kirchhoff problem of the following form

$$\begin{cases} m\left(\displaystyle\int_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}}dxdy\right)(-\Delta_{p(x)})^s u(x) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, $\lambda > 0$ is a real number, $0 < s < 1$, $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ is a continuous function with $sp(x,y) < d$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, $m : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition.

The nonlocal operator

$$(-\Delta_{p(x)})^s \phi(x) := P.V. \int_{\mathbb{R}^d} \frac{|\phi(x)-\phi(y)|^{p(x,y)-2}(\phi(x)-\phi(y))}{|x-y|^{d+sp(x,y)}}dy, \ \ (x,\phi) \in \mathbb{R}^d \times C_0^\infty(\mathbb{R}^d),$$

Date: Received: Dec 7, 2023; Accepted: May 10, 2024.
* Corresponding author.
2020 *Mathematics Subject Classification.* Primary 35R11, 47G20; Secondary 35S15, 35J60.
Key words and phrases. Variational methods, fractional $p(x)$-Laplacian, Kirchhoff-type problem, Cerami condition.

is called the fractional $p(x)$-Laplacian which is a natural generalization of the classical fractional p-Laplacian operator when $p(x,y) = p$ is constant function, where $P.V.$ stands for Cauchy principle value. A typical feature of this operator is the non-locality, i.e., the value of $(-\Delta_{p(x)})^s\phi(x)$ at any point $x \in \Omega$ depends not only on the values of ϕ on the domain Ω, but on the whole space $\mathbb{R}^d$.
Clearly, the nonlocal operator $(-\Delta_{p(x)})^s$ is the fractional version of the $p(x)$-Laplacian operator defined by $\Delta_{p(x)}\phi = \operatorname{div}(|\nabla\phi|^{p(x)-2}\phi)$, which is introduced by U. Kaufmann et al. in [16].

In the recent years, multiple authors have concentrate on the study of fractional $p(x)$-Laplacian problems, and a plethora of results have been obtained, see for instance [3, 4, 5, 18, 23] and the references therein. In [3], E. Azroul et al. considered the problem (1.1) in the particular case when $m \equiv 1$ and $\lambda f(x,u) = \lambda|u|^{r(x)-2}u(x) - |u|^{\bar{p}(x)-2}u(x)$:

$$\begin{cases} (-\Delta_{p(x)})^s u(x) = \lambda|u|^{r(x)-2}u(x) - |u|^{\bar{p}(x)-2}u(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{cases}$$

and by using Ekeland's variational principle, they proved that for any $\lambda > 0$ sufficiently small is an eigenvalue of the above problem.
By means of the symmetric Mountain Pass Theorem and the theory of the fractional Sobolev space, Z. Hao and H. Zheng in [18] investigated the existence and multiplicity of solutions for the problem (1.1) in the case when $f(x,u) = \lambda|u(x)|^{r(x)-2}u(x)$, where r is a continuous function satisfies certain conditions.
Recently, J. Zhang et al. in [23] proved the existence of nontrivial solution for the problem (1.1) in the case $\lambda = 1$ where f satisfies **(AR)**-condition introduced by A. Ambrosetti and P.H. Rabinowitz in [2], that is, there exist $T > 0$ and $\theta > l$ such that

$$0 < \theta F(x,t) \leq tf(x,t), \quad \text{for all } |t| > T \text{ and } a.e. x \in \Omega,$$

where $F(x,t) = \int_0^t f(x,s)ds$ and l is a precise constant.

As is known, the **(AR)**-condition plays a crucial role in the application of the variational method, which is used extensively to ensure the boundedness of the Palais-Smale sequences and the energy functional has a mountain pass geometry, but in many cases, cannot be satisfied. For example, in the case $p(x,y) = p$, the function $f(x,t) = |t|^{l-2}t\ln(1+|t|)$ does not satisfy **(AR)**-condition but it satisfies certain weaker conditions.

It is well known that the problem (1.1) is similar to the stationary problem introduced by Kirchhof in [17]:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|dx\right)\frac{\partial^2 u}{\partial x^2} = 0.$$

More precisely, Kirchhoff proposed this model as an extension of D'Alembert's classical wave equation, taking into account the effects of variations in string length during vibration. Furthermore, S. Woinowsky-Krieger in [22] considered the Kirchhoff-type evolution equation:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u = g(x,u),$$

which is a model for the deviation of an extensible beam.

Throughout this article, we assume that the continuous function $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ satisfies the following assumptions:

(P_1) $1 < p^- \le p(x, y) \le p^+ < +\infty$, where

$$p^- := \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y) \quad \text{and} \quad p^+ := \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y).$$

(P_2) p is symmetric, that is, $p(x, y) = p(y, x)$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.

In addition, we make the subsequent hypotheses on m and f:

(m_0) There exists $m^* > 0$ such that $\inf_{t\in\mathbb{R}_+} m(t) \ge m^*$.

(m_1) There exists $\mu \in \left[1, \frac{p_s^*(x)}{p^+}\right]$ such that for all $t \in \mathbb{R}_+$,

$$tm(t) \le \mu \widehat{m}(t),$$

where

$$\widehat{m}(t) = \int_0^t m(\tau)d\tau \quad \text{and} \quad p_s^*(x) := \frac{dp(x, x)}{d - sp(x, x)}.$$

(H_0) There exist $C > 0$ and $s \in C_+(\Omega)$ such that

$$|f(x, t)| \le C(1 + |t|^{s(x)-1}) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

(H_1) $1 < p^- \le p^+ < s^- \le s^+ < p_s^*(x) \quad$ for all $x \in \Omega$.

(H_2) $\liminf_{|t|\to\infty} \frac{F(x,t)}{|t|^{\mu p^+}} = +\infty$ uniformly a.e $x \in \Omega$,

where $F(x, t) = \int_0^t f(x, s)ds$ and μ comes from (m_1) above.

(H_3) There exist $c_1, r_1 \ge 0$ and $\theta > \frac{d}{sp^-}$ such as

$$|F(x, t)|^\theta \le c_1|t|^{\theta p^-}\mathcal{F}(x, t),$$

for all $(x, t) \in \Omega \times \mathbb{R}$, $|t| \ge r_1$ and $\mathcal{F}(x, t) := \frac{1}{\mu p^+} f(x, t)t - F(x, t) \ge 0$.

(H_4) $f(x, t) = \circ(|t|^{p^+-1})$ as $t \to 0$ uniformly for a.e. $x \in \Omega$.

Remark 1.1. The conditions (m_0) and (m_1) implies the following inequality:

$$\widehat{m}(t) \le \widehat{m}(1)(1 + t^\mu), \quad \text{for all } t \in \mathbb{R}_+.$$

Indeed, if $0 \le t \le 1$, since the function $\widehat{m}(.)$ is strictly increasing on $[0, +\infty)$, then $\widehat{m}(t) \le \widehat{m}(1)$, and if $t \ge 1$, we consider the function $K(t) = \frac{\widehat{m}(t)}{t^\mu}$. By direct calculation, it is clear that $K(.)$ is strictly decreasing on $[1, +\infty)$, then $\widehat{m}(t) \le \widehat{m}(1)t^\mu$.

Let us consider $\phi : E_0 \to \mathbb{R}$ the Euler functional corresponding to problem (1.1) which is defined as follows:

$$\phi(u) = \widehat{m}\left(\delta_{s,p}(u)\right) - \lambda \int_\Omega F(x, u)dx$$

where

$$\delta_{s,p}(u) = \int_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}} dxdy$$

and E_0 will be defined in preliminaries.
Then, it follows from the hypothesis (H_0) that the functional $\phi \in C^1(E_0, \mathbb{R})$, and its Fréchet derivative is

$$\langle \phi'(u), v\rangle = m\left(\delta_{s,p}(u)\right) \int_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+sp(x,y)}} dxdy$$
$$- \lambda \int_\Omega f(x,u)vdx,$$

for any $u, v \in E_0$. It is clear that any critical point of ϕ is a weak solution of problem (1.1).

Now, we present the main results of this paper.

Theorem 1.2. *Suppose that $(P_1)-(P_2)$, $(m_0)-(m_1)$ and $(H_0)-(H_4)$ hold. If $s^- > \mu p^+$, then for any $\lambda > 0$, the problem (1.1) has at least one nontrivial weak solution in E_0.*

Theorem 1.3. *Suppose that $(P_1)-(P_2)$, $(m_0)-(m_1)$ and $(H_0)-(H_3)$ hold. Then, there exists a positive constant λ^* such that the problem (1.1) admits at least two distinct weak solutions in E_0 for each $\lambda \in (0, \lambda^*)$.*

Theorem 1.4. *Suppose that $(P_1)-(P_2)$, $(m_0)-(m_1)$ and $(H_0)-(H_3)$ hold. Furthermore, we assume that*
(H_5) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$. Then, for any $\lambda > 0$, the problem (1.1) has infinitely many weak solutions (u_n) such that $\phi(u_n) \to +\infty$ as $n \to +\infty$.

2. Preliminaries

To study problem (1.1), we need some definitions and basic properties of generalized variable exponents Lebesgue spaces. For more details, see [10, 13, 14, 15] and references therein.

For $p \in C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}) : p^- := \inf_{x\in\overline{\Omega}} p(x) > 1\}$, we designate the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty \right\}$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\left\{ \alpha > 0 : \int_\Omega \left|\frac{u(x)}{\alpha}\right|^{p(x)} dx \leq 1 \right\}.$$

Proposition 2.1. [11]

(1) *The Lebesgue space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is defined as the dual space $L^{q(x)}(\Omega)$, where $q(x)$ is conjugate to $p(x)$, i.e., $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$*

and $v \in L^{q(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uv\ dx\right| \leq \left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}.$$

(2) *If* $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$, *for all* $x \in \overline{\Omega}$, *then* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is continuous.*

Now, we give the main properties of the fractional Sobolev space with variable exponent that we will use in the rest of this paper. Let $\overline{p}(x) = p(x,x)$ for all $x \in \overline{\Omega}$.

Via the Gagliardo approach, we designate the fractional Sobolev space with variable exponent by

$$W^{s,p(x,y)}(\Omega) = \Big\{u \in L^{\overline{p}(x)}(\Omega) : \int\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\alpha^{p(x,y)}|x-y|^{d+sp(x,y)}}dxdy, \text{ for some } \alpha > 0\Big\},$$

endowed with the norm

$$\|u\|_{s,p(x,y)} = [u]_{s,p(x,y)} + |u|_{\overline{p}(x)},$$

where

$$[u]_{s,p(x,y)} = \inf\Big\{\alpha > 0 : \int\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\alpha^{p(x,y)}|x-y|^{d+sp(x,y)}}dxdy \leq 1\Big\}$$

is a Gagliardo seminorm with variable exponent. With this norm, $W^{s,p(x,y)}(\Omega)$ is separable and reflexive Banach space [6].

For studying the problem (1.1), we need to define new variable order fractional Sobolev spaces with variable exponents. For more details we refer to [7]. For this, we set

$$E = \Big\{u : \mathbb{R}^d \to \mathbb{R} : u\mid_{\partial\Omega} \in L^{\overline{p}(x)}(\Omega) \text{ and } \int\int_R \frac{|u(x)-u(y)|^{p(x,y)}}{\alpha^{p(x,y)}|x-y|^{d+sp(x,y)}}dxdy, \text{ for some } \alpha > 0\Big\},$$

where $R = \mathbb{R}^d \times \mathbb{R}^d \setminus (\Omega^c \times \Omega^c)$. The space E is equipped with the following norm:

$$\|u\|_E = [u]_E + |u|_{\overline{p}(x)},$$

where

$$[u]_E = \inf\Big\{\alpha > 0 : \int\int_R \frac{|u(x)-u(y)|^{p(x,y)}}{\alpha^{p(x,y)}|x-y|^{d+sp(x,y)}}dxdy \leq 1\Big\}.$$

Since $\Omega \times \Omega \subset R$ and $\Omega \times \Omega \neq R$, then, the norms $\| \,.\, \|_{s,p(x,y)}$ and $\| \,.\, \|_E$ are different. This renders that $W^{s,p(x,y)}(\Omega)$ not sufficient for studying the problem (1.1). For this, we need to define the following linear subspace of E as

$$E_0 = \Big\{u \in E : u = 0 \ a.e. \text{ in } \ \mathbb{R}^d \setminus \Omega\Big\},$$

equipped with the norm

$$\begin{aligned}\|u\|_{E_0} &= \inf\Big\{\alpha > 0 : \int\int_R \frac{|u(x)-u(y)|^{p(x,y)}}{\alpha^{p(x,y)}|x-y|^{d+sp(x,y)}}dxdy \leq 1\Big\}\\ &= \inf\Big\{\alpha > 0 : \int\int_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^{p(x,y)}}{\alpha^{p(x,y)}|x-y|^{d+sp(x,y)}}dxdy \leq 1\Big\}.\end{aligned}$$

With this norm, E_0 is separable and reflexive Banach space [8].

Recall the following embedding results.

Proposition 2.2. *(see [7]) Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain and $0 < s < 1$. Suppose that $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ (P_1) and (P_2) such that $sp(x,y) < d$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Let $r \in C_+(\overline{\Omega})$ with*

$$1 < r(x) < p_s^*(x) = \frac{dp(x,x)}{d - sp(x,x)} \quad \text{for all} \ \ x \in \overline{\Omega}.$$

Then, here exists a positive constant $C^* = C^*(d,s,p,r)$ such that

$$|v|_{r(x)} \leq C^* \|v\|_{E_0}, \quad \text{for any} \ \ v \in E_0.$$

Moreover, there is a compact embedding $E_0 \hookrightarrow L^{r(x)}(\Omega)$.

We notice that the norms $\|.\|_E$ and $\|.\|_{E_0}$ are equivalent on E_0. We consider the modular functional $\rho_{E_0} : E_0 \to \mathbb{R}$ defined by

$$\rho_{E_0}(u) = \int\int_{\mathbb{R}^{2d}} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy.$$

The relationship between ρ_{E_0} and $|.|_{E_0}$ is established by the next result.

Proposition 2.3. *(see [8, 24]) For $u \in E_0$, $(u_n) \subset E_0$ and $\beta > 0$, we have*

(1) *For $u \neq 0$, $\|u\|_{E_0} = \beta \Longleftrightarrow \rho_{E_0}(\frac{u}{\beta}) = 1$;*
(2) $\|u\|_{E_0} < 1 \, (= 1, > 1) \Longleftrightarrow \rho_{E_0}(u) < 1 \, (= 1, > 1)$;
(3) $\|u\|_{E_0} > 1 \Longrightarrow \|u\|_{E_0}^{p^-} \leq \rho_{E_0}(u) \leq \|u\|_{E_0}^{p^+}$;
(4) $\|u\|_{E_0} < 1 \Longrightarrow \|u\|_{E_0}^{p^+} \leq \rho_{E_0}(u) \leq \|u\|_{E_0}^{p^-}$;
(5) (a) $\lim_{n\to+\infty} \|u_n\|_{E_0} = 0 \Leftrightarrow \lim_{n\to+\infty} \rho_{E_0}(u_n) = 0$;
 (b) $\lim_{n\to+\infty} \|u_n\|_{E_0} = +\infty \Leftrightarrow \lim_{n\to+\infty} \rho_{E_0}(u_n) = +\infty$.

Let $A : E_0 \to E_0^*$ be defined by

$$\langle A(u), v\rangle = \int\int_{R^{2d}} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{d+sp(x,y)}} dxdy, \quad (2.1)$$

for all $u, v \in E_0$, where E_0^* denotes the dual space of E_0. Similar to Lemma 4.2 in [6], we have the following Proposition.

Proposition 2.4. (1) *A is bounded and strictly monotone.*
(2) *A satisfies the (S_+)-property, i.e.,*

$$\text{if } u_n \rightharpoonup u \text{ in } E_0 \text{ and } \overline{\lim_{n\to+\infty}} \langle A(u_n) - A(u), u_n - u\rangle \leq 0, \text{ then } u_n \to u \text{ in } E_0.$$

(3) *A is a homeomorphism.*

Now, we can define the weak solution of problem (1.1).

Definition 2.5. We call $u \in E_0$ a **weak solution** of problem (1.1) if

$$m\left(\delta_{s,p}(u)\right) \int_{\mathbb{R}^{2d}} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{d+sp(x,y)}} dxdy - \lambda \int_\Omega f(x,u) v dx = 0,$$

for all $v \in E_0$, where

$$\delta_{s,p}(u) = \int_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}} dxdy.$$

Next, we recall the definition of the compactness condition of the Cerami which was introduced by G. Cerami in [9].

Definition 2.6. Let $(X, \|.\|)$ be a real Banach space and $\phi \in C^1(X, \mathbb{R})$. We say that ϕ satisfies the Cerami condition (we denote $(C)-$condition) in X, if any sequence $(u_n) \subset X$ such that

$$(\phi(u_n)) \text{ is bounded} \quad \text{and} \quad \|\phi'(u_n)\|_{X^*}(1+\|u_n\|) \to 0 \text{ as } n \to +\infty$$

has a strong convergent subsequence in X.

Remark 2.7.
(1) It is clear from the above definition that if ϕ satisfies the (PS)-condition, then it satisfies the (C)-condition. However, there are functionals that satisfy the (C)-condition but do not satisfy the (PS)-condition (see [9]). Consequently, the (C)-condition is weaker than the (PS)-condition.
(2) The (C)-condition and the (PS)-condition are equivalent if ϕ is bounded from below (see [20]).

Now, we present the following theorems which will play a fundamental role in the proof of main Theorems.

Theorem 2.8. *(see [2]) Let $(X, \|.\|)$ be a real Banach space, $\phi \in C^1(X, \mathbb{R})$ satisfies the (C)-condition, $\phi(0) = 0$, and the following conditions hold:*
(1) *There exist positive constant ρ and α such that $\phi(u) \geq \alpha$ for any $u \in X$ with $\|u\| = \rho$.*
(2) *There exists a function $e \in X$ such that $\|e\| > \rho$ and $\phi(e) \leq 0$.*

Then, the functional ϕ has a critical value $c \geq \alpha$, that is, there exists $u \in X$ such that $\phi(u) = c$ and $\phi'(u) = 0$ in X^*.

Theorem 2.9. [19, Theorem 2.6] *Let X be a real Banach space, $\varphi, \psi : X \to \mathbb{R}$ be two continuously Gateaus differentiable functionals such that φ is bounded from below and $\varphi(0) = \psi(0) = 0$. Let $\eta > 0$ be fixed, and it is assumed that for each*

$$\lambda \in \Gamma_0 := \left(0, \frac{\eta}{\sup\limits_{u \in \varphi^{-1}((-\infty,\eta))} \psi(u)}\right),$$

the functional $\phi = \varphi - \lambda\psi$ satisfies the (C)-condition for all $\lambda > 0$ and is unbounded from below. Then, for each $\lambda \in \Gamma_0$, the functional ϕ admits two distinct critical points.

In order to prove our multiplicity result, we will use the following Fountain Theorem [21, Theorem 3.6]. Let X be a real, reflexive, and separable Banach space, then there exist $\{e_j\}_{j\in\mathbb{N}} \subset X$ and $\{e_j^*\}_{j\in\mathbb{N}} \subset X^*$ such that

$$X = \overline{span\{e_j : j = 1, 2, ...\}}, \quad X^* = \overline{span\{e_j^* : j = 1, 2, ...\}},$$

and $\langle e_i^*, e_j\rangle = 1$ if $i = j$, $\langle e_i^*, e_j\rangle = 0$ if $i \neq j$.
We denote $X_j = span\{e_j\}$,$Y_k = \bigoplus_{j=1}^{k} X_j$ and $Z_k = \overline{\bigoplus_{j=k}^{+\infty} X_j}$.

Theorem 2.10. *Assume that $(X, \|.\|)$ is a Banach space, $\phi \in C^1(X,\mathbb{R})$ is an even functional fulfills the $(C)-$ condition and for every $k \in \mathbb{N}$, there exists $\gamma_k > \eta_k > 0$ such that*
(A_1) $b_k := \inf\{\phi(u) : u \in Z_k, \|u\| = \eta_k\} \to +\infty$ as $k \to +\infty$;
(A_2) $c_k := \max\{\phi(u) : u \in Y_k, \|u\| = \gamma_k\} \leq 0$.
Then, ϕ has a sequence of critical values which tends to $+\infty$.

All through this paper, the letters C_i, c_i, $i = 1, 2, 3...$ denote positive constants which may change from line to line.

3. Proofs of main results

First of all, we are going to show that the functional ϕ fulfills the (C)-condition.

Lemma 3.1. *If assumptions $(P_1)-(P_2)$, $(m_0)-(m_1)$ and $(H_0)-(H_3)$ hold, then for all $\lambda > 0$, the functional ϕ satisfies the (C)-condition.*

Proof. Let $(u_n) \subset E_0$ be a Cerami sequence for ϕ, namely,

$$(\phi(u_n)) \ \text{ is bounded } \ \text{and} \quad \|\phi'(u_n)\|_{E_0^*}(1 + \|u_n\|_{E_0}) \to 0, \tag{3.1}$$

which imply that

$$\sup|\phi(u_n)| \leq M \ \text{ and } \quad \langle\phi'(u_n), u_n\rangle = \circ_n(1), \tag{3.2}$$

where $\lim\limits_{n\to+\infty} \circ_n(1) = 0$ and $M > 0$.
We need to prove the boundedness of the sequence (u_n) in E_0. To this end, assume the contrary that the sequence (u_n) is unbounded in E_0. Without loss of generality, we can assume that $\|u_n\|_{E_0} > 1$. By virtue of (m_1) and (H_3), for n large enough, we have

$$\begin{aligned}
M + 1 &\geq \phi(u_n) - \frac{1}{\mu p^+}\langle\phi'(u_n), u_n\rangle \\
&= \widehat{m}\left(\delta_{s,p}(u_n)\right) - \lambda\int_\Omega F(x, u_n)dx - \frac{1}{\mu p^+}m\left(\delta_{s,p}(u_n)\right)\int_{\mathbb{R}^{2d}} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{d+sp(x,y)}}dxdy \\
&+ \frac{\lambda}{\mu p^+}\int_\Omega f(x, u_n)u_n dx \\
&\geq \frac{1}{\mu}m\left(\delta_{s,p}(u)\right)\delta_{s,p}(u_n) - \frac{1}{\mu p^+}m\left(\delta_{s,p}(u_n)\right)\int_{\mathbb{R}^{2d}} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{d+sp(x,y)}}dxdy \\
&+ \lambda\int_\Omega \left(\frac{1}{\mu p^+}f(x, u_n)u_n - F(x, u_n)\right)dx \\
&\geq \frac{1}{\mu p^+}m\left(\delta_{s,p}(u)\right)\int_{\mathbb{R}^{2d}} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{d+sp(x,y)}}dxdy \\
&- \frac{1}{\mu p^+}m\left(\delta_{s,p}(u_n)\right)\int_{\mathbb{R}^{2d}} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{d+sp(x,y)}}dxdy + \lambda\int_\Omega \mathcal{F}(x, u_n)dx.
\end{aligned}$$

Then, we obtain

$$M+1 \geq \lambda \int_{\Omega} \mathcal{F}(x,u_n)dx, \tag{3.3}$$

where

$$\mathcal{F}(x,u_n) := \frac{1}{\mu p^+} f(x,u_n)u_n - F(x,u_n)$$

and

$$\delta_{s,p}(u_n) = \int_{\mathbb{R}^{2d}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}} dxdy.$$

Using (m_0) and (m_1), it follows

$$\begin{aligned} M &\geq \phi(u_n) \\ &= \widehat{m}\left(\delta_{s,p}(u_n)\right) - \lambda \int_{\Omega} F(x,u_n)dx \\ &\geq \frac{m^*}{\mu p^+} \int_{\mathbb{R}^{2d}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy - \lambda \int_{\Omega} F(x,u_n)dx \\ &\geq \frac{m^*}{\mu p^+} \rho_{E_0}(u_n) - \lambda \int_{\Omega} F(x,u_n)dx. \end{aligned}$$

Because $\|u_n\|_{E_0} > 1$, using Proposition 2.3, we can obtain

$$M \geq \frac{m^*}{\mu p^+} \|u_n\|_{E_0}^{p^-} - \lambda \int_{\Omega} F(x,u_n)dx. \tag{3.4}$$

Since $\|u_n\|_{E_0} \to +\infty$ as $n \to +\infty$, we deduce that

$$\int_{\Omega} F(x,u_n)dx \geq \frac{m^*}{\lambda \mu p^+} \|u_n\|_{E_0}^{p^-} - \frac{M}{\lambda} \to +\infty \quad \text{as } n \to +\infty. \tag{3.5}$$

Furthermore, since $\widehat{m}(.)$is strictly increasing on $[0,+\infty[$, we have

$$\begin{aligned} \phi(u_n) &= \widehat{m}\left(\delta_{s,p}(u_n)\right) - \lambda \int_{\Omega} F(x,u_n)dx \\ &\leq \widehat{m}\left(\frac{1}{p^-} \int_{\mathbb{R}^{2d}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\right) - \lambda \int_{\Omega} F(x,u_n)dx. \end{aligned}$$

Then, we obtain

$$\phi(u_n) + \lambda \int_{\Omega} F(x,u_n)dx \leq \widehat{m}\left(\frac{1}{p^-} \int_{\mathbb{R}^{2d}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\right). \tag{3.6}$$

In view of condition (H_2), there exists a constant T_1 such that

$$F(x,t) > |t|^{\mu p^+} \quad \text{for all } \ x \in \Omega \ \text{ and } \ |t| > T_1.$$

Since $F(x,.)$ is continuous on $[-T_1,T_1]$, there exists a constant $C_0 > 0$ such that

$$|F(x,t)| \leq C_0 \quad \text{for all } \ (x,t) \in \Omega \times [-T_1,T_1].$$

Then, a real number K can be chosen such that $F(x,t) \geq K$ for all $(x,t) \in \Omega \times \mathbb{R}$. Hence,

$$\frac{F(x,u_n) - K}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\right)} \geq 0, \tag{3.7}$$

for all $(x,n) \in \Omega \times \mathbb{N}$.

Put $\beta_n = \frac{u_n}{\|u_n\|_{E_0}}$, so $\|\beta_n\|_{E_0} = 1$. Up to subsequences, for some $\beta \in E_0$, we have

$$\begin{aligned} \beta_n &\rightharpoonup \beta && \text{in } E_0, \\ \beta_n &\to \beta && \text{in } L^{\mu p^+}(\Omega), \\ \beta_n &\to \beta && \text{in } L^{r(x)}(\Omega) \ \text{ for all } \ 1 \leq r(x) < p_s^*(x) \ \text{ with } \ x \in \Omega, \\ \beta_n(x) &\to \beta(x) && \text{a.e. in } \Omega. \end{aligned} \tag{3.8}$$

Define the set $\Omega_0 = \{x \in \Omega : \beta(x) \neq 0\}$. Since $\|u_n\|_{E_0} \to +\infty$ as $n \to +\infty$, we have

$$|u_n(x)| = |\beta_n(x)| \|u_n\|_{E_0} \to +\infty,$$

for any $x \in \Omega_0$.
By (H_2) and Remark 1.1, for all $x \in \Omega_0$, we have

$$\begin{aligned} \frac{F(x,u_n)}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\right)} &\geq \frac{F(x,u_n)}{\widehat{m}(1)\left(1 + \frac{1}{(p^-)^\mu}\left(\int_{\mathbb{R}^{2d}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\right)^\mu\right)} \\ &\geq \frac{F(x,u_n)}{\widehat{m}(1)\left(\|u_n\|_{E_0}^{\mu p^+} + \frac{1}{(p^-)^\mu}\|u_n\|_{E_0}^{\mu p^+}\right)} \\ &\geq \frac{F(x,u_n)}{\widehat{m}(1)\left(1 + \frac{1}{(p^-)^\mu}\right)\|u_n\|_{E_0}^{\mu p^+}} \\ &\geq \frac{F(x,u_n)}{\widehat{m}(1)\left(1 + \frac{1}{(p^-)^\mu}\right)|u_n(x)|^{\mu p^+}} |\beta_n(x)|^{\mu p^+} \to +\infty, \end{aligned}$$

which imply that

$$\frac{F(x,u_n)}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\right)} \to +\infty \quad \text{as} \quad n \to +\infty. \tag{3.9}$$

Thus, $|\Omega_0| = 0$. In fact, suppose by contradiction that $|\Omega_0| \neq 0$. Uisng (3.5), (3.6), (3.9) and Fatou's lemma, we get

$$\begin{aligned}
\frac{1}{\lambda} &= \liminf_{n\to+\infty} \frac{\int_\Omega F(x,u_n)\,dx}{\phi(u_n)+\lambda\int_\Omega F(x,u_n)dx} \\
&\geq \liminf_{n\to+\infty}\int_\Omega \frac{F(x,u_n)}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}dxdy\right)}dx \\
&\geq \liminf_{n\to+\infty}\int_{\Omega_0} \frac{F(x,u_n)}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}dxdy\right)}dx \\
&\quad - \limsup_{n\to+\infty}\int_{\Omega_0} \frac{K}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}dxdy\right)}dx \\
&= \liminf_{n\to+\infty}\int_{\Omega_0} \frac{F(x,u_n)-K}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}dxdy\right)}dx \\
&\geq \int_{\Omega_0}\liminf_{n\to+\infty} \frac{F(x,u_n)-K}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}dxdy\right)}dx \\
&\geq \int_{\Omega_0}\liminf_{n\to+\infty} \frac{F(x,u_n(x))}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}dxdy\right)}dx \\
&\quad - \int_{\Omega_0}\limsup_{n\to+\infty} \frac{K}{\widehat{m}\left(\frac{1}{p^-}\int_{\mathbb{R}^{2d}}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}dxdy\right)}dx \\
&= +\infty,
\end{aligned}$$

which is a contradiction. Therefore, $\beta(x) = 0$ for a.e. $x \in \Omega$.
From (3.4) and (3.8), respectively, we can deduce that

$$\beta_n \to 0 \text{ in } L^{r(x)}(\Omega) \text{ for all } 1 \leq r(x) < p_s^*(x) \text{ with } x \in \overline{\Omega}, \quad \beta_n(x) \to 0 \quad \text{a.e. in } \Omega, \tag{3.10}$$

and

$$0 < \frac{m^*}{\lambda\mu p^+} \leq \limsup_{n\to+\infty}\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|_{E_0}^{p^-}}\,dx. \tag{3.11}$$

Using (H_0) and (H_1), we get

$$
\begin{aligned}
\int_{\{0\le|u_n(x)|\le r_1\}} \frac{|F(x,u_n)|}{\|u_n\|_{E_0}^{p^-}}\,dx &\le C\int_{\{0\le|u_n(x)|\le r_1\}} \frac{|u_n|+\frac{1}{s(x)}|u_n|^{s(x)}}{\|u_n\|_{E_0}^{p^-}}\,dx\\
&\le C\frac{|u_n|_1}{\|u_n\|_{E_0}^{p^-}}+\frac{C}{s^-}\int_{\{0\le|u_n(x)|\le r_1\}} |u_n|^{s(x)-p^-}|\beta_n|^{p^-}\,dx\\
&\le CC_0\frac{\|u_n\|}{\|u_n\|_{E_0}^{p^-}}+\frac{Cr_1^{s-p^-}}{s^-}\int_{\{0\le|u_n(x)|\le r_1\}} |\beta_n|^{p^-}\,dx\\
&\to 0, \quad \text{as } \ n\to+\infty
\end{aligned}
\tag{3.12}
$$

where $C_0>0$, s is either s^+ or s^- and r_1 comes from (H_3).
Put $\theta'=\frac{\theta}{\theta-1}$. Since $\theta>\frac{d}{sp^-}$, it follows that $\theta' p^- < p_s^*(x)$.
On the other hand, by virtue of hypothesis (H_3), (3.3) and (3.10), we deduce

$$
\begin{aligned}
\int_{\{|u_n(x)|\ge r_1\}} \frac{|F(x,u_n)|}{\|u_n\|_{E_0}^{p^-}}\,dx &\le 2\left[\int_{\{|u_n(x)|\ge r_1\}} \left(\frac{|F(x,u_n)|}{|u_n|^{p^-}}\right)^{\theta} dx\right]^{\frac{1}{\theta}}\left[\int_{\{|u_n(x)|\ge r_1\}} |\beta_n|^{\theta' p^-}\,dx\right]^{\frac{1}{\theta'}}\\
&\le 2c_1^{\frac{1}{\theta}}\left[\int_{\{|u_n(x)|\ge r_1\}} \mathcal{F}(x,u_n)dx\right]^{\frac{1}{\theta}}\left[\int_{\{|u_n(x)|\ge r_1\}} |\beta_n|^{\theta' p^-}\,dx\right]^{\frac{1}{\theta'}}\\
&\le 2c_1^{\frac{1}{\theta}}\left[\int_{\Omega} \mathcal{F}(x,u_n)dx\right]^{\frac{1}{\theta}}\left[\int_{\Omega} |\beta_n|^{\theta' p^-}\,dx\right]^{\frac{1}{\theta'}}\\
&\le 2c_1^{\frac{1}{\theta}}\left(\frac{M+1}{\lambda}\right)^{\frac{1}{\theta}}\left[\int_{\Omega} |\beta_n|^{\theta' p^-}\,dx\right]^{\frac{1}{\theta'}}\\
&\to 0 \quad \text{as } \ n\to+\infty.
\end{aligned}
\tag{3.13}
$$

Then, combining this with (3.12), we obtain

$$
\int_{\Omega} \frac{|F(x,u_n)|}{\|u_n\|_{E_0}^{p^-}}\,dx = \int_{\{0\le|u_n(x)|\le r_1\}} \frac{|F(x,u_n)|}{\|u_n\|_{E_0}^{p^-}}\,dx+\int_{\{|u_n(x)|\ge r_1\}} \frac{|F(x,u_n)|}{\|u_n\|_{E_0}^{p^-}}\,dx \longrightarrow 0, \ \text{as } n\to+\infty,
$$

which is a contradiction to (3.11). Thus, (u_n) is bounded in E_0.

Finally, we need to prove that any (C)-sequence has a convergent subsequence. Let $(u_n)E_0$ be a (C)-sequence. Then, (u_n) is bounded in E_0. Passing to the limit, if necessary, to a subsequence, from Proposition 2.2, we have

$$
u_n \rightharpoonup u \quad \text{in } E_0, \quad u_n\to u \ \text{ in } L^{r(x)}(\Omega), 1\le r(x)<p_s^*(x) \quad \text{and} \quad u_n(x)\to u(x) \ \text{ a.e. } x\in\Omega.
\tag{3.14}
$$

It is easy to check from (H_0), (3.14) and Hölder's inequality that

$$\Big|\int_\Omega f(x,u_n)(u_n-u)\;dx\Big| \leq C|1+|u_n|^{s(x)-1}|_{s'(x)}|u_n-u|_{s(x)} \longrightarrow 0 \quad \text{as } \; n\to+\infty, \tag{3.15}$$

where $\frac{1}{s(x)}+\frac{1}{s'(x)}=1$.
On the other hand, using (m_0), we obtain

$$m\left(\delta_{s,p}(u_n)\right)=m\left(\int_{\mathbb{R}^{2d}}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}}dxdy\right)\longrightarrow k\neq 0, \quad \text{as } \; n\to+\infty. \tag{3.16}$$

Next, since $u_n\rightharpoonup u$, from (3.1), we have

$$\langle \phi'(u_n),u_n-u\rangle \longrightarrow 0, \quad \text{as } \; n\to+\infty. \tag{3.17}$$

Then

$$\langle \phi'(u_n),u_n-u\rangle = m\left(\delta_{s,p}(u_n)\right)\langle A(u_n),u_n-u\rangle-\lambda\int_\Omega f(x,u_n)(u_n-u)dx\to 0, \quad \text{as } \; n\to+\infty,$$

where A is given in (2.1).
Finally, the combination of (3.15), (3.16) and (3.17) implies

$$\langle A(u_n),u_n-u\rangle \longrightarrow 0 \quad \text{as } \; n\to+\infty.$$

Since the operator A satisfies the (S_+) property in view of Proposition 2.4, we can obtain that $u_n\to u$ in E_0. The proof is complete. □

Proof of Theorem 1.2

By Lemma 3.1, ϕ satisfies the $(C)-$condition. According to definition of ϕ, we have $\phi(0)=0$. Then, to apply Theorem 2.8, we will show that ϕ has a mountain pass geometry.
First, we affirm that there exists $\alpha,\rho>0$ such that

$$\phi(u)\geqslant\alpha, \quad \forall u\in E_0 \quad \text{with} \;\; \|u\|_{E_0}=\rho. \tag{3.18}$$

In fact, by $(m_0)-(m_1)$ and Proposition 2.3, for $\|u|_{E_0}<1$, we obtain

$$\begin{aligned}\phi(u) &\geq \frac{m^*}{\mu p^+}\rho_{E_0}(u)-\lambda\int_\Omega F(x,u)dx\\ &\geq \frac{m^*}{\mu p^+}\|u\|_{E_0}^{p^+}-\lambda\int_\Omega F(x,u)dx.\end{aligned} \tag{3.19}$$

Since $\mu p^+<p_s^*(x)$ and $s(x)<p_s^*(x)$ for all $x\in\overline{\Omega}$ in view of conditions (m_1) and (H_1) respectively , we have from Proposition 2.2 that $E_0\hookrightarrow L^{\mu p^+}(\Omega)$ and $E_0\hookrightarrow L^{s(x)}(\Omega)$. So, there exist $c_6,c_7>0$ such that

$$|u|_{\mu p^+}\leq c_6\|u\|_{E_0} \quad \text{and} \quad |u|_{s(x)}\leq c_7\|u\|_{E_0}, \;\; \forall u\in E_0.$$

Let $\varepsilon>0$ such that $\varepsilon c_6^{p^+}<\frac{m^*}{\mu p^+}$. Using (H_0) and (H_4), it follows that

$$F(x,t)\leq\varepsilon|t|^{p^+}+C(\varepsilon)|t|^{s(x)}, \quad \forall(x,t)\in\Omega\times\mathbb{R}. \tag{3.20}$$

Therefore, by (3.19) and (3.20), for $\|u\|_{E_0} < 1$ sufficiently small, we get

$$\begin{aligned}\phi(u) &\geqslant \frac{m^*}{\mu p^+}\|u\|_{E_0}^{p^+} - \varepsilon \int_\Omega |u|^{p^+} dx - C(\varepsilon)\int_\Omega |u|^{s(x)} dx \\ &\geqslant \frac{m^*}{\mu p^+}\|u\|_{E_0}^{p^+} - \varepsilon c_6^{p^+}\|u\|_{E_0}^{p^+} - C(\varepsilon) c_7^{s^-}\|u\|_{E_0}^{s^-} \\ &\geq \left(\frac{m^*}{\mu p^+} - \varepsilon c_6^{p^+}\right)\|u\|_{E_0}^{p^+} - C(\varepsilon) c_7^{s^-}\|u\|_{E_0}^{s^-}.\end{aligned}$$

Since $s^- > p^+$, then there exists $\alpha, \rho > 0$ such that

$$\phi(u) \geqslant \alpha, \quad \forall u \in E_0 \quad \text{with} \quad \|u\|_{E_0} = \rho.$$

Next, we affirm that there exists $e \in E_0$ with $\|u\|_{E_0} > \rho$ such that

$$\phi(e) < 0. \tag{3.21}$$

In fact, From (H_2), it follows that for every $k > 0$, there exists a constant T_k such that

$$F(x,t) > k|t|^{\mu p^+} \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad |t| > T_k.$$

Since $F(x,.)$ is continuous on $[-T_k, T_k]$, there exists a constant $C_0 > 0$ such that

$$|F(x,t)| \leq C_0 \quad \text{for all} \quad (x,t) \in \Omega \times [-T_k, T_k].$$

Thus,

$$F(x,t) \geq k|t|^{\mu p^+} - C_0, \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}.$$

Let $w \in E_0 \setminus \{0\}$ such that $\|w\|_{E_0} = 1$ and $l > 1$ be large enough, using Remark 1.1 and the above inequality, we obtain

$$\begin{aligned}\phi(lw) &= \widehat{m}\left(\int_{\mathbb{R}^{2d}} \frac{|lw(x) - lw(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}} dxdy\right) - \lambda \int_\Omega F(x, lw)\, dx \\ &\leq \widehat{m}(1)\left(\int_{\mathbb{R}^{2d}} \frac{|lw(x) - lw(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}} dxdy\right)^\mu - \lambda k l^{\mu p^+}\int_\Omega |w|^{\mu p^+}\, dx + \lambda C_0|\Omega| \\ &\leq |l|^{\mu p^+}\left(\frac{\widehat{m}(1)}{(p^-)^\mu} - \lambda k\int_\Omega |w|^{\theta p^+}\, dx\right) + \lambda C_0|\Omega|\end{aligned}$$

where μ comes from (m_1).
As

$$\frac{\widehat{m}(1)}{(p^-)^\mu} - \lambda k\int_\Omega |w|^{\theta p^+}\, dx < 0,$$

for k large enough, we deduce

$$\phi(lw) \to -\infty, \quad \text{as} \quad l \to +\infty.$$

Thus, there exists $t_0 > 1$ and $e = t_0 w \in E_0 \setminus \overline{B_\rho(0)}$ such that $\phi(e) < 0$. The proof of Theorem 1.2 is complete.

Proof of Theorem 1.3

To prove Theorem 1.3, we need the following lemma.

Lemma 3.2. *Let hypotheses* (m_0) *and* (m_1) *be satisfied. Then, the functional* I *defined by*

$$\begin{aligned} I(u) &= \widehat{m}\left(\delta_{s,p}(u)\right) \\ &= \widehat{m}\left(\int_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}} dxdy\right), \end{aligned}$$

is bounded from below.

Proof. Using (m_0) and (m_1), we obtain

$$\begin{aligned} I(u) &= \widehat{m}\left(\int_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}} dxdy\right) \\ &\geq \frac{m^*}{\mu p^+}\left(\int_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\right) \\ &\geq \frac{m^*}{\mu p^+}\rho_{E_0}(u). \end{aligned}$$

According to Proposition 2.3 (5), we deduce that I is coercive. consequently, I is bounded from below. ☐

Now, we return to the proof of Theorem 1.3. Let $\varphi = I$ and $\psi(.) = \int_\Omega F(x,.)dx$. Obviously, by Lemma 3.2, the functional I is bounded from below. In view of the definition of I and ψ, we have $I(0) = \psi(0) = 0$. According to Lemma 3.1, ϕ satisfies the (C)- condition. To apply Theorem 2.9, it suffices to check that
(a_1) the functional ϕ is unbounded from below;
(a_2) for given $\eta = 1$, there exists $\lambda^* > 0$ such that

$$\sup_{u \in I^{-1}((-\infty,\eta))} \psi(u) < \frac{1}{\lambda^*}.$$

• Verification of (a_1). From (H_2), it follows that for every $k > 0$, there exists a constant T_k such that

$$F(x,t) > k|t|^{\mu p^+} \quad \text{for all} \;\; x \in \Omega \;\; \text{and} \;\; |t| > T_k.$$

Since $F(x,.)$ is continuous on $[-T_k, T_k]$, there exists a constant $C_0 > 0$ such that

$$|F(x,t)| \leq C_0 \quad \text{for all} \;\; (x,t) \in \Omega \times [-T_k, T_k].$$

Thus,

$$F(x,t) \geq k|t|^{\mu p^+} - C_0, \quad \text{for all} \;\; (x,t) \in \Omega \times \mathbb{R}. \tag{3.22}$$

Let $w \in E_0 \setminus \{0\}$ and $l > 1$ be large enough, using Remark 1.1 and the above inequality, we obtain

$$\begin{aligned} \phi(lw) &= \widehat{m}\left(\int_{\mathbb{R}^{2d}} \frac{|lw(x)-lw(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}} dxdy\right) - \lambda \int_\Omega F(x,lw)\; dx \\ &\leq \widehat{m}(1)\left(\int_{\mathbb{R}^{2d}} \frac{|lw(x)-lw(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}} dxdy\right)^\mu - \lambda k l^{\mu p^+} \int_\Omega |w|^{\mu p^+}\; dx + \lambda C_0|\Omega| \\ &\leq |l|^{\mu p^+}\left(\frac{\widehat{m}(1)}{(p^-)^\mu}\left(\int_{\mathbb{R}^{2d}} \frac{|w(x)-w(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\right)^\mu - \lambda k \int_\Omega |w|^{\theta p^+}\; dx\right) + \lambda C_0|\Omega| \end{aligned}$$

where μ comes from (m_1).
As

$$\frac{\widehat{m}(1)}{(p^-)^{\mu}}\left(\int_{\mathbb{R}^{2d}}\frac{|w(x)-w(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}dxdy\right)^{\mu}-\lambda k\int_{\Omega}|w|^{\theta p^+}\,dx<0,$$

for k large enough, we deduce

$$\phi(lw)\to-\infty,\qquad\text{as }\ l\to+\infty.$$

Consequently, ϕ is unbounded from below.
• Verification of (a_2). By virtue of (H_0) and Proposition 2.3, we get

$$\begin{aligned}\psi(u)&=\int_{\Omega}F(x,u)dx\\&\leq C\int_{\Omega}(|u|+\frac{1}{s(x)}|u|^{s(x)})\,dx\\&\leq Cc_4\|u\|_{E_0}+\frac{1}{s^-}\max\{|u|^{s^+}_{s(x)},|u|^{s^-}_{s(x)}\}\\&\leq Cc_4\|u\|_{E_0}+\max\{C_s^{-s^+},C_s^{-s^-}\}\max\{\|u\|^{s^+}_{E_0},\|u\|^{s^-}_{E_0}\}\end{aligned}\tag{3.23}$$

where $c_4>0$ and $C_s=\inf\limits_{u\in E_0\setminus\{0\}}\frac{\|u\|_{E_0}}{|u|_{s(x)}}$.

On the other hand, for all $u\in I^{-1}((-\infty,1))$, according to (m_0), (m_1) and Proposition 2.3, we get

$$\begin{aligned}\mu p^+\geq\mu p^+I(u)=\mu p^+\widehat{m}\left(\int_{\mathbb{R}^{2d}}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}}dxdy\right)\\&\geq\frac{m^*\mu p^+}{\mu p^+}\left(\int_{\mathbb{R}^{2d}}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}dxdy\right)\\&\geq m^*\|u_n\|^r_{E_0}\\&\geq\min\{m^*,\mu\}\|u_n\|^r_{E_0},\end{aligned}$$

where r is either p^- or p^+.
Because $\mu p^+>\mu\geq\min\{m^*,\mu\}$, we have $\frac{\mu p^+}{\min\{m^*,\mu\}}>1$.
Thus, we obtain

$$\|u\|_{E_0}\leq\max\left\{\left(\frac{\mu p^+}{\min\{m^*,\mu\}}\right)^{\frac{1}{p^-}},\left(\frac{\mu p^+}{\min\{m^*,\mu\}}\right)^{\frac{1}{p^+}}\right\}=\left(\frac{\mu p^+}{\min\{m^*,\mu\}}\right)^{\frac{1}{p^-}}.$$

In view of (H_1) and (3.23), we have

$$\begin{aligned}\sup_{u\in I^{-1}((-\infty,\eta))}\psi(u)&\leq Cc_4\left(\frac{\mu p^+}{\min\{m^*,\mu\}}\right)^{\frac{1}{p^-}}+\max\{C_s^{-s^+},C_s^{-s^-}\}\max\{\|u\|^{\frac{s^+}{p^-}}_{E_0},\|u\|^{\frac{s^-}{p^-}}_{E_0}\}\\&=Cc_4\left(\frac{\theta q^+}{\min\{m^*,\mu\}}\right)^{\frac{1}{p^-}}+\max\{C_s^{-s^+},C_s^{-s^-}\}\left(\frac{\mu p^+}{\min\{m^*,\mu\}}\right)^{\frac{s^+}{p^-}}.\end{aligned}\tag{3.24}$$

Let us denote

$$\lambda^* := \left(Cc_4 \left(\frac{\mu p^+}{\min\{m^*, \mu\}} \right)^{\frac{1}{p^-}} + \max\{C_s^{-s^+}, C_s^{-s^-}\} \left(\frac{\mu p^+}{\min\{m^*, \mu\}} \right)^{\frac{s^+}{p^-}} \right)^{-1}.$$

Taking into account (3.24), we assert that

$$\sup_{u \in I^{-1}((-\infty, \eta))} \psi(u) \leq \frac{1}{\lambda^*} < \frac{1}{\lambda}.$$

Finally, all assumptions of Theorem 2.9 are satisfied. Then, for all $\lambda \in (0, \lambda^*) \subset \Gamma_0$, the problem (1.1) admits at least two distinct weak solutions in E_0. This completes the proof.

Proof of Theorem 1.4

To prove Theorem 1.4, we need the following auxiliary lemmas.

Lemma 3.3. *(see [12]) For $s \in C_+(\overline{\Omega})$ such that $s(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Let*

$$\delta_k = \sup\{|u|_{s(x)} : \|u\|_{E_0} = 1, u \in Z_k\}.$$

Then, $\lim_{k \to +\infty} \delta_k = 0$.

Lemma 3.4. *(see [12]) For all $s \in C_+(\overline{\Omega})$ and $u \in L^{s(x)}(\Omega)$, there exists $y \in \Omega$ such that*

$$\int_\Omega |u|^{s(x)}\, dx = |u|_{s(x)}^{s(y)}. \tag{3.25}$$

Now, we return to the proof of Theorem 1.4. To this end, based on the Fountain Theorem 2.10, we will show that the problem (1.1) possesses infinitely many weak solutions with unbounded energy. Evidently, according to (H_5), ϕ is an even functional. By Lemma 3.1, we know that ϕ satisfies the (C)- condition. Then, to prove Theorem 1.4, it only remains to verify the following assertions:
(A_1) $b_k := \inf\{\phi(u) : u \in Z_k, \|u\|_{E_0} = \eta_k\} \to +\infty$ as $k \to +\infty$,
(A_2) $c_k := \max\{\phi(u) : u \in Y_k, \|u\|_{E_0} = \gamma_k\} \leq 0$.

(A_1) For any $u \in Z_k$ such that $\|u\|_{E_0} = \eta_k > 1$. It follows from (m_0), (m_1), (H_0), Proposition 2.2 and Lemma 3.4 that

$$\begin{aligned}
\phi(u) &= \widehat{m}\left(\int_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{d+sp(x,y)}}dxdy\right) - \lambda\int_\Omega F(x,u)dx \\
&\geq \frac{m^*}{\mu p^+}\|u\|_{E_0}^{p^-} - \lambda C\int_\Omega |u|\ dx - \lambda C\int_\Omega \frac{|u|^{s(x)}}{s(x)}dx \\
&\geq \frac{m^*}{\mu p^+}\|u\|_{E_0}^{p^-} - \lambda c_5\|u\|_{E_0} - \frac{\lambda c_6}{s^-}|u|_{s(x)}^{s(y)} \\
&\geq \begin{cases} \frac{m^*}{\mu p^+}\|u\|_{E_0}^{p^-} - \lambda c_5\|u\|_{E_0} - \frac{\lambda c_6}{s^-} & \text{if } |u|_{s(x)} \leq 1 \\ \frac{m^*}{\mu p^+}\|u\|_{E_0}^{p^-} - \lambda c_5\|u\|_{E_0} - \frac{\lambda c_6}{s^-}(\delta_k\|u\|_{E_0})^{s^+} & \text{if } |u|_{s(x)} > 1 \end{cases} \\
&\geq \frac{m^*}{\mu p^+}\|u\|_{E_0}^{p^-} - \lambda c_5\|u\|_{E_0} - \frac{\lambda c_6}{s^-}(\delta_k\|u\|_{E_0})^{s^+} - \frac{\lambda c_6}{s^-} \\
&\geq \eta_k^{p^-}\left(\frac{m^*}{\mu p^+} - \frac{\lambda c_6}{s^-}\delta_k^{s^+}\eta_k^{s^+-p^-}\right) - \lambda c_5\eta_k - \frac{\lambda c_6}{s^-}.
\end{aligned}$$

We fix η_k as follows

$$\eta_k = \left(\frac{\lambda\mu c_6}{m^*}\delta_k^{s^+}\right)^{\frac{1}{p^- - s^+}}.$$

Then, we obtain

$$\phi(u) \geq \frac{m^*}{\mu}\eta_k^{p^-}\left(\frac{1}{p^+} - \frac{1}{s^-}\right) - \lambda c_5\eta_k - \frac{\lambda c_6}{s^-}.$$

By (H_1), since $p^+ < s^-$ and $\delta_k \to 0$ as $k \to +\infty$, we conclude that $\eta_k \to +\infty$ as $k \to +\infty$.
Finally,

$$\phi(u) \to +\infty \text{ as } \ k \to +\infty.$$

Hence, (A_1) holds.

(A_2) Following the lines of the proof of Theorem 1.4 of [1]. Suppose by contradiction that (A_2) does not hold for some given k, then there exists a sequence (u_n) in Y_k such that

$$\|u_n\|_{E_0} \to +\infty \quad \text{and} \quad \phi(u_n) \geq 0. \tag{3.26}$$

Put $v_n = \frac{u_n}{\|u_n\|_{E_0}}$, then $\|v_n\|_{E_0} = 1$. Because Y_k has finite dimension, there exists $v \in Y_k \setminus \{0\}$ such that up to a subsequence,

$$\|v_n - v\|_{E_0} \to 0 \quad \text{and} \quad v_n(x) \to v(x) \quad a.e. \text{ in } \Omega.$$

If $v(x) \neq 0$, then $|u_n(x)| \to +\infty$. Hence, from (H_2), we have

$$\lim_{n\to+\infty}\frac{F(x,u_n(x))}{\|u_n\|_{E_0}^{\mu p^+}} = \lim_{n\to+\infty}\frac{F(x,u_n(x))}{|u_n(x)|^{\mu p^+}}|v_n(x)|^{\mu p^+} = +\infty,$$

for all $x \in W_0 = \{x \in \Omega : \omega(x) \neq 0\}$. According to proof of Lemma 3.1, we have

$$\int_{W_0} \frac{F(x, u_n(x))}{\|u_n\|_{E_0}^{\mu p^+}} \, dx \to +\infty \quad \text{as} \quad n \to +\infty. \tag{3.27}$$

Consequently, by Remark 1.1, for $\|u_n\|_{E_0} > 1$, we get

$$\begin{aligned}
\phi(u_n) &\leq \widehat{m}(1) \left(1 + \frac{1}{(p^-)^\mu} \left(\int_{\mathbb{R}^{2d}} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} dxdy\right)^\mu\right) - \int_{W_0} F(x, u_n)dx \\
&\leq \widehat{m}(1) + \frac{\widehat{m}(1)}{(p^-)^\mu} \|u_n\|_{E_0}^{\mu p^+} - \int_{W_0} F(x, u_n)dx \\
&\leq \widehat{m}(1) + \|u_n\|_{E_0}^{\mu p^+} \left(\frac{\widehat{m}(1)}{(p^-)^\mu} - \int_{W_0} \frac{F(x, u_n)}{\|u_n\|_{E_0}^{\mu p^+}} dx\right).
\end{aligned}$$

Using (3.27), we can deduce that

$$\phi(u_n) \longrightarrow -\infty \quad \text{as} \quad n \to +\infty,$$

which is a contradiction to (3.26). Finally, the assertion (A_2) is also valid. This completes the proof.

Acknowledgement. The authors wish to express their thanks to the referee for his or her very careful reading of the paper, giving valuable comments and helpful suggestions.

References

1. *Alves, C. O. & Liu, S. (2010). On superlinear $p(x)$-Laplacian equations in $\mathbb{R}^N$. Nonlinear Analysis: Theory, Methods & Applications, 73(8), 2566-2579.* https://doi.org/10.1016/j.na.2010.06.033
2. *Ambrosetti, A., & Rabinowitz, P. H. (1973). Dual variational methods in critical point theory and applications. Journal of functional Analysis, 14(4), 349-381.* https://doi.org/10.1016/0022-1236(73)90051-7
3. *Azroul, E., Benkirane, A., & Shimi, M. (2019). Eigenvalue problems involving the fractional $p(x)$-Laplacian operator, Adv. Oper. Theory, 4(2), 539-555.* https://doi.org/10.15352/aot.1809-1420
4. *Azroul, E., Benkirane, A., Shimi, M., & Srati, M. (2021). On a class of fractional $p(x)$-Kirchhoff type problems. Applicable Analysis, 100(2), 383-402.* https://doi.org/10.1080/00036811.2019.1603372
5. *Bahrouni, A. (2018). Comparison and sub-supersolution principles for the fractional $p(x)$-Laplacian. Journal of Mathematical Analysis and Applications, 458(2), 1363-1372.* https://doi.org/10.1016/j.jmaa.2017.10.025
6. *Bahrouni, A., & Rădulescu, V. (2018). On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent. Discrete Contin. Dyn. Syst. Ser. S, 11(3), 379-389.* https://doi.org/10.3934/dcdss.2018021
7. *Biswas, R., & Tiwari, S. (2021). On a class of Kirchhoff-Choquard equations involving variable-order fractional $p(\cdot)$-Laplacian and without Ambrosetti-Rabinowitz type condition. Topological Methods in Nonlinear Analysis, 58(2), 403-439.* https://doi.org/10.12775/TMNA.2020.072
8. *Biswas, R., & Tiwari, S. (2021). Variable order nonlocal Choquard problem with variable exponents. Complex Variables and Elliptic Equations, 66(5), 853-875.* https://doi.org/10.1080/17476933.2020.1751136

9. *Cerami, G. (1978). An existence criterion for the critical points on unbounded manifolds. Istit. Lombardo Accad. Sci. Lett. Rend. A., 112(2), 332-336.*
10. *Cruz-Uribe, D. V., & Fiorenza, A. (2013). Variable Lebesgue spaces: Foundations and harmonic analysis. Springer Science & Business Media.*
11. *Edmunds, D. E., & Rákosník, J. (1992). Density of smooth functions in $W^{k,p(x)}(\Omega)$. Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences, 437(1899), 229-236.* https://doi.org/10.1098/rspa.1992.0059
12. *El Ahmadi, M., Ayoujil, A., & Berrajaa, M. (2023). Existence and multiplicity of solutions for a class of double phase variable exponent problems with nonlinear boundary condition. Advanced Mathematical Models & Applications, 8, 401-414.*
13. *Fan, X. (2009). Remarks on eigenvalue problems involving the $p(x)$-Laplacian. Journal of Mathematical Analysis and Applications. 352(1), 85-98.* https://doi.org/10.1016/j.jmaa.2008.05.086
14. *Fan, X. L., & Zhang, Q. H. (2003). Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Analysis: Theory, Methods & Applications, 52(8), 1843-1852.* https://doi.org/10.1016/S0362-546X(02)00150-5
15. *Fan, X., & Zhao, D. (2001). On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. Journal of Mathematical Analysis and Applications, 263(2), 424-446.* https://doi.org/10.1006/jmaa.2000.7617
16. *Kaufmann, U., Rossi, J. D., Vidal, & R. E. (2017). Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians. Electronic Journal of Qualitative Theory of Differential Equation, 2017(76), 1-10.* https://doi.org/10.14232/ejqtde.2017.1.76
17. *Kirchhoff, G. (1883). Mechanik. Teubner, Leipzig, Germany.*
18. *Hao, Z., & Zheng, H. (2023). Existence and multiplicity of solutions for fractional $p(x)$-Kirchhoff-type problems. Electronic Research Archive, 31(6), 3309-3321.* https://doi.org/10.3934/era.2023167
19. *Lee, J., & Kim, Y. H. (2016). Multiplicity results for nonlinear Neumann boundary value problems involving p-Laplace type operators. Boundary Value Problems, 2016, 1-25.* https://doi.org/10.1186/s13661-016-0603-x
20. *Motreanu, D., Motreanu, V. V., & Papageorgiou, N. S. (2014). Topological and variational methods with applications to nonlinear boundary value problems, 1(7), Springer, New York.* https://doi.org/10.1007/978-1-4614-9323-5
21. *Willem, M. (1996). Minimax Theorems, Birkhäuser, Boston.*
22. *Woinowsky-Krieger, S. (1950). The effect of an axial force on the vibration of hinged bars. Journal of applied Mechanics, 17(1), 35-36.* https://doi.org/10.1115/1.4010053
23. *Zhang, J., Yang, D., & Wu, Y. (2021). Existence results for a Kirchhoff-type equation involving fractional $p(x)$-Laplacian. AIMS Mathematics, 6(8), 8390-8403.* https://doi.org/10.3934/math.2021486
24. *Zhang, C., & Zhang, X. (2020). Renormalized solutions for the fractional $p(x)$-Laplacian equation with L^1 data. Nonlinear Analysis,* ***190****, 111610.* https://doi.org/10.1016/j.na.2019.111610

[1] Department of Mathematics, Mohammed First University, Oujda, Morocco
Email address: elahmadi.mahmoud@ump.ac.ma

[2] Department of Mathematics, Mohammed First University, Oujda, Morocco
Email address: bouabdallah.mohamed@ump.ac.ma

[3] Department of Mathematics, Mohammed First University, Oujda, Morocco
Email address: lamaizi.anass@ump.ac.ma