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FUZZY BI-IDEALS AND FUZZY QUASI-IDEALS IN ORDERED SEMIRINGS

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ABSTRACT. The notions of fuzzy bi-ideals and fuzzy quasi-ideals in ordered semirings are introduced and some of their characterizations are obtained through regularity criterion.

1. INTRODUCTION

Semirings [2] which provide a common generalization of rings and distributive lattices arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, graph theory, automata theory, mathematical modelling and parallel computation systems etc.(for example, see [2], [3]). Semirings have also been proved to be an important algebraic tool in theoretical computer science, see for instance [3], for some detail and example. Many of the semirings, such as $\mathbb{N}$, have an order structure in addition to their algebraic structure and indeed the most interesting results concering them make use of the interplay between these two structures. Ideals of semirings play an important role in the structure theory of ordered semirings and useful for many purposes. Motivated by Kehayopulu and Tsingelis[4], in this paper we take an attempt to study how similar is the theory in terms of fuzzy set which was introduced by Zadeh [7], since nowadays fuzzy research concerns standardization, axiomatization, extensions to lattice-valued fuzzy sets, critical comparison of the different so-called soft computing models that have been launched during the past three decennia for the representation and processing of incomplete information [5].
Basically in this paper, we define fuzzy bi-ideals and fuzzy quasi-ideals in ordered semirings and study some of their related properties. At the end of the paper we characterize regular ordered semiring in terms of fuzzy bi-ideals and fuzzy quasi-ideals.

2. PRELIMINARIES

We recall the following definitions for subsequent use.

Definition 2.1. A hemiring (resp. semiring) is a non-empty set S on which operations addition and multiplication have been defined such that $(S, +)$ is a

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commutative monoid with identity 0, $(S, \cdot)$ is a semigroup (resp. monoid with identity 1_S), multiplication distributes over addition from either side, $1_S \neq 0$ and $0s = 0 = s0$ for all $s \in S$.

Definition 2.2. An ordered semiring is a semiring S equipped with a partial order $\leq$ such that the operation is monotonic and constant 0 is the least element of S.

Definition 2.3. A left ideal I of semiring S is a nonempty subset of S satisfying the following conditions:

(i) If $a, b \in I$ then $a + b \in I$;
(ii) If $a \in I$ and $s \in S$ then $sa \in I$;

A right ideal of S is defined in an analogous manner and an ideal of S is a nonempty subset which is both a left ideal and a right ideal of S.

Definition 2.4. A subset A of a semiring S is called a bi-ideal if A is closed under addition and $ASA \subseteq A$.
A subset A of a semiring S is called a quasi-ideal of S if A is closed under addition and $SA \cap AS \subseteq A$.

Now we recall the definition and example of ordered ideal from [1]

Definition 2.5. A left (resp. right) ideal I of S is called a left (resp. right) ordered ideal, if for any $a \in S$, $b \in I$, $a \leq b$ implies $a \in I$ (i.e. $(I] \subseteq I$). I is called an ordered ideal of S if it is both a left and a right ordered ideal of S.

Example 2.6. Let $S = ([0, 1], \vee, \cdot, 0)$ where [0,1] is the unit interval $a \vee b = \max\{a, b\}$ and $a \cdot b = (a + b - 1) \vee 0$ for $a, b \in [0, 1]$. Then it is easy to verify that S equipped with the usual ordering $\leq$ is an ordered semiring and $I = [0, \frac{1}{2}]$ is an ordered ideal of S.

Definition 2.7. [7] Let S be a non-empty set. A mapping $\mu : S \to [0, 1]$ is called a fuzzy subset of S.

Definition 2.8. The union and intersection of two fuzzy subsets μ and σ of a set S, denoted by $\mu \cup \sigma$ and $\mu \cap \sigma$ respectively, are defined by

$$(\mu \cup \sigma)(x) = \max\{\mu(x), \sigma(x)\} \text{ for all } x \in S$$
$$(\mu \cap \sigma)(x) = \min\{\mu(x), \sigma(x)\} \text{ for all } x \in S.$$

3. Main results

Throughout this paper unless otherwise mentioned S denote the ordered semiring and χ_S denote its characteristic function.

Definition 3.1. [6] Let μ be a non empty fuzzy subset of an ordered semiring S (i.e. $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a fuzzy left ideal [resp. fuzzy right ideal] of S if

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geq \mu(y)$ [resp. $\mu(xy) \geq \mu(x)$] and
(iii) $x \leq y$ implies $\mu(x) \geq \mu(y)$.

for all $x, y \in S$.
A fuzzy ideal of an ordered semiring S is a non empty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S.

Definition 3.2. [6] Let μ and ν be two fuzzy subsets of an ordered semiring S and $x,\ y,\ z \in S$. We define composition of μ and ν as follows:

$$\begin{aligned} \mu \circ_1 \nu(x) &= \sup_{x \leq yz}\{\min\{\mu(y), \nu(z))\}\} \\ &= 0,\ \text{if } x \text{ cannot be expressed as } x \leq yz \end{aligned}$$

Definition 3.3. A fuzzy subset μ of an ordered semiring S is called fuzzy bi-ideal if for all $x, y \in S$ we have

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$
(iii) $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$
(iv) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$

Definition 3.4. A fuzzy subset μ of an ordered semiring S is called fuzzy quasi-ideal if for all $x, y \in S$ we have

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $(\mu o_1 \chi_S) \cap (\chi_S o_1 \mu) \subseteq \mu$
(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y)$

Example 3.5. Let $S = \{0, a, b, c\}$ with the ordered relation $0 \prec c \prec b \prec a$. Define operations on S by following:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Then $(S, \oplus, \odot)$ forms an ordered semiring.
Now if we define a fuzzy subset μ of S by $\mu(0) = 1,\ \mu(c) = 0.3,\ \mu(b) = 0.2,\ $ and $\mu(a) = 0.1$, then μ will be a fuzzy bi-ideal of S.

Example 3.6. Let $S = \{0, a, b, c\}$ with the ordered relation $0 \prec a \prec b \prec c$. Define operations on S by following:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

Then $(S, \oplus, \odot)$ forms an ordered semiring.
Now if we define a fuzzy subset μ of S by $\mu(0) = 1,\ \mu(a) = 0.6,\ \mu(b) = 0.5,\ $ and $\mu(c) = 0.3$, then μ will be a fuzzy bi-ideal as well as a fuzzy quasi-ideal of S.

Proposition 3.7. *Intersection of a non-empty collection of fuzzy bi-ideals of S is also a fuzzy bi-ideal of S.*

Proof. Let $\{\mu_i : i \in I\}$ be a non-empty family of fuzzy bi-ideals of S and $x,\ y,\ z \in S$. Then

$$\begin{aligned}(\bigcap_{i\in I}\mu_i)(x+y) &= \inf_{i\in I}\{\mu_i(x+y)\} \geq \inf_{i\in I}\{\min\{\mu_i(x),\mu_i(y)\}\}\\ &= \min\{\inf_{i\in I}\mu_i(x), \inf_{i\in I}\mu_i(y)\} = \min\{(\bigcap_{i\in I}\mu_i)(x), (\bigcap_{i\in I}\mu_i)(y)\}.\end{aligned}$$

$$\begin{aligned}(\bigcap_{i\in I}\mu_i)(xy) &= \inf_{i\in I}\{\mu_i(xy)\} \geq \inf_{i\in I}\min\{\mu_i(x),\mu_i(y)\}\\ &= \min\{(\bigcap_{i\in I}\mu_i)(x), (\bigcap_{i\in I}\mu_i)(y)\}.\end{aligned}$$

and

$$\begin{aligned}(\bigcap_{i\in I}\mu_i)(xyz) &= \inf_{i\in I}\{\mu_i(xyz)\} \geq \inf_{i\in I}\min\{\mu_i(x),\mu_i(z)\}\\ &= \min\{(\bigcap_{i\in I}\mu_i)(x), (\bigcap_{i\in I}\mu_i)(z)\}.\end{aligned}$$

Suppose $x \leq y$. Then $\mu_i(x) \geq \mu_i(y)$ for all $i \in I$ which implies $(\bigcap_{i\in I}\mu_i)(x) \geq (\bigcap_{i\in I}\mu_i)(y)$. Hence $\bigcap_{i\in I}\mu_i$ is a fuzzy bi-ideal of S. □

Theorem 3.8. *Let $\{\mu_i : i \in I\}$ be a family of bi-ideals of S such that $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for $i, j \in I$. Then $\bigcup_{i\in I}\mu_i$ is a fuzzy bi-ideal of S.*

Proof. Let $\{\mu_i : i \in I\}$ be a family of bi-ideals of S and $x,\ y,\ z \in S$. Then we have

$$\begin{aligned}\bigcup_{i\in I}\mu_i(x+y) &= \max_{i\in I}\ \mu_i(x+y) \geq \max_{i\in I}\ \{\min\{\mu_i(x),\mu_i(y)\}\}\\ &= \min\{\max_{i\in I}\ \mu_i(x), \max_{i\in I}\ \mu_i(y)\} = \min\{\bigcup_{i\in I}\mu_i(x), \bigcup_{i\in I}\mu_i(y)\}\end{aligned}$$

In general max{ min{ } } $\leq$ min{ max{ } }.
Suppose for this problem $\max_{i\in I}\ \{\min\{\mu_i(x),\mu_i(y)\}\} \neq \min\{\max_{i\in I}\ \mu_i(x), \max_{i\in I}\ \mu_i(y)\}$.
Then there exists $r \in [0,1]$ such that

$$\max_{i\in I}\{\min\{\mu_i(x),\mu_i(y)\}\} < r < \min\{\max_{i\in I}\ \mu_i(x), \max_{i\in I}\ \mu_i(y)\}.$$

Since $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for $i, j \in I$ there exists k such that $r < \mu_k(x) \cap \mu_k(y)$. On the other hand $\mu_i(x) \cap \mu_i(y) < r$ for all $i \in I$, which is a contradiction. Hence $\max_{i\in I}\{\min\{\mu_i(x),\mu_i(y)\}\} = \min\{\max_{i\in I}\ \mu_i(x), \max_{i\in I}\ \mu_i(y)\}$.
Similarly we have

$$\begin{aligned}(\bigcup_{i\in I}\mu_i)(xy) &= \max_{i\in I}\{\mu_i(xy)\} \geq \max_{i\in I}\min\{\mu_i(x),\mu_i(y)\}\\ &= \min\{(\bigcup_{i\in I}\mu_i)(x), (\bigcup_{i\in I}\mu_i)(y)\}.\end{aligned}$$

and

$$\begin{aligned}(\bigcup_{i\in I}\mu_i)(xyz) &= \max_{i\in I}\{\mu_i(xyz)\} \geq \max_{i\in I}\ \min\{\mu_i(x),\mu_i(z)\}\\ &= \min\{(\bigcup_{i\in I}\mu_i)(x), (\bigcup_{i\in I}\mu_i)(z)\}.\end{aligned}$$

Thus $\bigcup_{i\in I}\mu_i$ is a fuzzy bi-ideal of S. □

Proposition 3.9. *If S is an ordered semiring then every fuzzy right (resp. left) ideals of S are fuzzy quasi ideals of S.*

Proof. Let μ be a fuzzy right ideal of an ordered semiring S. Then for $x,\ y \in S$, $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \mu(x)$. So, we only need to prove $(\chi_S o_1 \mu) \cap (\mu o_1 \chi_S) \subseteq \mu$. Excluding the trivial cases (i.e. $x \leq uv$ is not possible) we have the following:

$$x \leq uv \Rightarrow \mu(x) \geq \mu(uv) \geq \mu(u) = \min\{\mu(u), \chi_S(v)\}$$

and so

$$\mu(x) \geq \sup_{x \leq uv}\{\min\{\mu(u), \chi_S(v)\}\}$$

Therefore

$$\mu(x) \geq (\mu o_1 \chi_S)(x) \geq \min\{(\mu o_1 \chi_S)(x), (\chi_S o_1 \mu)(x)\} = ((\chi_S o_1 \mu) \cap (\mu o_1 \chi_S))(x).$$

Hence μ is a fuzzy quasi ideal of S. $\square$

Lemma 3.10. *In an ordered semiring every fuzzy quasi-ideals are fuzzy bi-ideals.*

Proof. Let μ be a fuzzy quasi ideal of S. It is sufficient to prove that $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$ for all $x,\ y,\ z \in S$. Since μ is a fuzzy quasi ideal of S,

$$\begin{aligned} \mu(xyz) &\geq ((\chi_S o_1 \mu) \cap (\mu o_1 \chi_S))(xyz) \\ &= \min\{(\mu o_1 \chi_S)(xyz), (\chi_S o_1 \mu)(xyz)\} \end{aligned}$$

Now

$$\begin{aligned} (\mu o_1 \chi_S)(xyz) &= \sup_{xyz \leq uv}\{\min\{\mu(u), \chi_S(v)\}\} \\ &= \min\{\mu(x), \chi_S(yz)\} = \mu(x) \end{aligned}$$

Also

$$\begin{aligned} (\chi_S o_1 \mu)(xyz) &= \sup_{xyz \leq uv}\{\min\{\chi_S(u), \mu(v)\}\} \\ &= \min\{\chi_S(xy), \mu(z)\} = \mu(z) \end{aligned}$$

Therefore $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$. $\square$

Definition 3.11. [6] An ordered semiring S is called regular (resp. intra-regular) if for each $a \in S$ there exist $x, y \in S$ such that $a \leq axa$ (resp. $a \leq xa^2y$).

Theorem 3.12. *In a regular ordered semiring, the fuzzy quasi ideals and the fuzzy bi-ideals coincide.*

Proof. It is sufficient to prove that every fuzzy bi-ideals are fuzzy quasi-ideals i.e. if μ be a fuzzy bi-ideal of S then $(\chi_S o_1 \mu) \cap (\mu o_1 \chi_S) \subseteq \mu \ \cdots (i)$.
Let $x \in S$ and excluding the trivial case, suppose $(\mu o_1 \chi_S)(x) \leq \mu(x)$. Then we have

$$\begin{aligned} \mu(x) &\geq (\mu o_1 \chi_S)(x) \geq \min\{(\chi_S o_1 \mu)(x), (\mu o_1 \chi_S)(x)\} \\ &= ((\chi_S o_1 \mu) \cap (\mu o_1 \chi_S))(x) \end{aligned}$$

and so (i) is satisfied.
Now suppose $(\mu o_1 \chi_S)(x) > \mu(x)$. Then there exist $u,\ v \in S$ with $x \leq uv$ such that $\min\{\mu(u), \chi_S(v)\} > \mu(x)$ i.e. $\mu(u) \geq \mu(x) \ \cdots (ii)$.
We will now prove that $(\chi_S o_1 \mu)(x) \leq \mu(x)$ so that condition (i) is satisfied. Since $(\chi_S o_1 \mu)(x) = \sup_{x \leq yz}\{\min\{\chi_S(y), \mu(z)\}\}$, it is enough to show that

$$\min\{\chi_S(y), \mu(z)\} \leq \mu(x) \text{ for some } y,\ z \in S \text{ with } x \leq yz.$$

As S is regular there exists $s \in S$ such that $x \leq xsx \leq uvsyz$. Then since μ is a fuzzy bi-ideal, we have

$$\mu(x) \geq \mu(uvsyz) \geq \min\{\mu(u), \mu(z)\}.$$

If $\min\{\mu(u), \mu(z)\} = \mu(u)$, then $\mu(u) \leq \mu(x)$ which is not possible by (ii). Thus $\min\{\mu(u), \mu(z)\} = \mu(z)$ and then

$$\mu(x) \geq \mu(z) = \min\{\chi_S(y), \mu(z)\}.$$

Hence the proof. □

Proposition 3.13. *Let μ and ν be respectively the fuzzy right and fuzzy left ideal of an ordered semiring S. Then $\mu \cap \nu$ is a fuzzy quasi ideal of S.*

Proof. Let μ and ν be respectively the fuzzy right and fuzzy left ideal of an ordered semiring S. Then $(\mu \cap \nu)(x+y) \geq \min\{(\mu \cap \nu)(x), (\mu \cap \nu)(y)\}$. Now

$$(\mu o_1 \chi_S)(x) = \sup_{x \leq ab}\{\min\{\mu(a), \chi_S(b)\}\} = \mu(a) \leq \mu(ab) \leq \mu(x)$$

and

$$(\chi_S o_1 \nu)(x) = \sup_{x \leq ab}\{\min\{\chi_S(a), \nu(b),\}\} = \nu(b) \leq \nu(ab) \leq \nu(x).$$

Therefore $((\mu \cap \nu) o_1 \chi_S) \cap (\chi_S o_1 (\mu \cap \nu)) \subseteq (\mu o_1 \chi_S) \cap (\chi_S o_1 \nu) \subseteq \mu \cap \nu$.
Now suppose $x \leq y$. Then $\mu(x) \geq \mu(y)$, $\nu(x) \geq \nu(y)$ and so $\min\{\mu(x), \nu(x)\} \geq \min\{\mu(y), \nu(y)\}$ which implies $(\mu \cap \nu)(x) \geq (\mu \cap \nu)(y)$.
Hence $\mu \cap \nu$ is a fuzzy quasi ideal of S. □

Theorem 3.14. *A fuzzy subset μ of an ordered semiring S is a fuzzy left(resp. right) ideal of S if and only if for all $x,\ y \in S$, we have*

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\chi_S o_1 \mu \subseteq \mu$ *(resp.* $\mu o_1 \chi_S \subseteq \mu$*)*

Proof. Assume that μ is a fuzzy left ideal of S. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in S$. If $(\chi_S o_1 \mu)(x) = 0$, it is clear that $(\chi_S o_1 \mu)(x) \leq \mu(x)$. Otherwise, there exist elements $y,\ z \in S$ such that $x \leq yz$. Then we have

$$\begin{aligned}(\chi_S o_1 \mu)(x) &= \sup_{x \leq yz}\{\min\{\chi_S(y), \mu(z)\}\} \\ &\leq \sup_{x \leq yz}\{\mu(z)\} \leq \mu(yz) \leq \mu(x).\end{aligned}$$

This implies that $\chi_S o_1 \mu \subseteq \mu$.
Conversely, assume that the given conditions hold. Then it is sufficient to show that $\mu(xy) \geq \mu(y)$ for all $x, y \in S$. Now

$$\begin{aligned}\mu(xy) &\geq (\chi_S o_1 \mu)(xy) = \sup_{xy \leq ab}\{\min\{\chi_S(a), \mu(b)\}\} \\ &\geq \min\{\chi_S(x), \mu(y)\} = \mu(y).\end{aligned}$$

Hence μ is a fuzzy left ideal of S.
The case for fuzzy right ideal can be proved similarly. □

Lemma 3.15. *Let S be an ordered semiring . Then the following conditions are equivalent.*

(i) *S is regular.*
(ii) *$B = BSB$ for every bi-ideal B of S.*
(iii) *$Q = QSQ$ for every quasi-ideal Q of S.*

Proof. (i)⇒(ii) Assume that (i) holds. Let B be any bi-ideal of S and x be any element of B. Then there exists $a \in S$ such that $x \leq xax$. Then it is easy to see that $x \leq xax \in BSB$. Hence $B \subseteq BSB$. On the other hand, since B is an bi-ideal of S, we have $BSB \subseteq B$ and so $B = BSB$.
(ii)⇒(iii) This is straightforward.
(iii)⇒(i) Assume that (iii) holds. Let R and L be any right and left ideal of S respectively. Then $R \cap L$ is a quasi-ideal of S. Also, $R \cap L = (R \cap L)S(R \cap L) \subseteq RSL \subseteq RL \subseteq R \cap L$. Therefore $RL = R \cap L$. Hence from Proposition 6.35 of [2] S is regular. □

Theorem 3.16. *Let S be an ordered semiring. Then the following conditions are equivalent.*

(i) *S is regular.*
(ii) *$\mu \subseteq \mu o_1 \chi_S o_1 \mu$ for every fuzzy bi-ideal μ of S.*
(iii) *$\mu \subseteq \mu o_1 \chi_S o_1 \mu$ for every fuzzy quasi-ideal μ of S.*

Proof. (i)⇒(ii) Assume that (i) holds. Let μ be any fuzzy bi-ideal of S and x be any element of S. Since S is regular there exists $a \in S$ such that $x \leq xax$.

$$\begin{aligned}(\mu o_1 \chi_S o_1 \mu)(x) &= \sup_{x \leq yz}\{\{(\mu o_1 \chi_S)(y), \mu(z)\}\} \\ &\geq \min_{x \leq xax}\{(\mu o_1 \chi_S)(xa), (\mu(x)\} \\ &= \min\{\sup_{xa \leq cd}\{\min\{\mu(c), \chi_S(d), \mu(x)\}\}\} \\ &\geq \min\{\mu(x), \chi_S(a), \mu(x)\} = \mu(x)\end{aligned}$$

This implies that $\mu \subseteq \mu o_1 \chi_S o_1 \mu$.
(ii)⇒(iii) This is straight forward from Lemma 3.10
(iii)⇒(i) Assume that (iii) holds and Q be any quasi-ideal of S. Then the characteristic function χ_Q of Q is a fuzzy quasi-ideal of S. Now $\chi_Q \subseteq \chi_Q o_1 \chi_S o_1 \chi_Q = \chi_{QSQ}$. Therefore $Q \subseteq QSQ$. On the other hand, since Q is any quasi-ideal of S, we have $QSQ \subseteq SQ \cap QS \subseteq Q$ and so $QSQ = Q$. Therefore S is regular by Lemma 3.15. □

Theorem 3.17. *Let S be an ordered semiring. Then the following conditions are equivalent.*

(i) *S is regular.*
(ii) *$\mu \cap \nu \subseteq \mu o_1 \nu o_1 \mu$ for every fuzzy bi-ideal μ and every fuzzy ideal ν of S.*
(iii) *$\mu \cap \nu \subseteq \mu o_1 \nu o_1 \mu$ for every fuzzy quasi-ideal μ and every fuzzy ideal ν of S.*

Proof. (i)⇒(ii) Assume that (i) holds. Let μ and ν be any fuzzy bi-ideal and fuzzy ideal of S, respectively and x be any element of B. Since S is regular, there exists $a \in S$ such that $x \leq xax$. Then

$$\begin{aligned}
(\mu o_1 \nu o_1 \mu)(x) &= \sup_{x \leq yz} \{\{(\mu o_1 \nu)(y), \mu(z)\}\} \\
&\geq \min_{x \leq xax} \{(\mu o_1 \nu)(xa), \mu(x)\} \\
&= \min\{ \sup_{xa \leq cd} \{\min\{\mu(c), \nu(d)\}\}, \mu(x)\} \\
&\geq \min_{xa \leq xaxa} \{\mu(x), \nu(axa), \mu(x)\} \\
&\geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x)
\end{aligned}$$

(ii)⇒(iii) This is straight forward from Lemma 3.10.
(iii)⇒(i) Assume that (iii) holds. Let μ be any fuzzy quasi-ideal of S. Then since χ_S is a fuzzy ideal of S, we have $\mu = \mu \cap \chi_S \subseteq \mu o_1 \chi_S o_1 \mu$. Therefore S is regular by Theorem 3.16. □

Conclusion: In this paper we introduce and study fuzzy bi-ideals and fuzzy quasi-ideals in ordered semirings. Basically we characterize multiplicative regular ordered semirings in terms of fuzzy bi-ideals and fuzzy quasi-ideals. Our next work on this topic will focus on (the work is on progress) fuzzy h-bi-ideals, fuzzy h-quasi-ideals, fuzzy h-interior-ideals etc. with more generalized form of compositions of fuzzy ideals of ordered semirings (resp. duo ordered semirings) and generalized form of regularity, intra-regularity as stated in the conclusion of [6].

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