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ON THE NUMERICAL SOLUTION OF FRACTIONAL-ORDER FREDHOLM-VOLTERRA INTEGRO-DIFFERENTIAL AND MULTI-FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATIONS

TAYEBEH NASIRI[1] AND AZIM AMINATAEI[2*]

Abstract. In this work, we have applied the homotopy perturbation method (HPM) and variational iteration method (VIM) for solving fractional Fredholm and Volterra integrodifferential equation. Further, we have shown that multi-fractional order integrodifferential equation is a special case of fractional Fredholm and Volterra integrodifferential equation. For this, the fractional derivatives are described in the Caputo sense. Some numerical experiments are solved by using the aforesaid methods wherein the high accuracy and efficiency of both methods are observed.

1. Introduction and preliminaries

Consider the fractional integrodifferential equation of the form

$$D^{\alpha}y(x)=\sum_{i=0}^{n-1}p_i(x)y^{(i)}(x)+f(x)+\lambda_1\int_0^1 k_1(x,t)F(y(t))dt+\lambda_2\int_0^x k_2(x,t)G(y(t))dt, \tag{1.1}$$

$x\in[0,1],\ n-1<\alpha\leq n$, and $n\in\mathbb{N}$, with the initial conditions:

$$y^{(j)}(0)=c_j,\qquad j=0,1,\cdots,n-1, \tag{1.2}$$

where D^{α} denotes a differential operator with fractional order α and $F(y(t))$, and $G(y(t))$ are nonlinear functions.

Such integrodifferential equations are generalizations of fractional differential equations, Fredholm integral equations, and Volterra integral equations. These kinds of equations arise in various physical processes such as acoustics, chemistry and biology, electromagnetism, heat conduction in materials with memory, viscoelastic materials, signal processing and etc [1]–[4].

In the above equation, for known β ($k-1<\beta\leq k,\quad k\in\mathbb{N}$), when $\lambda_1=0$, $\lambda_2=\frac{1}{\Gamma(\beta)}$, and $k_2(x,t)=(x-t)^{\beta-1}$, the equation reduces to the following

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* Corresponding author.
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equation:

$$\begin{aligned} D^{\alpha}y(x) &= \sum_{i=0}^{n-1} p_i(x)y^{(i)}(x) + f(x) + \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}G(y(t))dt \\ &= \sum_{i=0}^{n-1} p_i(x)y^{(i)}(x) + f(x) + J^{\beta}(G(y(x)), \end{aligned} \tag{1.3}$$

that is a multi-fractional order integro-differential equation and J^{β} is Riemann-Liouville integral of fractional order β.

In this work, we apply the HPM and VIM to solve Eq. (1.1) and the special case of it, i.e. Eq. (1.3). The HPM and VIM are proposed by the Chinese mathematician He [5]–[8]. HPM is a coupling of traditional perturbation method and homotopy in topology [5] and VIM is a modification of a general Lagrange multiplier method [9]. These methods solve the problem without any need to linearization, restrictive assumptions or discretization of the variables. Therefore, those are not affected by computation round off errors and are not faced with necessity of large computer memory and time. These methods have been successfully applied by many authors [10]–[13] for finding the approximate solutions of fractional equations which arise in scientific and engineering problems.

The remainder of the paper proceeds as follows: In section 2, some necessary definitions and mathematical preliminaries of the fractional calculus theory are given. In sections 3 and 4, applications of the HPM and VIM are described for Eqs. (1.1) and (1.2), respectively. In section 5, to illustrate the methods, some experiments are provided. Finally, we monitor a short conclusion in section 6.

2. Basic definitions

In this section, we give some necessary definitions and mathematical preliminaries of the fractional calculus theory which are used further in this work.

Definition 2.1. A real function $f(x),\ x>0$ is said to be in space $c_{\mu},\ \mu\in\mathbb{R}$ if there exists a real number $p>\mu$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in c(0,\infty)$, and it is said to be in the space c_{μ}^n if and only if $f^{(n)}\in c_{\mu},\ n\in\mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha>0$, of a function $f\in c_{\mu},\ \mu\geq -1$, is defined as:

$$\begin{aligned} &J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \qquad \alpha>0, \\ &J^0 f(x) = f(x). \end{aligned} \tag{2.1}$$

There are various types of definition for the fractional derivative of order $\alpha>0$. The most commonly used definitions are the Riemann-Liouville and Caputo formulas. But Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional equations. Therefore, the Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as:

$$D^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x(x-t)^{m-\alpha-1}f^{(m)}(t)dt, \tag{2.2}$$

for $m-1<\alpha\leq m,\ m\in\mathbb{N},\ x>0,\ f\in c_{-1}^{m}$.

We shall state some properties of the fractional integral J^{α} and fractional differential D^{α} operators.

Lemma 2.4. *For $f\in c_{\mu},\ \mu\geq -1,\ \alpha,\beta\geq 0$ and $\gamma\geq -1$, the following properties hold:*

$$\begin{aligned}
&1)\ J^{\alpha}J^{\beta}f(x)=J^{\alpha+\beta}f(x),\\
&2)\ J^{\alpha}J^{\beta}f(x)=J^{\beta}J^{\alpha}f(x),\\
&3)\ J^{\alpha}x^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}.
\end{aligned} \tag{2.3}$$

Lemma 2.5. *If $m-1<\alpha\leq m,\ m\in\mathbb{N},\ f\in c_{\mu}^{m},\ \mu\geq -1$, then the following two properties hold:*

$$\begin{aligned}
&1)\ D^{\alpha}J^{\alpha}f(x)=f(x),\\
&2)\ (J^{\alpha}D^{\alpha})f(x)=f(x)-\sum_{k=0}^{m-1}f^{(k)}(0^{+})\frac{x^{k}}{k!}.
\end{aligned} \tag{2.4}$$

For more details on the mathematical properties of fractional derivatives and integrals, see [1].

3. Application of the HPM

Consider the fractional-order integro-differential equation (1.1) with the initial conditions (1.2), where we assume that the functions $k_i(x,t),\ (i=1,2)$ can be approximated by Taylor polynomial.
In view of HPM [5, 6], we construct the following homotopy of Eq. (1.1):

$$\begin{aligned}
(1-p)D^{\alpha}y(x)+p\Big(D^{\alpha}y(x)-\sum_{i=0}^{n-1}p_i(x)y^{(i)}(x)-f(x)-\lambda_1\int_0^1k_1(x,t)F(y(t))dt\\
-\lambda_2\int_0^x k_2(x,t)G(y(t))dt\Big)=0,\quad p\in[0,1],
\end{aligned} \tag{3.1}$$

or

$$\begin{aligned}
D^{\alpha}y(x)=p\Big(\sum_{i=0}^{n-1}p_i(x)y^{(i)}(x)+f(x)+\lambda_1\int_0^1k_1(x,t)F(y(t))dt\\
+\lambda_2\int_0^x k_2(x,t)G(y(t))dt\Big),\quad p\in[0,1],
\end{aligned} \tag{3.2}$$

where $p\in[0,1]$ is an embedding parameter. The homotopy parameter p always changes from zero to unity. In case $p=0$, Eq. (3) becomes the linearized equation: $D^{\alpha}y(x) = 0$, and when it is one, Eq. (3) turns out to be the original equation (1.1). Using parameter p, we expand the solution of Eq. (3) in the following form:

$$y(x) = y_0(x) + py_1(x) + p^2 y_2(x) + p^3 y_3(x) + \cdots . \tag{3.3}$$

Setting $p = 1$ in Eq. (3.3), yields the solution of Eq. (1.1) as:

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots . \tag{3.4}$$

The convergence of series (3.4) has been proved by [14]. Substitution of Eq. (3.3) into Eq. (3), and then equating the terms with identical power of p, we obtain the following series of linear equations:

$$p^0 : D^{\alpha} y_0 = 0, \tag{3.5}$$

$$\begin{aligned} p^1 : D^{\alpha} y_1 &= \sum_{i=0}^{n-1} p_i(x) y_0^{(i)}(x) + f(x) + \lambda_1 \int_0^1 k_1(x,t) F_1(y_0(t))dt \\ &+ \lambda_2 \int_0^x k_2(x,t) G_1(y_0(t))dt, \end{aligned} \tag{3.6}$$

$$\begin{aligned} p^2 : D^{\alpha} y_2 &= \sum_{i=0}^{n-1} p_i(x) y_1^{(i)}(x) + \lambda_1 \int_0^1 k_1(x,t) F_2(y_1(t))dt \\ &+ \lambda_2 \int_0^x k_2(x,t) G_2(y_1(t))dt, \end{aligned} \tag{3.7}$$

$$\vdots$$

$$\begin{aligned} p^n : D^{\alpha} y_n &= \sum_{i=0}^{n-1} p_i(x) y_{n-1}^{(i)}(x) + \lambda_1 \int_0^1 k_1(x,t) F_n(y_{n-1}(t))dt \\ &+ \lambda_2 \int_0^x k_2(x,t) G_n(y_{n-1}(t))dt, \ n \geq 2, \end{aligned} \tag{3.8}$$

where the functions $F_1, F_2, F_3, \cdots$; and $G_1, G_2, G_3, \cdots$; satisfy the following conditions:

$$F\left(y_0(t) + py_1(t) + p^2 y_2(t) + \cdots\right) = F_1\left(y_0(t)\right) + pF_2\left(y_1(t)\right) + p^2 F_3\left(y_2(t)\right) + \cdots ,$$
$$G\left(y_0(t) + py_1(t) + p^2 y_2(t) + \cdots\right) = G_1\left(y_0(t)\right) + pG_2\left(y_1(t)\right) + p^2 G_3\left(y_2(t)\right) + \cdots .$$

For Eq. (3.5), the initial approximation can be chosen in the following way:

$$y_0 = \sum_{j=0}^{n-1} c_j \frac{x^j}{j!}. \tag{3.9}$$

Eqs. (3.5)-(3.8), can be solved by applying the operator J^{α}, which is the inverse of the operator D^{α}, and then by simple computation, we can express $\phi_N(x) = \sum_{n=0}^{N-1} y_n(x)$, which denotes the $N-$terms approximation to $y(x)$. To show the efficiency of the present method in comparison with the exact solution, we evaluate the absolute error defined by $Ey_N(x) = |y(x) - \phi_N(x)|$, where $y(x)$ is the exact solution.

4. Application of the VIM

To illustrate the basic ideas of VIM, consider the fractional-order integro-differential equation (1.1) with the initial conditions (1.2), where we assume that the functions $k_i(x,t)$, $(i=1,2)$ can be approximated by Taylor polynomial. According to the VIM [7, 8], we construct the correction functional for Eq. (1.1), as:

$$\begin{aligned} y_{k+1}(x) &= y_k(x) + J^{\gamma}\Big[\lambda\Big(D^{\alpha}y_k(x) - \sum_{i=0}^{n-1} p_i(x)\tilde{y}_k^{(i)}(x) - f(x) \\ &\quad - \lambda_1\int_0^1 k_1(x,t)F(\tilde{y}_k(t))dt - \lambda_2\int_0^x k_2(x,t)G(\tilde{y}_k(t))dt\Big)\Big] \\ &= y_k(x) + \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}\Big[\lambda\Big(D^{\alpha}y_k(s) - \sum_{i=0}^{n-1} p_i(s)\tilde{y}_k^{(i)}(s) - f(s) \\ &\quad - \lambda_1\int_0^1 k_1(s,t)F(\tilde{y}_k(t))dt - \lambda_2\int_0^s k_2(s,t)G(\tilde{y}_k(t))dt\Big)\Big], \end{aligned} \tag{4.1}$$

where J^{α} is Riemann-Liouville fractional integral operator of order $\gamma=\alpha+1-n$, λ is a general Lagrange multiplier and $\tilde{y}_k$ denotes restricted variation, i.e. $\delta\tilde{y}_k=0$. In order to make approximation for the identification of an approximate Lagrange multiplier, we write the correctional function (4.1) as:

$$\begin{aligned} y_{k+1}(x) &= y_k(x) + \int_0^x \lambda(s)\Big[D^n y_k(s) - \sum_{i=0}^{n-1} p_i(s)\tilde{y}_k^{(i)}(s) - f(s) \\ &\quad - \lambda_1\int_0^1 k_1(s,t)F(\tilde{y}_k(t))dt - \lambda_2\int_0^s k_2(s,t)G(\tilde{y}_k(t))dt\Big]ds. \end{aligned} \tag{4.2}$$

Making the above functional stationary, noticing that $\delta\tilde{y}_k=0$, we have:

$$\delta y_{k+1}(x) = \delta y_k(x) + \delta\int_0^x \lambda(s)\left[D^n y_k(s) - f(s)\right]ds, \tag{4.3}$$

yields the following Lagrange multipliers,

$$\lambda = -1, \qquad \text{for} \quad n=1;$$

$$\lambda = s-x, \qquad \text{for} \quad n=2;$$

and in general,

$$\lambda = \frac{(-1)^n}{(n-1)!}(s-x)^{n-1}, \qquad n\geq 1. \tag{4.4}$$

Therefore, by substituting (4.4) into functional (4.2), we obtain the following iteration formula:

$$y_{k+1}(x) = y_k(x) - \frac{1}{(n-1)!\Gamma(\alpha+1-n)} \int_0^x (x-s)^{\alpha-n}(x-s)^{n-1} \Bigg[D^\alpha y_k(s) - \sum_{i=0}^{n-1} p_i(s) y_k^{(i)}(s) - f(s) - \lambda_1 \int_0^1 k_1(s,t) F(y_k(t))dt - \lambda_2 \int_0^s k_2(s,t) G(y_k(t))dt \Bigg]. \tag{4.5}$$

Consequently, we obtain the following iteration formula:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{(n-1)!} J^\alpha \Bigg[D^\alpha y_k(x) - \sum_{i=0}^{n-1} p_i(x) y_k^{(i)}(x) - f(x) - \lambda_1 \int_0^1 k_1(x,t) F(y_k(t))dt - \lambda_2 \int_0^x k_2(x,t) G(y_k(t))dt \Bigg]. \tag{4.6}$$

Now, we can choose the initial approximation as:

$$y_0(x) = \sum_{j=0}^{n-1} c_j \frac{x^j}{j!}, \tag{4.7}$$

and then by iteration formula (4.6), we can obtain $y_1, y_2, y_3, \cdots$.
Generally one iteration leads to accurate solution by the VIM, if the initial approximation is carefully chosen. To show the efficiency of the VIM in comparison with the exact solution, we evaluate the absolute error defined by $Ey_k(x) = |y(x) - y_k(x)|$, where $y(x)$ is the exact solution, and $y_k(x)$ is the kth-iteration approximate solution.

5. Numerical experiments

In this section, four experiments including two experiments of fractional-order Fredholm-Volterra integro-differential equation and two experiments of multi-fractional order integro-differential equation are solved. The computations associated with the experiments given below were performed in Maple 14 on a PC with a CPU of 2.2 GHZ.

Example 5.1. Consider the following linear fractional-order integro-differential equation:

$$D^\alpha y(x) + xy'(x) - xy(x) = e^x - \sin(x) + \frac{1}{2} x \cos x + \int_0^1 \sin(x) e^{-t} y(t)dt - \frac{1}{2} \int_0^x \cos(x) e^{-t} y(t)dt, \tag{5.1}$$

$1 < \alpha \le 2$, $0 < x < 1$, with the initial conditions:

$$y(0) = 1, \quad y'(0) = 1. \tag{5.2}$$

For $\alpha = 2$, the exact solution of Eq. (5.1) is [15]:

$$y(x) = e^x.$$

According to the HPM, we construct the following homotopy:

$$\begin{aligned} D^\alpha y(x) = p\big(& -xy'(x) + xy(x) + e^x - \sin(x) + \frac{1}{2}x\cos x + \int_0^1 \sin(x)e^{-t}y(t)dt \\ & -\frac{1}{2}\int_0^x \cos(x)e^{-t}y(t)dt\big). \end{aligned} \tag{5.3}$$

By substituting of Eq. (3.3) into Eq. (5.3), and then equating the coefficients of p to zero, we obtain the following series of linear equations:

$$\begin{aligned} p^0 &: D^\alpha y_0 = 0, \\ p^1 &: D^\alpha y_1 = xy_0(x) - xy_0'(x) + e^x - \sin(x) + \frac{1}{2}x\cos x + \int_0^1 \sin(x)e^{-t}y_0(t)dt \\ &\qquad -\frac{1}{2}\int_0^x \cos(x)e^{-t}y_0(t)dt, \\ p^2 &: D^\alpha y_2 = xy_1(x) - xy_1'(x) + \int_0^1 \sin(x)e^{-t}y_1(t)dt - \frac{1}{2}\int_0^x \cos(x)e^{-t}y_1(t)dt, \\ &\quad \vdots \\ p^n &: D^\alpha y_n = xy_{n-1}(x) - xy_{n-1}'(x) + \int_0^1 \sin(x)e^{-t}y_{n-1}(t)dt \\ &\qquad -\frac{1}{2}\int_0^x \cos(x)e^{-t}y_{n-1}(t)dt, \ n \geq 2. \end{aligned} \tag{5.4}$$

Applying the operator J^α to the above series of linear equations and using initial conditions (5.2), we obtain:

$$y_0(x) = 1 + x, \tag{5.5}$$

$$\begin{aligned} y_1(x) = J^\alpha\big[& xy_0(x) - xy_0'(x) + e^x - \sin(x) + \frac{1}{2}x\cos x + \int_0^1 \sin(x)e^{-t}y_0(t)dt \\ & -\frac{1}{2}\int_0^x \cos(x)e^{-t}y_0(t)dt\big], \end{aligned} \tag{5.6}$$

$$y_2(x) = J^\alpha\big[xy_1(x) - xy_1'(x) + \int_0^1 \sin(x)e^{-t}y_1(t)dt - \frac{1}{2}\int_0^x \cos(x)e^{-t}y_1(t)dt\big],$$

$$\vdots$$

In order to avoid difficult fractional integration, we can take the truncated Taylor expansion of $e^x, \sin(x), \cos(x)$, and e^{-x} as follows:

$$\begin{aligned} e^x &\simeq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, & e^{-x} &\simeq 1 - x + \frac{x^2}{2} - \frac{x^3}{6}, \\ \sin(x) &\simeq x - \frac{x^3}{6} + \frac{x^5}{120}, & \cos(x) &\simeq 1 - \frac{x^2}{2} + \frac{x^4}{24}. \end{aligned} \tag{5.7}$$

Therefore, by solving Eqs. (??)-(??), we have:

$$\begin{aligned} y_1(x) = x^\alpha \Bigg[&\frac{1}{\Gamma(\alpha+1)} + \frac{53x}{60\Gamma(\alpha+2)} + \frac{3x^2}{\Gamma(\alpha+3)} + \frac{97x^3}{60\Gamma(\alpha+4)} - \frac{x^4}{\Gamma(\alpha+5)} \\ &- \frac{427x^5}{60\Gamma(\alpha+6)} + \frac{15x^6}{\Gamma(\alpha+7)} - \frac{49x^7}{2\Gamma(\alpha+8)} - \frac{70x^8}{\Gamma(\alpha+9)} + \frac{252x^9}{\Gamma(\alpha+10)} \Bigg], \end{aligned} \tag{5.8}$$

$$\begin{aligned} y_2(x) = &\frac{x^{2\alpha+1}(\alpha+1)}{\Gamma(2\alpha+2)} + \frac{53x^{2\alpha+2}(\alpha+2)}{6\Gamma(2\alpha+3)} + \frac{3x^{2\alpha+3}(\alpha+3)}{\Gamma(2\alpha+4)} + \frac{97x^{2\alpha+4}(\alpha+4)}{60\Gamma(2\alpha+5)} \\ &- \frac{x^{2\alpha+5}(\alpha+5)}{\Gamma(2\alpha+6)} - \frac{427x^{2\alpha+6}(\alpha+6)}{60\Gamma(2\alpha+7)} + \frac{153x^{2\alpha+7}(\alpha+7)}{\Gamma(2\alpha+8)} - \frac{7x^{2\alpha+9}(\alpha+9)}{\Gamma(2\alpha+10)} \\ &- \frac{49x^{2\alpha+8}(\alpha+8)}{2\Gamma(2\alpha+9)} + \frac{252x^{2\alpha+10}(\alpha+10)}{\Gamma(2\alpha+11)} - \frac{x^{2\alpha}\alpha}{\Gamma(2\alpha+1)} - \cdots . \end{aligned} \tag{5.9}$$

Now, we form the 2-terms approximation as follows:

$$\phi_2(x) = y_0(x) + y_1(x) + y_2(x). \tag{5.10}$$

Table 1 shows the approximation values of $y(x)$ for different values of α by HPM. According to the VIM, the iteration formula (4.6) of Eq. (5.1) can be written in the following form:

$$\begin{aligned} y_{k+1}(x) = y_k(x) - (\alpha-1)J^\alpha \Bigg[&D^\alpha y_k(x) + xy_k'(x) - xy_k(x) - e^x + \sin(x) - \frac{1}{2}x\cos x \\ &- \int_0^1 \sin(x)e^{-t}y_k(t)dt + \frac{1}{2}\int_0^x \cos(x)e^{-t}y_k(t)dt \Bigg]. \end{aligned} \tag{5.11}$$

In order to avoid difficult fractional integration, we use Eqs. (5.7), and assume the initial approximation as the following:

$$y_0(x) = 1 + x.$$

Therefore, by iteration formula (5.11), the first two iterations are as the following:

$$\begin{aligned} y_1(x) = 1 + x - (\alpha-1)x^\alpha \Bigg[&- \frac{1}{\Gamma(\alpha+1)} - \frac{53x}{60\Gamma(\alpha+2)} - \frac{3x^2}{\Gamma(\alpha+3)} - \frac{97x^3}{60\Gamma(\alpha+4)} \\ &+ \frac{x^4}{\Gamma(\alpha+5)} + \frac{427x^5}{60\Gamma(\alpha+6)} - \frac{15x^6}{\Gamma(\alpha+7)} + \frac{49x^7}{2\Gamma(\alpha+8)} + \frac{70x^8}{\Gamma(\alpha+9)} - \frac{252x^9}{\Gamma(\alpha+10)} \Bigg], \end{aligned} \tag{5.12}$$

$$
\begin{aligned}
y_2(x) = (2-\alpha)\Bigg[& 1 + x - (\alpha-1)x^{\alpha}\Bigg(-\frac{1}{\Gamma(\alpha+1)} - \frac{53x}{60\Gamma(\alpha+2)} - \frac{3x^2}{\Gamma(\alpha+3)} \\
& - \frac{97x^3}{60\Gamma(\alpha+4)} + \frac{x^4}{\Gamma(\alpha+5)} + \frac{427x^5}{60\Gamma(\alpha+6)} - \frac{15x^6}{\Gamma(\alpha+7)} + \frac{49x^7}{2\Gamma(\alpha+8)} + \frac{70x^8}{\Gamma(\alpha+9)} \\
& - \frac{252x^9}{\Gamma(\alpha+10)}\Bigg)\Bigg] - (\alpha-1)\Bigg[-1 - x - \frac{x^{\alpha}}{\Gamma(\alpha+1)} - \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} \\
& + (\alpha-1)\Bigg(-\frac{x^{2\alpha+1}\Gamma(\alpha+2)}{\Gamma(2\alpha+2)\Gamma(\alpha+1)} - \frac{53x^{2\alpha+2}\Gamma(\alpha+3)}{60\Gamma(2\alpha+3)\Gamma(\alpha+2)} - \frac{3x^{2\alpha+3}\Gamma(\alpha+4)}{\Gamma(2\alpha+4)\Gamma(\alpha+3)} \\
& - \frac{97x^{2\alpha+4}\Gamma(\alpha+5)}{60\Gamma(2\alpha+5)\Gamma(\alpha+4)} + \cdots\Bigg) + \cdots\Bigg]. \qquad (5.13)
\end{aligned}
$$

Table 2 shows the approximate values of $y(x)$ according to the second iteration of VIM for different values of α. It is clear that in both methods, the accuracy can be improved by computing more terms of approximated solutions and/or by taking more terms in the Taylor expansion of $e^x, \sin(x), \cos(x)$, and e^{-x}.

TABLE 1. Numerical results of the HPM for different values of x, α and absolute error of experiment 1.

x	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$	Ey_2 in $\alpha = 2$
0	1.00000000	1.00000000	1.00000000	1.00000000	$0.00000000e+00$
0.2	1.32540354	1.27206557	1.23987818	1.22138756	$1.51930000e-05$
0.4	1.71761450	1.61762887	1.54362471	1.49168016	$1.44538000e-04$
0.6	2.16554166	2.02786661	1.91289338	1.82134571	$7.73090000e-04$
0.8	2.67378963	2.50633991	2.35481325	2.22226747	$3.27345400e-03$
1.0	3.25012426	3.05934645	2.87798167	2.70712230	$1.11595310e-02$

TABLE 2. Numerical results of the VIM for different values of x, α and absolute error of experiment 1.

x	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$	Ey_2 in $\alpha = 2$
0	1.00000000	1.00000000	1.00000000	1.00000000	$0.00000000e+00$
0.2	1.25632571	1.25450766	1.23743932	1.22138757	$1.51850000e-05$
0.4	1.54891323	1.56846581	1.53600323	1.49168146	$1.43238000e-04$
0.6	1.87888903	1.94183556	1.90038446	1.82136793	$7.50871000e-04$
0.8	2.25521900	2.38588561	2.34228032	2.22243392	$3.10701200e-03$
1.0	2.68909394	2.91610352	2.87690177	2.70791595	$1.03658810e-02$

Example 5.2. Consider the following nonlinear fractional-order integro-differential equation:

$$
D^{\alpha}y(x) = f(x) - xy(x) + \int_0^1 (x+t)y(t)dt + \int_0^x (x-t)[y(t)]^2dt, \qquad 0 < \alpha \leq 1,\ 0 < x < 1, \qquad (5.14)
$$

with the initial condition:

$$y(0) = -2, \tag{5.15}$$

wherein $f(x)$ is the following:

$$f(x) = -\frac{1}{30}x^6 + \frac{1}{3}x^4 + x^3 - 2x^2 + \frac{5}{3}x + \frac{3}{4}.$$

For $\alpha = 1$, the exact solution of Eq. (5.14) is [16]:

$$y(x) = x^2 - 2.$$

According to the HPM, we construct the following homotopy:

$$D^\alpha y(x) = p\left(f(x) - xy(x) + \int_0^1 (x+t)y(t)dt + \int_0^x (x-t)[y(t)]^2 dt\right). \tag{5.16}$$

The same to previous experiment, by substitution of Eq. (3.3) into Eq. (5.16), and then by equating the coefficients of p to zero, we obtain the following series of linear equations:

$$\begin{aligned}
&p^0 : D^\alpha y_0 = 0,\\
&p^1 : D^\alpha y_1 = f(x) - xy_0(x) + \int_0^1 (x+t)y_0(t)dt + \int_0^x (x-t)[y_0(t)]^2 dt,\\
&p^2 : D^\alpha y_2 = -xy_1(x) + \int_0^1 (x+t)y_1(t)dt + 2\int_0^x (x-t)y_0(t)y_1(t)dt,\\
&p^3 : D^\alpha y_3 = -xy_2(x) + \int_0^1 (x+t)y_2(t)dt + \int_0^x (x-t)[2y_0(t)y_2(t) + y_1^2(t)]dt,\\
&\vdots
\end{aligned}$$

Applying the operator J^{α} to the above series of linear equations and using initial condition (5.15), we obtain:

$$y_0(x) = -2, \tag{5.17}$$

$$y_1(x) = -\frac{24x^{\alpha+6}}{\Gamma(\alpha+7)} + \frac{8x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{6x^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{5x^{\alpha+1}}{4\Gamma(\alpha+1)}, \tag{5.18}$$

$$\begin{aligned} y_2(x) = {} & \frac{92x^{2\alpha+8}}{\Gamma(2\alpha+9)} + \frac{24x^{2\alpha+7}(\alpha+7)}{\Gamma(2\alpha+8)} - \frac{32x^{2\alpha+6}}{\Gamma(2\alpha+7)} - \frac{8x^{2\alpha+5}(\alpha+5)}{\Gamma(2\alpha+6)} \\ & - \frac{6x^{2\alpha+4}(\alpha+4)}{\Gamma(2\alpha+5)} - \frac{x^{2\alpha+2}(5\alpha+7)}{3\Gamma(2\alpha+3)} - \frac{20x^{2\alpha+3}}{3\Gamma(2\alpha+4)} + \frac{x^{2\alpha+1}(\alpha+1)}{4\Gamma(2\alpha+2)} \\ & + \frac{(54144 + 9580\alpha^2 + 39948\alpha - 23\alpha^3 - 385\alpha^4 - 61\alpha^5 - 3\alpha^6)x^{\alpha+1}}{12\Gamma(\alpha+8)\Gamma(\alpha+2)} \\ & + \frac{(47136 + 10222\alpha^2 + 38488\alpha + 201\alpha^3 - 61\alpha^5 - 367\alpha^4 - 3\alpha^6)x^{\alpha}}{12(\alpha+8)\Gamma(\alpha+7)\Gamma(\alpha+1)}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} y_3(x) = {} & -\frac{96x^{3\alpha+9}(2\alpha+9)}{\Gamma(3\alpha+10)} - \frac{24x^{3\alpha+8}(\alpha+7)(2\alpha+8)}{\Gamma(3\alpha+9)} + \frac{32x^{3\alpha+7}(2\alpha+7)}{\Gamma(3\alpha+8)} \\ & + \frac{8x^{3\alpha+6}(\alpha+5)(2\alpha+6)}{\Gamma(3\alpha+7)} + \cdots . \end{aligned} \tag{5.20}$$

Now, we can form the 3-terms approximation as:

$$\phi_2(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x). \tag{5.21}$$

Table 3 shows the approximate values of $y(x)$ for different values of α obtained by HPM. According to VIM, the iteration formula (4.6) of Eq. (5.14) can be written as the following:

$$\begin{aligned} y_{k+1}(x) = y_k(x) - J^{\alpha}\Big[& D^{\alpha}y_k(x) - f(x) + xy_k(x) - \int_0^1 (x+t)y_k(t)dt \\ & - \int_0^x (x-t)[y_k(t)]^2 dt\Big]. \end{aligned} \tag{5.22}$$

Assume that initial approximation is as:

$$y_0(x) = -2.$$

Therefore, by iteration formula (5.22), the first three iterations are as the following:

$$y_1(x) = -2 - x^\alpha\left(\frac{24x^6}{\Gamma(\alpha+7)} + \frac{8x^4}{\Gamma(\alpha+5)} + \frac{6x^3}{\Gamma(\alpha+4)} + \frac{5x}{3\Gamma(\alpha+2)} - \frac{1}{4\Gamma(\alpha+1)}\right), \tag{5.23}$$

$$\begin{aligned} y_2(x) &= -2 - \frac{24x^{\alpha+6}}{\Gamma(\alpha+7)} + \frac{8x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{6x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\Big(- \frac{5}{3} + \frac{24}{\Gamma(\alpha+8)} \\ &- \frac{8}{\Gamma(\alpha+6)} - \frac{6}{\Gamma(\alpha+5)} - \frac{5}{3\Gamma(\alpha+3)} + \frac{1}{4\Gamma(\alpha+2)}\Big) \\ &- \frac{x^\alpha}{\Gamma(\alpha+1)}\Big(\frac{1}{4} + \frac{24}{(\alpha+8)\Gamma(\alpha+7)} - \frac{8}{(\alpha+6)\Gamma(\alpha+5)} - \frac{6}{(\alpha+5)\Gamma(\alpha+4)} \\ &- \frac{5}{3(\alpha+3)\Gamma(\alpha+2)} + \frac{1}{4(\alpha+2)\Gamma(\alpha+1)}\Big) + \frac{96x^{2\alpha+8}}{\Gamma(2\alpha+9)} - \frac{32x^{2\alpha+6}}{\Gamma(2\alpha+7)} - \cdots, \end{aligned} \tag{5.24}$$

$$\begin{aligned} y_3(x) &= -2 - \frac{24x^{\alpha+6}}{\Gamma(\alpha+7)} + \frac{8x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{6x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{4x^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} \\ &+ \frac{3x^\alpha}{4\Gamma(\alpha+1)} + \frac{24x^{2\alpha+7}(\alpha+7)}{\Gamma(2\alpha+8)} - \frac{8x^{2\alpha+5}(\alpha+5)}{\Gamma(2\alpha+6)} - \frac{6x^{2\alpha+4}(\alpha+4)}{\Gamma(2\alpha+5)} - \cdots. \end{aligned} \tag{5.25}$$

Tabel 4 shows the approximate values obtained by third iteration of VIM. It is clear that in both methods, the accuracy can be improved by computing more terms of approximated solutions.

TABLE 3. Numerical results of the HPM for different values of x, α and absolute error of experiment 2.

x	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1.0$	Ey_3 in $\alpha = 1$
0	-2.00000000	-2.00000000	-2.00000000	-2.00000000	$0.00000000e+00$
0.2	-1.48116559	-1.73676128	-1.89064124	-1.96316810	$3.16809500e-03$
0.4	-1.18730520	-1.45147420	-1.69031934	-1.84574442	$5.74442200e-03$
0.6	-0.98566194	-1.15974792	-1.42249001	-1.64611592	$6.11591900e-03$
0.8	-0.81307080	-0.87696497	-1.10500104	-1.36373062	$3.73062500e-03$
1.0	-0.46483341	-0.56487336	-0.74336986	-0.99898694	$1.01306450e-03$

Example 5.3. Consider the following multi-fractional order equation:

$$D^{0.5}y(x) = \frac{6x^{2.5}}{\Gamma(3.5)} - \frac{\Gamma(4)x^{4.5}}{\Gamma(5.5)} + J^{1.5}y(x), \qquad x \in [0,1], \tag{5.26}$$

with the initial condition:

$$y(0) = 0. \tag{5.27}$$

TABLE 4. Numerical results of the VIM for different values of x, α and absolute error of experiment 2.

x	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1.0$	Ey_3 in $\alpha = 1$
0	-2.00000000	-2.00000000	-2.00000000	-2.00000000	$0.00000000e+00$
0.2	-1.48220753	-1.73774341	-1.89097879	-1.96325638	$3.25638500e-03$
0.4	-1.18359162	-1.45257916	-1.69093847	-1.84594475	$5.94475300e-03$
0.6	-0.96343305	-1.15869456	-1.42325985	-1.64649726	$6.49726500e-03$
0.8	-0.75690382	-0.86953884	-1.10570133	-1.36460668	$4.60667700e-03$
1.0	-0.39575960	-0.55229398	-0.74531808	-1.00167703	$1.67702700e-03$

The exact solution of Eq. (5.26) is [17]:

$$y(x) = x^3.$$

According to the HPM, we construct the following homotopy:

$$D^{0.5}y(x) = p\left(\frac{6x^{2.5}}{\Gamma(3.5)} - \frac{\Gamma(4)x^{4.5}}{\Gamma(5.5)} + J^{1.5}y(x)\right), \tag{5.28}$$

By substitution of Eq. (3.3) into Eq. (5.28), and then by equating the coefficients of p to zero, we obtain the following series of linear equations:

$$\begin{aligned}
&p^0 : D^{0.5}y_0 = 0,\\
&p^1 : D^{0.5}y_1 = \frac{6x^{2.5}}{\Gamma(3.5)} - \frac{\Gamma(4)x^{4.5}}{\Gamma(5.5)} + J^{1.5}y_0(x),\\
&p^2 : D^{0.5}y_2 = J^{1.5}y_1(x),\\
&\vdots\\
&p^n : D^{0.5}y_n = J^{1.5}y_{n-1}, \qquad n \geq 2.
\end{aligned}$$

Applying the operator $J^{0.5}$ to the above series of linear equations and using initial condition (5.27), we obtain:

$$y_0(x) = 0, \tag{5.29}$$

$$y_1(x) = J^{0.5}\left(\frac{6x^{2.5}}{\Gamma(3.5)} - \frac{\Gamma(4)x^{4.5}}{\Gamma(5.5)}\right) + J^2y_0(x), \tag{5.30}$$

$$y_2(x) = J^2y_1(x), \tag{5.31}$$

$$\vdots$$

$$y_n(x) = J^2y_{n-1}(x), \qquad n \geq 2. \tag{5.32}$$

By solving Eqs. (5.29)-(5.32), we can obtain $y_1, y_2, y_3, \cdots$. Therefore, we can form the 3-terms and 6-terms approximations as the following:

$$\phi_3(x) = x^3 - \frac{x^9}{60480}, \tag{5.33}$$

$$\vdots$$

$$\phi_6(x) = x^3 - \frac{x^{15}}{217945728000}. \tag{5.34}$$

According to VIM, the iteration formula (4.6) of Eq. (5.26) can be written in the following form:

$$y_{k+1}(x) = y_k(x) - J^{0.5}\left(D^{0.5}y_k(x) - \frac{6x^{2.5}}{\Gamma(3.5)} + \frac{\Gamma(4)x^{4.5}}{\Gamma(5.5)} - J^{1.5}y_k(x)\right). \tag{5.35}$$

Assume that initial approximation is as:

$$y_0(x) = 0.$$

Now, by iteration formula (5.35), we can obtain $y_1, y_2, y_3, \cdots$ as the following forms:

$$y_1(x) = x^3 - \frac{x^5}{20}, \tag{5.36}$$

$$y_2(x) = x^3 - \frac{x^7}{840}, \tag{5.37}$$

$$y_3(x) = x^3 - \frac{x^9}{60480}, \tag{5.38}$$

$$\vdots$$

$$y_6(x) = x^3 - \frac{x^{15}}{217945728000}, \tag{5.39}$$

$$\vdots$$

Table 5 shows the approximate values of $y(x)$ for HPM and VIM.

Example 5.4. Consider the following multi-fractional order equation:

$$D^{0.75}y(x) = \frac{x^{0.25}}{\Gamma(1.25)} - \frac{2x^{2.75}}{\Gamma(3.75)} + J^{0.75}\left(y^2(x)\right), \qquad x \in [0,1], \tag{5.40}$$

with the initial condition:

$$y(0) = 0. \tag{5.41}$$

The exact solution of Eq. (5.40) is [17]:

$$y(x) = x.$$

According to the HPM, we construct the following homotopy:

$$D^{0.75}y(x) = p\left(\frac{x^{0.25}}{\Gamma(1.25)} - \frac{2x^{2.75}}{\Gamma(3.75)} + J^{0.75}\left(y^2(x)\right)\right). \tag{5.42}$$

By substitution of Eq. (3.3) into Eq. (5.42), and then by equating the coefficients of p to zero, we obtain the following series of linear equations:

$$
\begin{aligned}
&p^0 : D^{0.75}y_0 = 0,\\
&p^1 : D^{0.75}y_1 = \frac{x^{0.25}}{\Gamma(1.25)} - \frac{2x^{2.75}}{\Gamma(3.75)} + J^{0.75}\left(y_0^2(x)\right),\\
&p^2 : D^{0.75}y_2 = J^{0.75}\left(2y_0(x)y_1(x)\right),\\
&p^3 : D^{0.75}y_3 = J^{0.75}\left(2y_0(x)y_1(x) + y_1^2(x)\right),\\
&\vdots
\end{aligned}
$$

Applying the operator $J^{0.75}$ to the above series of linear equations and using initial condition (5.41), we obtain:

$$y_0(x) = 0, \tag{5.43}$$

$$y_1(x) = J^{0.75}\left(\frac{x^{0.25}}{\Gamma(1.25)} - \frac{2x^{2.75}}{\Gamma(3.75)}\right) + J^{1.5}\left(y_0^2(x)\right), \tag{5.44}$$

$$y_2(x) = J^{1.5}\left(2y_0(x)y_1(x)\right), \tag{5.45}$$

$$\vdots$$

$$
\begin{aligned}
y_{2n+1}(x) = J^{1.5}\big(&2y_0(x)y_{2n-0}(x) + 2y_1(x)y_{2n-1}(x) + \cdots + 2y_{n-1}(x)y_{2n-(n-1)}(x)\\
&+ y_n^2(x)\big), n = 1, 2, 3, \cdots ,
\end{aligned} \tag{5.46}
$$

$$
\begin{aligned}
y_{2n+2}(x) = J^{1.5}&\left(2y_0(x)y_{2n+1-0}(x) + 2y_1(x)y_{2n+1-1}(x) + \cdots + 2y_n(x)y_{2n+1-n}(x)\right),\\
&n = 1, 2, 3, \cdots .
\end{aligned} \tag{5.47}
$$

Thus, by solving Eqs. (5.43)-(5.47), we obtain $y_1, y_2, y_3, \cdots$ as the following:

$$y_1(x) = x - \frac{2x^{3.5}}{\Gamma(4.5)}, \tag{5.48}$$

$$y_2(x) = 0, \tag{5.49}$$

$$y_3(x) = \frac{2x^{3.5}}{\Gamma(4.5)} - \frac{\Gamma(5.5)x^6}{180\Gamma(4.5)} + \frac{4\Gamma(8)x^{8.5}}{(\Gamma(4.5))^2\Gamma(9.5)}. \tag{5.50}$$

$$\vdots$$

Now, we can form the 3-terms approximation as the following:

$$\phi_3(x) = x - \frac{\Gamma(5.5)x^6}{180\Gamma(4.5)} + \frac{4\Gamma(8)x^{8.5}}{(\Gamma(4.5))^2\Gamma(9.5)}. \tag{5.51}$$

Applying to the VIM, the iteration formula (4.6) of Eq. (5.40), can be written in the following form:

$$y_{k+1}(x) = y_k(x) - J^{0.75}\left(D^{0.75}y_k(x) - \frac{x^{0.25}}{\Gamma(1.25)} + \frac{2x^{2.75}}{\Gamma(3.75)}\right) + J^{1.5}\left(y_k^2(x)\right). \tag{5.52}$$

Assume that initial approximation is as:

$$y_0(x) = 0.$$

Now, by iteration formula (5.52), we can obtain $y_1, y_2, y_3, \cdots$ as the following form:

$$y_1(x) = x - \frac{2x^{3.5}}{\Gamma(4.5)}, \tag{5.53}$$

$$y_2(x) = x - \frac{2x^{3.5}}{\Gamma(4.5)} + \frac{2x^3}{\Gamma(4.5)} - \frac{4\Gamma(5.5)x^6}{\Gamma(4.5)\Gamma(7)} + \frac{4\Gamma(8)x^{8.5}}{(\Gamma(4.5))^2\,\Gamma(9.5)}, \tag{5.54}$$

$$\begin{aligned} y_3(x) &= x - 0.000863378902x^{8.5} - 0.0499999999x^6 + 0.001516735689x^{7.5} \\ &+ 0.02866866858x^{5.5} - 0.002744187429x^8 + 2.00163815 \times 10^{-8}x^{18.5} \\ &- 0.000008814106948x^{13.5} - 9.994819626 \times 10^{-7}x^{16} + 0.0002446067134x^{11} \\ &+ 0.000009439482570x^{13} - 0.000262175964x^{10.5}. \end{aligned} \tag{5.55}$$

$$\vdots$$

Table 6 shows the approximate values of $y(x)$ for HPM and VIM. It is clear that in the both methods, the accuracy can be improved by computing more terms of approximated solutions.

TABLE 5. Absolute errors of different order of the HPM and VIM for $x \in [0, 1]$ experiment 3.

x	HPM		VIM	
	Ey_3	Ey_6	Ey_3	Ey_6
0	0	0	0	0
0.2	$8.00000000e-12$	0	$8.00000000e-12$	0
0.4	$4.33000000e-09$	0	$4.33000000e-09$	0
0.6	$1.66600000e-07$	0	$1.66600000e-07$	0
0.8	$2.21920000e-06$	0	$2.21920000e-06$	0
1.0	$1.65344000e-05$	0	$1.65344000e-05$	0

TABLE 6. Absolute errors of the HPM and VIM for $x \in [0, 1]$ experiment 4.

x	HPM Ey_3	VIM Ey_3
0	0	0
0.2	$1.59860000e-06$	$2.50340000e-06$
0.4	$1.01882300e-04$	$8.26766000e-05$
0.6	$1.15014920e-03$	$5.35615900e-04$
0.8	$6.36616360e-03$	$1.53921770e-03$
1.0	$2.37509236e-02$	$1.55991500e-03$

6. Conclusions

Finding the numerical solution of fractional-order Fredholm-Volterra integro-differential equation is the main objective of this work. In order to achieve this, we apply homotopy perturbation method and variational iteration method. These methods provide the solutions in terms of convergent series with easily computable components in a direct way. It is shown that multi-fractional order integro-differential equation is a special case of fractional-order Fredholm-Volterra integro-differential equation. Four typical experiments including two experiments of fractional-order Fredholm-Volterra integro-differential equations and two experiments of multi-fractional order integro-differential equations have been demonstrated in order to illustrate the reliability and accuracy of the methods. It is evident from the solution process that the present methods provide a good approximate solution in comparison with the exact solutions.

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[1] Faculty of Mathematics, Department of Applied Mathematics, K. N. Toosi University of Technology, P. O. Box: 16315-1618, Tehran, Iran.
E-mail address: t_nasiri2008@yahoo.com

[2] Faculty of Mathematics, Department of Applied Mathematics, K. N. Toosi University of Technology, P. O. Box: 16315-1618, Tehran, Iran.
E-mail address: ataei@kntu.ac.ir