Gulf Journal of Mathematics
Vol 2, Issue 4 (2014) 1-13

POSTULATIONS IN PROJECTIVE SPACES OF UNIONS OF A FIXED SCHEMES AND SEVERAL HIGH DEGREE GENERAL CURVES

E. BALLICO[1]

ABSTRACT. Fix a subscheme $Z \subset \mathbb{P}^r$, $r \geq 5$, with $\dim(Z) \leq r-5$ and an integer $g \geq 0$. We prove the existence of an integer Δ (depending only on r, g and the Hilbert polynomial of Z) with the following property. Fix positive integers $c > 0$ $d_i > 0$, $0 \leq g_i \leq g$, $1 \leq i \leq c$, with $\sum_i d_i \geq \Delta$; if $g_i > 0$, then assume, say, $d_i \geq r + rg_i/2$. Let $X \subset \mathbb{P}^r$ be a general union of c disjoint curves $X_1, \dots, X_c$ with $\deg(X_i) = d_i$ and $p_a(X_i) = g_i$. Then $Z \cup X$ has maximal rank, i.e. its Hilbert function is the expected one.

1. Introduction

Let $X \subset \mathbb{P}^r$ be a closed subscheme. The projective scheme X is said to have *maximal rank* if for every integer t either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$, i.e. if for every integer t the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(\mathcal{O}_X(t))$ is either injective or surjective. Set $\delta_r(0) := 1$ and $\delta_r(1) := r+1$. For all integers $g \geq 2$ such that $2 \leq g \leq r-3$ set $\delta_r(g) := r + \lceil g(r-2)/2 \rceil$. For all integers g, r such that $g \geq r-2 \geq 2$ and gr even (resp. gr odd) set $\delta_r(g) := 2 + gr/2$ (resp. $7/2 + gr/2$). For all integers $r \geq 3$, $g \geq 0$ and $d \geq 2g+1$ let $Z(r; d, g)$ denote the set of all smooth curves $X \subset \mathbb{P}^r$ with degree d and genus g. Each set $Z(r; d, g)$ is a non-empty and irreducible open subset of the Hilbert scheme of $\mathbb{P}^r$. For any integer $s \geq 1$ let $Z(r; (d_1, g_1), \dots, (d_s, g_s))$ be the set of all smooth curves $Y \subset \mathbb{P}^r$ with s connected components, say $Y = Y_1 \sqcup \cdots \sqcup Y_s$, with $Y_i \in Z(r; d_i, g_i)$ for all i. This observation explains the word “ general ” in the statement of Theorems 1.1 and 1.2 below. Indeed, each c-ple $(Y_1, \dots, Y_c)$ must be a general element of the irreducible algebraic set $\prod_{i=1}^{c} Z(r; g_i, d_i)$.

Let $p \in \mathbb{Q}[t]$ be an integer valued polynomial. We say that p is *admissible* if it is the Hilbert function of some closed subscheme $Z \subseteq \mathbb{P}^r$ (if Z exists, then $\deg(Z) = \deg(p)$). We only need that each closed scheme $Z \subset \mathbb{P}^r$ has a Hilbert polynomial p and in the statements of Theorems 1.1 and 1.2 we may take the same Δ and k_0 for two closed subscheme of $\mathbb{P}^r$ with the same Hilbert polynomial. Moreover Δ and k_0 are bounded (in terms of r and g) by the degree of p and the sums of the absolute values of the coefficients of p.

Date: Received: May 18, 2014; Accepted: Jul 22, 2014.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 14N05; Secondary 14H99.
Key words and phrases. postulation, Hilbert function, unions of curves.

Theorem 1.1. *Fix integers $r \ge 5$, $g \ge 0$ and an admissible polynomial $p \in \mathbb{Q}[t]$ with $\deg(p) \le r-5$. Fix a closed subscheme $Z \subset \mathbb{P}^r$ with p as its Hilbert polynomial. Then there exists an integer Δ (depending only on r, p and g) with the following property. Fix an integers $c > 0$ and d_i, g_i, $1 \le i \le c$, such that $d_i \ge \delta_r(g_i)$ for all i and $\sum_i d_i \ge \Delta$. Then a general union in $\mathbb{P}^r$ of Z and a general element of $Z(r;(d_1,c_1),\dots,(d_c,g_c))$ has maximal rank.*

Take integers $c > 0$, d_i, g_i, $1 \le i \le c$, such that $g_i \ge 0$ and $d_i \ge \delta_r(g_i)$ for i. The *critical value* of $(r;(d_1,g_1),\dots,(d_c,g_c))$ with respect to the admissible polynomial p or the *critical value* of $(r;p;(d_1,g_1),\dots,(d_c,g_c))$ is the first integer $k \ge \max\{\rho, 2\}$ such that $p(k)+\sum_i(k_id_1+1-g_i) \le \binom{r+k}{r}$, where ρ is the Gotzmann number of p (see [6], [11] or section 2 for more details).

Theorem 1.2. *Fix integers $r \ge 5$, $g \ge 0$ and an admissible polynomial p with $\deg(p) \le r-5$. Fix a closed subscheme $Z \subset \mathbb{P}^r$ with p as its Hilbert polynomial. Then there exists an integer k_0 (depending only on r, p and g) with the following property. Fix an integers $c > 0$ and d_i, g_i, $1 \le i \le c$, such that $d_i \ge \delta_r(g_i)$ for all i and $(d_1,g_1),\dots,(d_c,g_c)$ has critical value $k \ge k_0$. Then a general union in $\mathbb{P}^r$ of Z and a general element X of $Z(r;(d_1,c_1),\dots,(d_c,g_c))$ has maximal rank, i.e. $h^1(\mathcal{I}_{Z\cup X}(t)) = 0$ for all $t \ge k$ and $h^0(\mathcal{I}_{Z\cup X}(k-1)) = 0$.*

In the statement of Theorem 1.2 we always take $k_0 \ge 2$ and k_0 bigger than the Gotzmann number of p.

See [1], [2] and [3] for previous attempts to similar asymptotic statements.

2. Preliminaries

Notation 2.1. For all integers $d > 0$, $k \ge 0$ and $g \ge 0$ set $[d,k,g] := kd+1-g$. Set $[d,k] := [d,k,0]$ and $[0,k] := 0$.

Notice that $[a,k] = 0$ if $a = 0$ and $[a,k] = ak+1$ if $a > 0$.

Fix integers d,g,r with $d \ge \max\{2g+1, g+r\}$ and any $X \in Z(r;d,g)$. Since $d \ge 2g+1$, we have $h^1(\mathcal{O}_X(1)) = 0$. Hence $h^1(\mathcal{O}_X(t)) = 0$ for all $t > 0$. Hence $h^0(\mathcal{O}_X(k)) = kd+1-g$ for all $k > 0$. We obviously have $h^0(\mathcal{O}_X) = 1$. Hence $h^0(\mathcal{O}_X) = 0\cdot d+1-g$ if and only if $g = 0$. Since $d \ge g+r$, a general $W \in Z(r;d,g)$ spans $\mathbb{P}^r$.

For any irreducible and projective variety X and any $P \in X_{\mathrm{reg}}$ let $(2P,X)$ denote the closed subscheme of X with $(\mathcal{I}_{P,X})^2$ as its ideal sheaf. The scheme $(2P,X)$ is a zero-dimensional scheme, $(2P,X)_{\mathrm{red}} = \{P\}$ and $\deg((2P,X)) = 1+\dim(X)$. We say that $(2P,X)$ is a 2-point of X.

Fix an admissible polynomial $p \in \mathbb{Q}[t]$, i.e. the Hilbert polynomial of some closed subscheme of $\mathbb{P}^r$. There is a minimal integer x (called the Gotzmann number of p) such that $h^i(\mathcal{I}_Z(x-i))$ for all $i > 0$ and all schemes $Z \subset \mathbb{P}^r$ with Hilbert polynomial p ([6], [11]); by Castelnuovo-Mumford's lemma we also have $h^i(\mathcal{I}_Z(t-i))$ for all $i > 0$ and all $t \ge x$. We say that 0 is the Gotzmann number of the polynomial $p \equiv 0$. Let ρ be the Gotzmann number of p. Take any $X \in Z(r;(d_1,g_1),\dots,(d_c,g_c))$ and fix an integer $t > 0$ such that $h^1(\mathcal{I}_X(t)) = 0$; if $g_i > 0$ for some i assume $t \ge 2$. Since $\dim(X) = 1$, we have $h^i(\mathcal{O}_X(z)) = 0$ for all

$z \in \mathbb{Z}$ and all $i \geq 2$. Therefore the long cohomology exact sequence associated to the exact sequence

$$0 \to \mathcal{I}_X(y) \to \mathcal{O}_{\mathbb{P}^r}(y) \to \mathcal{O}_X(y) \to 0$$

gives $h^i(\mathcal{I}_X(y)) = 0$ for all $i \geq 3$ and all $y \geq -r$. Since $h^1(\mathcal{O}_X(t-1)) = 0$, we also get $h^2(\mathcal{I}_X(t-1)) = 0$. Hence X has at most regularity t. Now assume that t is at least the Gotzmann number ρ of p and that $X \cap Z = \emptyset$. We get $h^i(\mathcal{I}_{Z\cup X}(t+1-i)) = 0$ for all $i > 0$. In this case we say that the critical value of $(r; p; (d_1, g_1), \dots, (d_c, g_c))$ is $\leq t$. The *critical value* of $(r; p; (d_1, g_1), \dots, (d_c, g_c))$ is the minimal integer $k \geq \max\{\rho, 2\}$ such that $\binom{r+k}{r} \geq p(k) + \sum_i (kd_i + 1 - g_i)$. If $d_1 + \cdots + d_c \geq (r+1)\binom{r+\epsilon}{r}$ (much less is needed), then $h^0(\mathcal{O}_X(\epsilon)) > \binom{r+\epsilon}{r}$ for every $X \in Z(r; (d_1, g_1), \dots, (d_c, g_c))$. Hence $h^1(\mathcal{I}_X(\epsilon)) > 0$ for every $X \in Z(r; (d_1, g_1), \dots, (d_c, g_c))$. Hence in the set-up of Theorems 1.1 and 1.2 with this additional restriction $d_1 + \cdots + d_c \geq (r+1)\binom{r+\epsilon}{r}$ and with $Z \cap X = \emptyset$ the critical value is the minimal integer $k \geq 0$ such that $h^0(\mathcal{O}_{X\cup Z}(k)) \leq h^0(\mathcal{O}_{\mathbb{P}^r}(k))$.

Let $Z'(r; d, g)$ (resp. $Z'(r; (d_1, g_1), \dots, (d_s, g_s))$) denote the closure of $Z(r; d, g)$ (resp. $Z(r; (d_1, g_1), \dots, (d_s, g_s))$) in the Hilbert scheme of $\mathbb{P}^r$. Since the algebraic set $Z(r; (d_1, g_1) \dots, (d_s, g_s))$ is irreducible, we get that the algebraic set $Z'(r; (d_1, g_1) \dots, (d_s, g_s))$ is irreducible. If $g_i = 0$ for all i, then we often write $Z(r; d_1, \dots, d_s)$ (resp. $Z'(r; d_1, \dots, d_c)$) instead of $Z(r; (d_1, 0), \dots, (d_s, 0))$ (resp. $Z'(r; (d_1, g_1), \dots, (d_s, g_s))$). We use the short-hand $Z(r; d_1, \dots, d_s)$ even if $d_i = 0$ for some i, taking $Y_i = \emptyset$, i.e. we set $Z(r; 0, 0) := \{\emptyset\}$.

Remark 2.2. Fix $X \in Z(r; d, c)$. Let $E \subset \mathbb{P}^r$ be a smooth rational curve such that the curve $X \cup E$ is nodal and connected and $x := \sharp(X \cap E) \leq 2$. Set $e := \deg(E)$. The curve $X \cup E$ has arithmetic genus $g + x - 1$ and degree $d + e$. Since $X \cup E$ is nodal, its normal sheaf is locally free. We claim that $X \cup E \in Z'(r; d + e, g + e - 1)$. This statement is almost contained in both [9] and [13] (both papers handle more general and difficult cases); it can be proved first degenerating E to a connected union E' of e lines with $E' \cup X$ nodal and then applying to $X \cup E'$ either [13, Theorem 5.2] or [9, Theorem 4.1, Remark 4.1.2 and Corollary 4.2]. A Mayer-Vietoris exact sequence shows that $h^1(\mathcal{O}_{X\cup E}(1)) = 0$. Hence for any $Y \in Z(r; d + e, g + e - 1)$ and any integer $t \geq 0$ we have $h^0(\mathcal{O}_{X\cup E}(t)) = td + 1 - (g + x - 1) = h^0(\mathcal{O}_Y(t))$. Let $Z \subset \mathbb{P}^r$ be any closed subscheme with $\dim(Z) \leq r - 2$ and $Z \cap (X \cup E) = \emptyset$. Fix $t \in \mathbb{N}$. By the semicontinuity theorem for cohomology ([7, Theorem III.12.8]) to prove that either $h^1(\mathcal{I}_{Z\cup Y}(t)) = 0$ or $h^0(\mathcal{I}_{Z\cup Y}(t)) = 0$ for a general $Y \in Z(r; d + e, g + x - 1)$ it is sufficient to prove that either $h^1(\mathcal{I}_{Z\cup X\cup E}(t)) = 0$ or $h^0(\mathcal{I}_{Z\cup X\cup E}(t)) = 0$.

For each smooth curve $X \subset \mathbb{P}^r$ let N_X denote its normal bundle.

Lemma 2.3. *Fix integers $r \geq 4$, $g \geq 0$ and $D \geq \delta_r(g)$. Let $X \subset \mathbb{P}^r$ be a general smooth curve of degree D and genus g. Then $h^1(N_X(-1)) = 0$.*

Proof. Since X is smooth, $N_C(-1)$ is a quotient of $T\mathbb{P}^r(-1)|X$. Call $\mathcal{F}$ the kernel of the surjection $T\mathbb{P}^r(-1)|X \to N_X(-1)$. Since $\dim(X) = 1$, we have $h^2(\mathcal{F}) = 0$. Hence $h^1(N_X(-1)) \leq h^1(T\mathbb{P}(-1)|X)$.

If $g = 0$, then $h^1(T\mathbb{P}(-1)|X) = 0$, because $T\mathbb{P}(-1)|X$ is a quotient of $\mathcal{O}_X^{r+1}$ by the Euler's sequence of $T\mathbb{P}^r$.

Now assume $g = 1$. We claim that $h^1(T\mathbb{P}(-1)|X) = 0$ by the Atiyah's classification of vector bundles on X. Indeed, by the universal property of the Grassmannian we may take as $T\mathbb{P}^r(-1)|X$ any spanned rank r vector bundle E on C with $\deg(E) = \deg(C)$; for an elliptic curve X we take E with no trivial factor and hence $h^1(T\mathbb{P}(-1)|X) = 0$.

Now assume $g \geq 2$. By the semicontinuity theorem it is sufficient to prove the existence of a smooth curve $X \subset \mathbb{P}^r$ of genus g and degree D such that $h^1(N_X(-1)) = 0$.

First assume $g \geq r - 2$ and gr even. Set $t := g(r-2)/2 + 1$, $x := 1 + (2t - 2)/(r-2) = g + 1$ and $d := D - t$. Since $D \geq \delta_r(g)$, we have $d \geq x$. Apply [5, Lemma 3] with $n := r$, $z := g$, $d := D - t$, $g := 0$, the integers x, t and z just defined, starting with a general rational curve of degree $D - t$; our assumption on g gives $D - t \geq g + 2 \geq r$.

Now assume $g \geq r-2$ and gr odd. Set $t := 1 + \lceil g(r-2)/2 \rceil = 3/2 + g(r-2)/2$. We have $(2t-2)/(r-2) = [1 + g(r-2)]/(r-2) \leq g + 1$. Apply [5, Lemme 3] with $n := r$, $z := g$, $g := 0$, the integers x, t and z just defined, starting with a general rational curve of degree $d := D - t$; our assumption on g gives $D - t \geq 1 + (2t-2)/(r-2) \geq g + 2 \geq r$.

Now assume $2 \leq g \leq r - 3$. In this case in [5, Lemme 3] we take $d := r$, $g := 0$, $t := \lceil g(r-2)/2 \rceil$, $x := 1 + (2t-2)/(r-2)$, and $z := g + 1$. We need to check that $r \geq 1 + (2t-2)/(r-2)$, i.e. that $(r-2)(r-1) \geq 2t - 2$. This inequality is satisfied, because $t \leq g(r-2)/2 + 1/2 \leq (r-3)(r-2)/2 + 1/2$ and hence $2t - 2 \leq (r-3)(r-2)$. □

Remark 2.4. Fix integers $r \geq 3$, $g \geq 0$, $d \geq g + r$ and $x > 0$. Fix any $C \in Z(r; d, g)$ and any $S \subset C$ such that $\sharp(S) = x$. To prove that for a general $B \subset \mathbb{P}^r$ there is $X \in Z(r; d, g)$ with $B \subset X$ and that the set of all such curves X has the expected dimension $\dim(Z(r; d, g)) - x(r-1)$ it is sufficient to prove that $h^1(N_C(-S)) = 0$ (use the proof of [12, Theorem 1.5] or [5]). Moreover the set of all such curves C has pure dimension $h^0(N_C(-S)) = (r+1)d + (r-3)(1-g) - x(r-1)$. Since C is smooth, N_C is a quotient of $T\mathbb{P}^r|C$ and hence it is a quotient of $\mathcal{O}_C(1)^{\oplus(r+1)}$ by the Euler's sequence of $T\mathbb{P}^r$. Hence $h^1(N_C(-S)) = 0$ if $h^1(\mathcal{O}_C(1)(-S)) = 0$. This is true if $d - x \geq 2g - 1$. In particular if $g = 0$, then it is sufficient to assume $d \geq x + 1$. In the case $g = 1$ we may use Atiyah's classification of vector bundles on elliptic curves and conclude (since $N_C(-1)$ is spanned and with no trivial factor if $x \leq d$. In the case $g = 1$ we may do a proof which works for all $d \geq x$ in the following way. We first reduce to the case $x = d = r$. Then we use that any two $(r+2)$-ples of points of $\mathbb{P}^r$ in linearly general position are projectively equivalent and that any $r + 2$ general points of an integral and non-degenerate curve $C \subset \mathbb{P}^r$ are linearly independent. Fix a general $S \subset H$ such that $\sharp(S) = d$. To prove the existence of $X \in Z(r; d, g)$ such that $X \cap H = S$ it is sufficient to prove the existence of $C \in Z(r; d, g)$ with $h^1(N_C(-1)) = 0$. If it exists, we also know that $A \cap H$ is a general subset of H with cardinality d for a general $A \in Z(r; d, g)$.

The first part of Remark 2.4 and an obvious dimensional count give the following two lemmas.

Lemma 2.5. *Fix integers $n \ge 3$ and $x > 0$. Fix any closed subscheme $T \subset \mathbb{P}^n$ such that $\dim(T) \le n-2$. Let $S \subset \mathbb{P}^n$ be a general subset with $\sharp(S) \le x+1$. Then there exists a smooth, connected and rational curve $C \subset \mathbb{P}^n$ such that $S \subset C$, $\deg(C) = x$ and $C \cap T = \emptyset$.*

Lemma 2.6. *Let $Z \subset \mathbb{P}^n$, $n \ge 3$, be a closed subscheme with $\dim(Z) \le n-2$. Fix integers $y > 0$, $x_i > 0$, $z_i \ge x_i$, $1 \le i \le y$, $f \ge 0$, $z_1 + \cdots + z_y \ge x_1 + \cdots + x_y$ and a general set $S \cup S_1 \cup \cdots S_y \subset \mathbb{P}^n$ such that $\sharp(S_i) = x_i+1$, $\sharp(S) = f$, $S \cap S_i = \emptyset$ for all i and $S_i \cap S_j = \emptyset$ for all $i \ne j$. Then there are smooth rational curves C_i, $1 \le i \le y$, such that $\deg(C_i) = z_i$, $S_i \subset C_i$, $C_i \cap Z = \emptyset$ for all i, $C_i \ne C_j$ for all $i \ne j$, $C_1 \cup \cdots \cup C_y$ is nodal, and $S = \mathrm{Sing}(C_1 \cup \cdots \cup C_y)$.*

Lemma 2.7. *Fix integer $r \ge 3$ and an integer valued polynomial p such that $\deg(p) \le r-2$ and with positive leading coefficient. Fix an integer $x > 0$ such that $p(y) \le p(y+1)$ for all $y \ge x-1$ and $p(x) \le \binom{r+x}{r}$ (x exists, because $\deg(p) < r$ and the leading coefficient of p is > 0). Then there is an positive integer $a \ge \binom{r+x}{r}/x$ with the following property. Fix integers $c > 0$ and $d_i > 0$ such that $\sum_{i=1}^{c} d_i \ge a$. Let k be the minimal integer $\ge x$ such that $p(k) + \sum_{i=1}^{c}[d_i,k] \le \binom{r+k}{r}$ (it exists, because $\deg(p) < r$). Then $p(y) + \sum_{i=1}^{c}[d_i,y] \le \binom{r+y}{r}$ for all $y > k$ and the function $y \mapsto \binom{r+y}{r} - p(y) + \sum_{i=1}^{c}[d_c,y]$ is non-decreasing.*

Proof. Since $a \ge \binom{r+x}{r}/x$ and $p(x) \ge 0$, we have $p(x) + \sum_{i=1}^{c}[d_c,x] > \binom{r+x}{r}$ and hence $k > x$. It is sufficient to prove the last property. Call $\psi(y)$ this function and fix $y > k$. We have $\psi(y) - \psi(y-1) = \binom{r+y-1}{r-1} - p(y) + p(y-1) - (d_1 + \cdots + d_c)$. Since $p(k) + \sum_{i=1}^{c}[d_i,k] \le \binom{r+k}{r}$, we have $d_1 + \cdots + d_c \le (\binom{r+k}{r} - p(k))/k$. Since $\deg(p) \le r-2$, we have $x_1 + \cdots + x_z \sim \binom{r+k}{r}/k \sim r^{k-2}/r!$ if $k \gg k_o(r,p)$ (with $k_o(r,p)$ depending only on r and p) for all positive integers $z, x_1, \dots, x_z$ with critical value k and hence $d_1 + \cdots d_c \sim k^{r-2}/r! \ll \binom{r+k-1}{r-1}$. Use that $\deg(p) \le r-2$. $\square$

The integer k is often called the *critical value* of $d_1, \dots, d_c$ with respect to r and the polynomial p (in the case $p \equiv 0$ used in [4] it was called the *crucial value*).

Lemma 2.8. *Fix r, p and g for which Theorem 1.2 is true for all integers $k \ge k_0$. Set $\Delta := \binom{r+k_0}{r}$. Then Theorem 1.1 is true for r, p and g with this integer Δ.*

Proof. Fix integers $c > 0$, d_i, g_i, $1 \le i \le c$, such that $d_i \ge \delta_r(g_i)$ for all i. It is sufficient to note that $(r; q; (d_1, g_1), \dots, (d_c, g_c))$ has critical value $\ge k_0$ even with respect to the polynomial $q(t) \equiv 0$ and that $q(k) \ge 0$ if k is at least the Gotzmann number of q. $\square$

Remark 2.9. By Lemma 2.8 to prove Theorem 1.1 it is sufficient to prove Theorem 1.2.

3. Trees and the case $g = 0$ of Theorem 1.2

A degree x tree T of $\mathbb{P}^r$, $r \ge 3$, is a connected and nodal curve $T \subset \mathbb{P}^r$ such that $\deg(T) = x$, $p_a(X) = 0$ and every irreducible component of T has degree one. We call $\emptyset$ a tree of degree zero. Let $W(r;d_1,\dots,d_c)$ denotes the set of all curves $Y\subset\mathbb{P}^r$ with c connected components, say $Y=Y_1\sqcup\cdots\sqcup Y_c$ with each Y_i a tree of degree d_i. If $x\ge 4$ the algebraic set $W(r;x)$ is not irreducible (it is equidimensional and its connected components are irreducible). For any integer valued admissible polynomial p we define the following 5 assertions:

Assertion $T(r,p,k)$. Fix an integer $r\ge 4$ and the admissible polynomial $p\in\mathbb{Q}[t]$ with $p\equiv 0$ if $r=4$ and $\deg(p)\le r-5$ if $r\ge 5$. Fix integers $c>0$, $a_i\ge 0$ such that $p(k)+\sum_{i=1}^{c}[a_i,k]\le\binom{r+k}{r}$ and any $Z\subset\mathbb{P}^r$ with p as its Hilbert polynomial. Then there exists $Y\in W(r;a_1,\dots,a_c)$ such that $Z\cap Y=\emptyset$ and $h^1(\mathcal{I}_{Z\cup Y}(k))=0$.

Assertion $T'(r,p,k)$ Fix an integer $r\ge 4$ and the admissible polynomial $p\in\mathbb{Q}[t]$ with $p\equiv 0$ if $r=4$ and $\deg(p)\le r-5$ if $r\ge 5$. Fix integers $c>0$ and $a_i\ge 0$, $1\le c$, and $p(k)+\sum_{i=1}^{c}[a_i,k]\ge\binom{r+k}{r}$. Fix any $Z\subset\mathbb{P}^r$ with p as its Hilbert polynomial. Then there exists $Y\in W(r;a_1,\dots,a_c)$ such that $Z\cap Y=\emptyset$ and $h^0(\mathcal{I}_{Z\cup Y}(k))=0$.

Assertion $T(r,p,k)_s$. Fix an integer $r\ge 4$ and the admissible polynomial $p\in\mathbb{Q}[t]$ with $p\equiv 0$ if $r=4$ and $\deg(p)\le r-5$ if $r\ge 5$. Fix integers $c\ge s$, $a_i\ge 0$, $1\le i\le c$, such that $a_i=1$ for all $i\le s$ and $p(k)+\sum_{i=1}^{c}[a_i,k]\le\binom{r+k}{r}$. Fix any $Z\subset\mathbb{P}^r$ with p as its admissible polynomial. Then there exists $Y\in W(r;a_1,\dots,a_c)$ such that $Z\cap Y=\emptyset$ and $h^1(\mathcal{I}_{Z\cup Y}(k))=0$.

Assertion $T'(r,p,k)_s$. Fix integers $r\ge 4$, $s\ge 0$, and the admissible polynomial $p\in\mathbb{Q}[t]$ with $p\equiv 0$ if $r=4$ and $\deg(p)\le r-5$ if $r\ge 5$. Fix integers $c\ge s$, and $a_i\ge 0$, $1\le i\le c$, such that $a_i=1$ for all $i\le s$ and $p(k)+\sum_{i=1}^{c}[a_i,k]\ge\binom{r+k}{r}$. Fix any $Z\subset\mathbb{P}^r$ with p as its admissible polynomial. Then there exists $Y\in W(r;a_1,\dots,a_c)$ such that $Z\cap Y=\emptyset$ and $h^0(\mathcal{I}_{Z\cup Y}(k))=0$.

Assertion $H(x,N)$, $N\ge 4$, $x\ge 1$. Take any positive integers c and d_i, $1\le i\le c$, such that $\sum_{i=1}^{c}x(d_i+1)\le\binom{N+x}{N}$. Then there is $Y\in W(r;d_1,\dots,d_c)$ such that $h^1(\mathcal{I}_Y(x))=0$.

In [4, page 588] there is a similar assertion with the same name; it gives a stronger thesis, but it make a mild restriction on the integers c and d_i, $1\le i\le c$, i.e. in [4] the authors required that x is the critical value of $d_1,\dots,d_c$, i.e. that $\sum_{i=1}^{c}(x-1)(d_i+1)>\binom{N+x-1}{N}$. Take $d_1,\dots,d_c$ and assume that it has critical value $y<x$, i.e. that y is the first positive integer y such that $\sum_{i=1}^{c}y(d_i+1)\le\binom{N+y}{N}$. By [4] there is $Y\in W(r;d_1,\dots,d_c)$ such that $h^1(\mathcal{I}_Y(y))=0$. Castelnuovo-Mumford's lemma gives $h^1(\mathcal{I}_Y(x))=0$. Hence [4] proves $H(x,N)$ in the form we stated here for all $N\ge 4$ and all $x\ge 1$.

Remark 3.1. Since $p\equiv 0$ if $r=4$, $T(4,p,k)$ is true by $H(k,4)$. Hence $T(4,p,k)_s$ is true for any $s\ge 0$.

Remark 3.2. Fix a projective scheme $Z\subset\mathbb{P}^r$, an integer $k>0$ and a disjoint union $A\subset\mathbb{P}^r$ of c trees. Assume $A\cap Z=\emptyset$.

(i) Let $B\subset A$ be any subcurve. We have $B\cap Z$. B is a disjoint union of finitely many trees. If $h^0(\mathcal{I}_{Z\cup B}(k))=0$, then obviously $h^0(\mathcal{I}_{Z\cup A}(k))=0$.

(ii) Assume $h^1(\mathcal{I}_{Z\cup A}(k)) = 0$, i.e. assume that the restriction map $\rho : H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(\mathcal{O}_{Z\cup A}(k))$ is surjective. Take a line $L \subset A$ which is a final line of a connected component of A and let B be the closure of $A \setminus L$. If L is a connected component of A, then $h^0(\mathcal{O}_A(k)) = h^0(\mathcal{O}_B(k)) + k + 1$. If L is not a connected component of A, then B is a disjoint union of c trees and $h^0(\mathcal{O}_B(k)) + k$ (use a Mayer-Vietoris exact sequence). In both cases the restriction map $\rho' : H^0(\mathcal{O}_{Z\cup A}(k)) \to H^0(\mathcal{O}_{Z\cup B}(k))$ is surjective (use a Mayer-Vietoris exact sequence). Since $\rho' \circ \rho$ is surjective, we get $h^1(\mathcal{I}_{Z\cup B}(k)) = 0$.

(iii) Fix $B \in Z(r; d, q)$ and a line $L \subset \mathbb{P}^r$ such that $1 \le \sharp(L \cap B) \le 2$ and L meets quasi-transversally B. We have $B \cup L \in Z'(r; d+1, g + \sharp(L\cap B) - 1)$ ([9], [13]). Fix an integer $k > 0$ and a scheme $Z \subset \mathbb{P}^r$ such that $Z \cap (B \cup L) = \emptyset$. A Mayer-Vietoris exact sequence gives that the restriction map $H^0(\mathcal{O}_{B\cup L}(k)) \to H^0(\mathcal{O}_B(k))$ is surjective. Hence if $h^1(\mathcal{I}_{Z\cup B\cup L}(k)) = 0$, then $h^1(\mathcal{I}_{Z\cup B}(k)) = 0$. Obviously if $h^0(\mathcal{I}_{Z\cup B}(k)) = 0$, then $h^0(\mathcal{I}_{Z\cup B\cup L}(k)) = 0$.

For all integer-valued polynomials p and all positive integers r, k define the integers $a_{r,p,k}$ and $b_{r,p,k}$ by the following relations:

$$p(k) + (k+1)a_{r,p,k} - b_{r,p,k} = \binom{r+k}{k},\ 0 \le b_{r,p,k} \le k \tag{3.1}$$

For each integer $k \ge \eta$ set $a'_{r,p,k} := a_{r,p,k}$ if $b_{r,p,k} = 0$ and $a'_{r,p,k} := a_{r,p,\eta} - 1$ if $b_{r,p,k} > 0$. Set $b'_{r,p,k} := \binom{r+k}{r} - p(k) - (k+1)a'_{r,p,k}$. We have $0 \le b'_{r,p,k} \le k$. If $b_{r,p,k} = 0$, then $b'_{r,p,k} = 0$. If $b_{r,p,k} > 0$, then $b'_{r,p,k} = k + 1 - b_{r,p,k}$.

Assertion $H_{r,p,k}$: We have $a_{r,p,k} \ge 2b_{r,p(k),k}$ and $h^i(\mathcal{I}_{Z\cup Y}(k)) = 0$, $i = 0, 1$, where $Y \subset \mathbb{P}^r$ be a general union of $a_{r,p,k} - 2b_{r,p,k}$ lines and $b_{r,p,k}$ reducible conics whose singular point is contained in H.

We recall the following lemma (Assertion B'_t and Claim 1 of [2, page 15]) (we only need the case $\deg(p(t)) \le r - 5$).

Lemma 3.3. *Fix an integer $r \ge 4$ and a closed subscheme $Z \subset \mathbb{P}^r$ such that $\dim(Z) \le r - 3$. Then there is an integer $\eta(r, p)$ (only depending on r and the Hilbert polynomial p) of such that $H_{r,p,k}$ is true for all $k \ge \eta(r, p)$.*

Increasing if necessary $\eta(r, p)$ we may also assume that $\eta(r, p) - r - 2$ is at least the Gotzmann number of p and that $a_{r,p,t} - a_{r,p,t-1} \ge 2t$ for all $t \ge \eta(r, p)$ (here we use that $r \ge 4$ and the asymptotic estimates in the proof of Lemma 2.7). Hence $h^i(\mathcal{O}_Z(t - r)) = 0$ for all $t \ge \eta(r, p) - 2$. We also assume that $\eta(r, p)$ is so large that the asymptotic bounds works for the proof of Lemma 3.5 below (Claims 1 and 2).

Take $\gamma := a_{r,p,\eta(r,p)+5}$. Fix integers $c > 0$ and $d_i > 0$, $1 \le i \le c$, such that $d_i = 1$ for at least γ indices i. Let k be the minimal integer $k \ge \eta(r, p)$ such that $p(k) + \sum_{i=1}^{c}(d_ik + 1) \le \binom{r+k}{k}$.

Lemma 3.4. *We have $k \ge \eta(r, p) + 5$ and $p(k-1) + \sum_{i=1}^{c}(d_i(k-1)+1) > \binom{r+k-1}{r}$.*

Proof. Since $a_{r,p,t} - a_{r,p,t-1} \ge 2t$ for all $t \ge \eta(r, p)$ and $\sum_{i=1}^{c}(d_ik + 1) \ge \gamma(k + 1)$, we get $k \ge \eta(r, p) + 5$. The second assertion follows from the inequality $k >$

$\eta(r,p)$ and the definition of k as the minimal integer $\ge \eta(r,p)$ with a certain property. $\square$

In this section we prove that for each $r \ge 5$ and any admissible polynomial p with $\deg(p(t)) \le r-5$ there in integer ψ (depending only on r and p such that $T(r,p,k)$ and $T'(r,p,k)$ are true for all $k \ge \psi$ (see Corollary 3.8).

Standing Assumption: We fix an integer $r \ge 5$ and an admissible polynomial p with $\deg(p(t)) \le r-5$. We assume the existence of an integer μ (depending only on $r-1$ and p_1 and hence only on r and p) such that $T(r-1,p_1,k)$ is true for all $k \ge \mu$. Since $p_1 = 0$ if $r=5$, we are allowed to make this inductive proof starting with the case $r=4$ (Remark 3.1).

Lemma 3.5. *Assertion $T(r,p,k)_\gamma$ is true for all k.*

Proof. Let $Z \subset \mathbb{P}^r$ be a closed subscheme with p as its Hilbert polynomial. Fix integers $c \ge \gamma$ and $d_i > 0$, $1 \le i \le c$, such that $d_i = 1$ for at least γ indices i and $p(k) + \sum_i [d_i,k] \le \binom{r+k}{k}$. We need to find $Y \in W(r;d_1,\dots,d_c)$ such that $Y \cap Z = \emptyset$ and $h^1(\mathcal{I}_{Z\cup Y}(k)) = 0$. We use induction on k, the case $k \le \eta+4$ being true, because no such $c,d_1,\dots,d_c$ exists in this case by the definition of the integer γ (Lemma 3.4). Now assume $k \ge \eta+5$ and that the lemma is true for all data with critical value $\le k-1$. Since this datum has critical value $> k-1$, we have $p(k-1) + \sum_i [d_i,k-1] > \binom{r+k-1}{r}$. Since $\gamma \ge a_{r,p,\eta+4}$, there are integers a_i such that $a_i = 1$ for all $i \le \gamma$, $0 \le a_i \le d_i$ for all i, and $0 \le f \le k-1$, where $f := \binom{r+k-1}{r} - \sum_{i=1}^{c}[a_i,k-1]$. By the inductive assumption there is $W \in W(r;a_1,\dots,a_c)$ with $h^1(\mathcal{I}_{Z\cup W}(k-1)) = 0$ and hence $h^0(\mathcal{I}_{Z\cup W}(k)) = f$. Deforming if necessary W we may assume that $W \cap H$ is a general subset of H with cardinality $\deg(W)$.

Claim 1: We have $d_1 + \cdots + d_c \ge a_1 + \cdots + a_c + 2f$.

Proof of Claim 1: Assume $d_1 + \cdots + d_c \le a_1 + \cdots + a_c + 2f - 1$. Let x be the number of integers i with $a_i > 0$. Since $d_i > 0$ for all i, we may assume $a_i \ne 0$ if and only if $i \le x$. Since $p(k-1) + \sum_{i=1}^{c}[a_i,k-1] = \binom{r+k-1}{r} - f$, $p(k) + c + k(d_1 + \cdots + d_k) \ge \binom{r+k}{r} - k$ and $f \le k-1$, we get $p(k) - p(k-1) + (c-x) + (a_1 + \cdots + a_x) + k(2k-1) \ge \binom{r+k-1}{r-1}$. Since $a_1 + \cdots + a_x \le \binom{r+k-1}{r}/k$, for large k we get a contradiction.

Claim 2: We have $3(a_1 + \cdots + a_c)/2 \ge d_1 + \cdots + d_c$.

Proof of Claim 2: Indeed, recall that $\deg(p) \le r-5$; we only need here that $\deg(p) \le r-3$. Since $p(k-1) + (k-1)(a_1 + \cdots + a_c) \ge \binom{r+k-1}{r} - k + 1$, we have $(a_1 + \cdots + a_c) \ge (\binom{r+k-1}{r} - p(k-1))/k$, $p(k-1) \le k^{r-2}$ for $k \gg 0$, $p(k) \ge 0$, and $d_1 + \cdots + d_c \le (\binom{r+k}{k} - c - p(k))/k$.

By Claim 2 we may also assume that if $d_i \ge 2$, then $a_i \ge d_i/2$. Let $W_1,\dots,W_c$ be the connected components of W with $\deg(W_i) = a_i$. Fix a general $S \subset H$ such that $\sharp(S) = f$. For all i with $(d_i,a_i) \ne (1,0)$ fix $S_i \subseteq W_i \cap H$ such that $\sharp(S_i) = d_i - a_i$. For all i with $(d_i,a_i) = (1,0)$ set $S_i = \emptyset$.

Let $E \subset H$ be a general union of $d_1 + \cdots + d_s - a_1 - \cdots - a_s - 2f$ lines and f reducible conics, with the only restriction that $S = \mathrm{Sing}(Y)$, $E \cap (W \cap H) = S_1 \cup \cdots \cup S_c$ and that each irreducible component of E contains at most one point of $W \cap H$ (if $a_i > 0$ for all i we get that each irreducible component of H contains exactly one point of $W \cap H$). For each $P \in \mathrm{Sing}(E)$ take any 3-dimensional linear space $V_P \subset \mathbb{P}^r$ such that $V_P \cap H$ is the plane containing the reducible conic of E containing P. Set $v_P := \{2P, V_P\}$ and $\chi := \cup_{P \in \mathrm{Sing}(Y)} v_P$. The scheme $E \cup \chi$ is a flat limit of a family of $2f$ disjoint lines of $\mathbb{P}^r$ containing $\mathrm{Sing}(E)$ ([8], [10]). We may deform $E \cup \chi$ to a family of disjoint lines keeping fixed the points of $E \cap (W \cap H)$. Hence $W \cup E \cup \chi \in W'(r; d_1, \dots, d_c)$. The set $S = \mathrm{Sing}(E)$ is formed by f general points of H. Since $\mathrm{Res}_H(Z \cup Y \cup E \cup \chi) = Z \cup Y \cup S$, we have $h^i(\mathcal{I}_{\mathrm{Res}_H(Z \cup Y \cup E \cup \chi)}(k-1)) = 0$, $i = 0, 1$. By the definition of the integer f we have $h^0(H, \mathcal{O}_{(Z \cap H) \cup (Y \cap H) \cup E}(k)) = \binom{r+k-1}{r} - (\binom{r+k}{r} - p(k) - \sum_i [d_i, k]) \le \binom{r+k-1}{r-1}$. By the inductive assumption on r we may assume $h^1(H, \mathcal{I}_{(Z \cap H) \cup E}(k)) = 0$. Since the set $Y \cap H \setminus (Y \cap H) \cup E$ is general in H we get $h^1(H, \mathcal{I}_{(Z \cap H) \cup (Y \cap H) \cup E}(k)) = 0$. Hence the Castelnuovo's inequality gives $h^1(\mathcal{I}_{Z \cup W \cup E \cup \chi}(k)) = 0$. □

Minimal modifications of the proof of Lemma 3.5 gives the following result.

Lemma 3.6. *Assertion $T'(r, p, k)_\gamma$ is true for all k.*

Lemma 3.7. *There are integers η_i, $0 \le i \le \gamma$, (depending only on r and p) such that $T(r, p, k)_{\gamma - i}$ and $T'(r, p, k)_{\gamma - i}$ are true for all $k \ge \eta_i$.*

Proof. Set $\eta_0 := \eta + 5$. It works for the case $i = 0$ by Lemmas 3.5 and 3.6. For each integer i such that $1 \le i \le \gamma$ we want to define an integer η_i such that $T(r, p, k)_{\gamma - i}$ and $T'(r, p, k-1)_{\gamma - i}$ are true for all integers $k \ge \eta_i$. We always take $\eta_i \le \eta_{i+1}$ if $i < \gamma$. Hence $\eta_i > \eta$ for all i. By the definition of the integer η for all $k \ge \eta_i - 2$ we may use the inductive assumption in $\mathbb{P}^{r-1}$ for the Hilbert polynomial $p_1(t)$ and hence for the scheme $Z \cap H$. We fix $i \in \{1, \dots, \gamma\}$ and we assume that we have proved the existence of η_{i-1}. We work with integers $k \ge \eta_{i-1} + 2$ and hence implicitly we impose that the integer η_i satisfies $\eta_i \ge \eta_{i-1} + 2$. We work in the set-up of $T(r, p, k)_{\gamma - 1}$, the other case being similar (for $T'(r, p, k)_{\gamma - i}$ we reduce to the case $\binom{r+k}{r} \le p(k) + \sum_i [d_i, k] \le \binom{r+k}{r} + k$). Fix integers $c > 0$ and $d_i > 0$, $1 \le i \le c$, with critical value $k \ge \eta_{i-1} + 2$ and such that $d_i = 1$ for at least $\gamma - i$ indices. If $d_i = 1$ for at least $\gamma - i + 1$ indices, then the definition of η_{i-1} gives the existence of $A, B \in W(r; d_1, \dots, d_c)$ such that $A \cap Z = B \cap Z = \emptyset$, $h^1(\mathcal{I}_A(k)) = 0$ and $h^0(\mathcal{I}_B(k-1)) = 0$. Hence to check if $T(r, p, k)$ and $T'(r, p, k-1)$ are satisfied by the datum $d_1, \dots, d_c$, we may assume that $d_i = 1$ for exactly $\gamma - i$ indices, say $d_i = 1$ if and only if $1 \le i \le \gamma - i$, and that d_c is the largest d_j's. By part (ii) of Remark 3.2 iterated several times we may assume that $p(k) + \sum_i [d_i, k] \le \binom{r+k}{r} - k + 1$.

(a) In this step we assume $d_c \ge 3$. Write $d'_i := d_i$ for $i < c$, $d'_c = d_i - 2$ and $d'_c c + 1 = 1$. We have $\sum_i [d_i, k] = \sum_i^{c+1} [d'_i, k] + k - 1$ and hence $p(k) + \sum_{i=1}^{c+1} [d', k] \ge \binom{r+k}{r} - 2k + 1$. As in the proof of Lemma 3.5 we see the existence of integers a_i, $1 \le i \le c+1$, such that $a_i = 1$ if either $i \le \gamma - i$ or $i = c+1$, $0 \le a_i \le d'_i$ for all i and $0 \le f \le k - 1$, where $f := \binom{r+k-1}{r} - p(k-1) - \sum_i [a_i, k-1]$. We may also assume that Claims 1 and 2 of the proof of Lemma 3.5 are true for our $a_1, \dots, a_{c+1}$. Since $k - 1 \ge \eta_{i-1}$ and $a_i = 1$ for $\gamma - i + 1$ indices, we get $h^1(\mathcal{I}_{Z \cup Y}(k-1)) = 0$ for a general $Y \in W(r; a_1, \dots, a_{c+1})$. We continue as in the proof of Lemma 3.5.

(b) In this step we assume $d_c = 2$ and hence $d_j = 2$ for all $i \ge \gamma - i$. Set $d_i'' = 1$ for all $i \le \gamma - i + 1$ and $d_i'' = 2$ for all $i > \gamma - i$. We have $\sum_i [d_i, k] = \sum_i [d_i', k] + k - 1$ and hence $p(k) + \sum_i [d_i', k-1] \ge \binom{r+k-1}{r} - 2k$. We use the proof of Lemma 3.5 with d_i'' instead of d_i. □

Set $\psi := \eta_\gamma$. Lemma 3.7 gives the following result.

Corollary 3.8. *There is an integer ψ (depending only on r and p) such that $T(r,p,k)$ and $T'(r,p,k)$ are true for all $k \ge \psi$.*

Proof of Theorem 1.2 when $g = 0$: Set $k_0 := \psi + 1$, where ψ is the integer appearing in Corollary 3.8. Fix $Z \subset \mathbb{P}^r$ with p as its Hilbert polynomial and and fix $c > 0$ and $d_i > 0$, $1 \le i \le c$, and with critical value $k \ge k_0$ with respect to p. Since $Z(r; d_1, \dots, d_c)$ is irreducible, by the semicontinuity theorem it is sufficient to find $A, B \in Z(r; d_1, \dots, d_c)$ such that $A \cap Z = B \cap Z = \emptyset$, $h^1(\mathcal{I}_{Z \cup B}(k)) = 0$ and $h^0(\mathcal{I}_{Z \cup A}(k-1)) = 0$. Since $T(r,p,k)$ is true, there is $W \in W(r; d_1, \dots, d_r)$ such that $Z \cap W = \emptyset$ and $h^1(\mathcal{I}_{Z \cup W}(k)) = 0$. By the semicontinuity theorem we have $h^1(\mathcal{I}_{Z \cup B}(k)) = 0$ for a general $B \in Z(r; d_1, \dots, d_c)$. Since $k - 1 \ge \psi$, $T'(r,p,k-1)$ is true. Hence here is $E \in W(r; d_1, \dots, d_r)$ such that $Z \cap E = \emptyset$ and $h^1(\mathcal{I}_{Z \cup E}(k)) = 0$. By the semicontinuity theorem we have $h^0(\mathcal{I}_{Z \cup A}(k-1)) = 0$ for a general $A \in Z(r; d_1, \dots, d_c)$. □

In the next section we will also use the case $n = r - 1$ and $q = p_1$ of the next lemma.

Lemma 3.9. *Fix an integer $n \ge 4$ and an admissible polynomial q with $\deg(q) \le n - 5$ (case $n \ge 5$) and $q \equiv 0$ (case $n = 4$). Then there exists an integer k_0 (depending only on r and q) with the following property. Fix integers $k \ge k_0$, $c > 0$, $d_i > 0$, $1 \le i \le c$ and f such that $0 \le f \le k + 1$ and $q(k) + \sum_i [d_i, k] + f(2k+1) \le \binom{r+k}{r}$. Let $W \subset \mathbb{P}^n$ be a closed scheme with q as its Hilbert polynomial. Take a general $E \in Z(n; d_1, \dots, d_c)$ and a general union $G \subset \mathbb{P}^n$ of f reducible conics. Then $W \cap (E \cup F) = \emptyset$ and $h^1(\mathcal{I}_{W \cup E \cup F}(k)) = 0$.*

Proof. There is an integer k_0 such that $T(n,q,k)$ for all $k \ge k_0$. Recall that $W(n; d_1, \dots, d_c) \subseteq Z'(n; d_1, \dots, d_c)$. For any $k \ge k_0$ there is $E' \in W(n; d_1, \dots, d_c)$ and a union $G \subset \mathbb{P}^n$ of f reducible conics such that $E' \cap G = \emptyset$, G has f connected components, $W \cap (E' \cup G)) = \emptyset$, and $h^1(\mathcal{I}_{W \cup E' \cup F}(k)) = 0$. Use the semicontinuity theorem for cohomology. □

4. Proof of Theorem 1.2 for $g > 0$

In this section we conclude the proof of Theorem 1.2 (and hence of Theorem 1.1 by Remark 2.9) proving the case $g > 0$ of Theorem 1.2.

Let k_0' be an integer such that the case $g = 0$ of Theorem 1.2 is true for all $k \ge k_0'$, $T(r-1, p_1, k)$ is true for all $k \ge k_0'$, $T'(r-1, p_1, k)$ is true for all $k \ge k_0' - 1$ and the case $n = r - 1$ and $q = p_1$ holds for all $k \ge k_0'$. We also assume that the asymptotic numerical inequalities used in the proofs of Claim 1 and 2 of Lemma 3.5 are satisfied. With these restrictions we claim that we may take $k_0 = k_0' + 2$. Fix an integer $k \ge k_0' + 1$ and fix c, d_i, g_i, $1 \le i \le c$, with critical value k with respect to p and $g_i \le g$ for all i. Let $Z \subset \mathbb{P}^r$ be a closed subscheme with p as its Hilbert polynomial. Fix a general hyperplane $H \subset \mathbb{P}^r$. Since $Z(r;(d_1,g_1),\dots,(d_c,g_c))$ is irreducible, it is sufficient to find $A, B \in Z(r;(d_1,g_1),\dots,(d_c,g_c))$ such that $A \cap Z = B \cap Z = \emptyset$, $h^1(\mathcal{I}_{Z\cup B}(k)) = 0$ and $h^0(\mathcal{I}_{Z\cup A}(k-1)) = 0$.

(a) Since $(r;p;(d_1,g_1),\dots,(d_c,g_c)))$ has critical value k, we have $p(k-1) + \sum_i [d_i, k-1, g_i] > \binom{r+k-1}{r}$. Set $\alpha := \binom{n+r}{r} - \sum_i [d_i,k,g_i]$. Adding several disjoint lines, i.e. adding data $(d_j,g_j) = (1,0)$ we reduce to the case $0 \le \alpha \le k$ (part (ii) of Remark 3.2). Set $e_i := 0$ if $g_i := 0$ and $e_i = g_i + 1$ if $g_i > 0$. Set $\epsilon_i := 0$ if $g_i = 0$ and $\epsilon_i := 1$ if $g_i > 0$.

Claim 1: We have $p(k-1) + \sum_{i=1}^c [e_i, k-1, 0] \le \binom{r+k-1}{r} - k$.

Proof of Claim 1: If $g_i = 0$, then $[e_i, k-1, 0] = 0$ and hence $2[e_i, k-1, 0] \le [d_i, k, 0]$. Now assume $g_i > 0$. Since $d_i \ge \delta_r(g_i)$ and $r \ge 5$, we have $[d_i, k, g_i] \ge [\delta_r(g_i), k, g_i] \ge 2[e_i, k, 0]$. Use the asymptotic estimates in the proof of Claim 2 in the proof of Lemma 3.5.

Claim 2: We have $p(k-1) + \sum_{i=1}^c [d_i - g_i - \epsilon_i, k-1, 0] \ge \binom{r+k-1}{r} + 4k^2$.

Proof of Claim 2: Since $\alpha \le k$ and $\deg(p) \le r-2$, we have $\sum_i [d_i, k-1, g_i] \sim \binom{r+k}{r}(k-1)/k$. If $x > q > 0$, then $[x-q-1, k-1, 0] = [x, k-1, q] - (qk + k - 1 - 2q)$. Now assume $x \ge \delta_r(q)$. We want to check that in all cases we have $[x-q-1, k-1, 0] \ge (r-4)[x, k-1, 0]/r$. It is sufficient to check the last inequality when $x = \delta_r(q)$. In this case it is sufficient to check that $(qk + k - 1 - 2q) \le 4(\delta_r(q)(k-1) + 1 - q)/r$. First assume $q = 1$. We have $\delta_r(1) = r+1$; since $r(2k-3) \le 4(r+1)(k-1)$, we are done in this case. Now assume $2 \le q \le r-3$ and hence $\delta_r(q) = r + \lceil q(r-2)/2 \rceil$. We have $r(qk + k - 1 - 2q) \le 2[(q(r-2) + 2r)(k-1) + 2 - 2q]$. Now assume $q \ge r-2$. Since $\delta_r(q) \ge 2 + qr/2$, in this case it is sufficient to use that $r(qk+k-1-2q) \le 2[(2q + 4 + q(r-2))(k-1) + 2 - 2q]$. Since $\binom{r+k}{r} = \binom{r+k-1}{r}(r+k)/r$ and $(r+k)/r > rk/(k-1)(r-4)$, we get Claim 2.

By Claims 1 and 2 there are integers a_i such that $e_i \le a_i \le d_i - g_i - \epsilon_i$ and $0 \le f \le k$, where $f := \binom{r+k-1}{r} - p(k-1) - \sum_i [d_i - g_i, k-1, 0] - \alpha$. Let $Y := Y_1 \sqcup \cdots \sqcup Y_c$ be a general element of $Z(r; a_1, \dots, a_c)$. In particular $Z \cap Y = \emptyset$. Since $k - 1 \ge k_0'$, the case $g = 0$ of Theorem 1.2 gives $h^1(\mathcal{I}_{Z\cup Y}(k-1)) = 0$ and hence $h^0(\mathcal{I}_{Z\cup Y}(k-1)) = f + \alpha$. Hence $h^1(\mathcal{I}_{Z\cup Y\cup S}(k-1)) = 0$ and $h^0(\mathcal{I}_{Z\cup Y\cup S}(k-1)) = \alpha$ for a general $S \subset H$ such that $\sharp(S) = f$. If $a_i > 0$ fix any $S_i \subset Y \cap H$ with $\sharp(S_i) = g_i + 1$. If $a_i = 0$ (and hence $g_i = 0$) set $S_i = \emptyset$.

Claim 3: We have $\sum_i (d_i - a_i) \ge 2f + g_1 + \cdots + g_c + e$, where e is the number of indices i with $a_i \ne d_i$.

Proof of Claim 3: Notice that $e = \sum_i \epsilon_i$. Since $a_i \le d_i - g_i - \epsilon_i$, we have $d_i - a_i \ge g_i + \epsilon_i$ for all i. We have $2f \le 2k$ and hence $2fk \le 2k^2$. Claim 2 gives that $\sum_i [a_i, k-1, 0] \le \sum_i [d_i - g_i - \epsilon_i, k-1, 0] + 4k^2$. Since $f = \binom{r+k-1}{r} - p(k-1) - \sum_i [d_i - g_i, k-1, 0] - \alpha$ and $\alpha = \binom{r+k}{r} - p(k) - \sum_i [d_i, k, g_i]$, we have

$$p_1(k) + \sum_i ([d_i - a_i, k] - f - \sum_i g_i = \binom{r+k-1}{r-1} \tag{4.1}$$

We apply $T(r-1,p_1,k)$ in H with the data $c := \sum_i(d_i-a_i)-f$, $d_j=2$ for exactly f indices j and with $d_i=1$ for the other $\sum_i(d_i-a_i)-2f$ indices i. Since $\sum_i a_i \sim k^{r-1}/r!$, $\sum_i g_i+\epsilon_i$ has at most order $k^{r-1}/r\cdot r!$ (we use the inequality $d_i \ge \delta_r(g_i)$) and $f\le k$, there are integers $x_i\ge 0$, $1\le i\le c$, such that $\sum_i x_i = 2f$ and $a_i \le g_i+\epsilon_i+x_i$. By (4.1) and the inequality $\sum_i g_i \ge 0$ we get the existence of $E\in W(r;2,\cdots,1)$ with $E\cap Z=\emptyset$ and $h^1(\mathcal{I}_{Z\cup E}(k))=0$. By the semicontinuity theorem for cohomology there is $F=F_1\sqcup\cdots\sqcup F_{c'}$ with $F_i\in Z(r-1;d_i-a_i-x_i,0)$ if $i\le c$ (hence $F_i=\emptyset$ if $a_i=d_i$ and hence $g_i=0$) and F_i a reducible conic if $c+1\le i\le c+f$; alternatively, quote Lemma 3.9. If $i\le c$ and $F_i\ne\emptyset$, then we may find F_i with the additional condition that it contains a fixed subset $S_i\subset H$ with $\sharp(S_i)=g_i+1$, because $\deg(F_i)=d_i-a_i\ge g_i+1$ (even if $g_i=0$, by the assumption $a_i<d_i$) (Lemmas 2.5 and 2.6; another proof may be given using trees and then smoothing them). Fix a general $S\cup S'\cup S''\subset H$ such that $\sharp(S)=\sharp(S')=\sharp(S'')$ and $S\cap S'=S\cap S''=S'\cap S''=\emptyset$. We may find F with the additional condition that $S=\mathrm{Sing}(F)$ and for any reducible conic $F_i\subset F$ one of the lines of F_i contains a point of S', while the other line of F_i contains a point of S'' (remember that we also have $F_i\cap F_j=\emptyset$ for all $i\ne j$ and hence that each point of $S'\cup S''$ is contained in F). Even with these additional conditions we may find F with $F\cap(Z\cap H)=\emptyset$ and $h^1(H,\mathcal{I}_{(Z\cap H)\cup F}(k))=0$. By Claim 3 we have $(\sum_i \sharp(S_i))+\sharp(S')+\sharp(S'')=g_1+\cdots+g_c+e\le\sum_i(d_i-a_i)$. If $Y_i\ne\emptyset$, then $a_i\ge g_i+1$. Since $Y\cap H$ is general in H, we may take as S_i a subset of $Y_i\cap H$ (as announced before Claim 3). For each $P\in S$ let $V_P\subset\mathbb{P}^r$ be any 3-dimensional linear subspace such that $H\cap V_P$ is the plane spanned by the connected component of F containing P. Set $\chi := \cup_{P\in S}\{2P,V_P\}$. For each $i=1,\dots,c$ we have $Y_i\cup F_i\in Z'(r;d_i-x_i,g_i)$. Taking $\sharp((S'\cup S'')\cap(Y_i\cap H))=x_i$ for all i (this is possible, because $\sum_i a_i\ge g_1+\cdots+e$ by Claim 3 and $2(\sum_i a_i)\ge d_i$ (Claim 2 in the proof of Lemma 3.5), we get $Y\cup F\cup\chi\in Z'(r;(d_1,g_1),\dots,(d_c,g_c))$. Hence by the Castelnuovo's sequence it is sufficient to prove that $h^1(H,\mathcal{I}_{((Z\cup Y)\cap H)\cup F}(k))=0$. Since $Y\cap H$ is general in H and $h^0(\mathcal{O}_{(Z\cup Y)\cap H)\cup F}(k))=\binom{r+k-1}{r-1}$ by (4.1), it is sufficient to prove that $h^1(H,\mathcal{I}_{(Z\cap H)\cup F}(k))=0$. If $r=5$, then we may $H(k,4)$, because $Z\cap H=\emptyset$. If $r\ge 6$ we use Proposition 3.9 with $n=r-1$ and $q=p_1$.

(b) Now we prove the existence of A. Set $\beta=p(k-1)+\sum_i[d_i,k-1,g_i]-\binom{r+k}{r}$. By the definition of critical value we have $\beta>0$. Decreasing if necessary c and d_i (use Remark 2.2) we may assume that either $\beta\le k-1$ or $g_i>0$ and $d_i=\delta_r(g_i)$ for all i. Since $L\cup A\in Z'(r;d_i,g_i)$ for any $A\in Z'(r;d_i-1,g_i-1)$ and any secant line L of A, even in the latter case we reduce to the case $\beta\le k-2$ (part (iii) of Remark 3.2). Hence we may assume $1\le\beta\le k-1$. We copy the proof of step (a) using $k-2$ instead of k and $k-1$ instead of k (e.g. we use $[d_i-g_i-\epsilon_i,k-2,0]$ instead of $[d_i-g_i-\epsilon_i,k-1,0]$) and h^0 instead of h^1 in the Castelnuovo's inequalities. Even in this step we use that $h^1(H,\mathcal{I}_{(Z\cap H)\cup F}(k-1))=0$ and hence $h^0(H,\mathcal{I}_{(Z\cap H)\cup F}(k-1))=\binom{r+k-2}{r-1}-p_1(k-1)-h^0(\mathcal{O}_F(k-1))$. To get $h^0(H,\mathcal{I}_{((Z\cup Y)\cap H)\cup F}(k-1))=0$ we use that $Y\cap H$ is general in H and that □

A concluding remark 1. The key part of the proof is the case $g=0$ and we emphasize that even for smooth rational curves our proof heavily use reducible curves (trees). We may also use Theorem 1.2 to prove the case $g>0$ of Lemma

1.2. Our proof of the case $g > 0$ only use the case $g = 0$ and some asymptotic inequalities which depends on the shape of the integers $\delta_r(q)$. If $g \geq 2$ we could use induction on g making minimal modifications.

Acknowledgement. The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

1. E. Ballico, *Postulation of a union $X \subset \mathbb{P}^r$, $r \geq 4$, of a given zero-dimensional scheme and several general lines*, ISRN Geometry Volume 2013, Article ID 391721, 6 pages http://dx.doi.org/10.1155/2013/391721
2. E. Ballico, *Postulation in projective spaces of a union of a low dimensional scheme and general rational curves or general unions of lines*, Gulf J. Math. **2** (2014), no. 3, 1–18.
3. E. Ballico, *Adding a general union of a prescribed number of curves with high sum of their degrees improve the Hilbert function of any scheme*, Gen. Math. Notes (to appear).
4. E. Ballico and Ph. Ellia, *On the postulation of many disjoint rational curves in $\mathbf{P}_N$, $N \geq 4$*, Boll. Un. Mat. Ital. B (6) **4** (1985), no. 2, 585–599.
5. E. Ballico and Ph. Ellia, *Bonnes petites composantes des schémas de Hilbert de courbes lisses de $\mathbb{P}^m$*, C. R. Acad. Sci. Paris **306** (1988), 187–190.
6. G. Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. **158** (1978), 61–70.
7. R. Hartshorne, Algebraic Geometry, Springer-Verlag, Berlin–Heidelberg–New York, 1977.
8. R. Hartshorne and A. Hirschowitz, *Droites en position générale dans $\mathbb{P}^n$*, Algebraic Geometry, Proceedings, La Rábida 1981, 169–188, Lect. Notes in Math. 961, Springer, Berlin, 1982.
9. R. Hartshorne and A. Hirschowitz, *Smoothing algebraic space curves*, Algebraic Geometry, Sitges 1983, 98–131, Lecture Notes in Math. 1124, Springer, Berlin, 1985.
10. A. Hirschowitz, *Sur la postulation générique des courbes rationnelles*, Acta Math. **146** (1981), 209–230.
11. A. Iarrobino and S. L. Kleiman, *The Gotzmann theorems and the Hilbert Scheme* (Appendix C of A. Iarrobino and V. Kanev, Power sums, Gorenstein algebras, and determinantal loci), Lecture Notes in Mathematics, pages 289–312, Springer-Verlag, Berlin, 1999.
12. D. Perrin, *Courbes passant par m points généraux de $\mathbf{P}^3$*, Bull. Soc. Math. France, Mem. **28/29**, 1987.
13. E. Sernesi, *On the existence of certain families of curves*, Invent. Math. **75** (1984), no. 1, 25–57.

[1] Dept. of Mathematics, University of Trento, 38123 Povo (TN), Italy
E-mail address: ballico@science.unitn.it