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ON THE FINE SPECTRUM OF GENERALIZED THIRD ORDER DIFFERENCE OPERATOR Δ_{vuw} ON THE BANACH SPACE c_0

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ABSTRACT. In this paper we examine the fine spectrum of Δ_{vuw} operator in Banach space c_0.

1. INTRODUCTION AND PRELIMINARIES

By ω, we shall denote the space of all real or complex valued sequences. Any vector subspace of ω is called a sequence space. Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk}x_k \tag{1.1}$$

Let X be a Banach space and $T : X \to X$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in X : y = Tx, x \in X\}. \tag{1.2}$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual space X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.
Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T we associate the operator

$$T_\lambda = T - \lambda I, \tag{1.3}$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1}, \tag{1.4}$$

and call it the *resolvent* operator of T. Many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we

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shall be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. The boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in X.

Definition 1.1. Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A *regular value* λ of T is a complex number such that :
(R1) T_λ^{-1} exists,
(R2) T_λ^{-1} is bounded,
(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows: The *point(discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an *eigenvalue* of T.
The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and satisfies (R3) but not (R2), that is, T_λ^{-1} is unbounded.
The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but does not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in X.
Hence if $(T - \lambda I)x = \theta$ for some $x \neq \theta$, then $\lambda \in \sigma_p(T, X)$, by definition, that is, λ is an eigenvalue of T. The vector x is then called an *eigenvector* of T corresponding to the eigenvalue λ.

Lemma 1.2 ([4]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if the supremum of c_0 norms of the columns of A is bounded.*

Lemma 1.3 ([4]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 1.4 ([8]). *T has a dense range if and only if T^* is one to one, where T^* denotes the adjoint operator of the operator T.*

Lemma 1.5 ([8]). *The adjoint operator T^* of T is onto if and only if T has a bounded range.*

The present work is a continuation of the previous works, which are [1], [2], [3], [5], [6], [7], [9], [10], [11], [12].

2. Main results

In this section, we compute the spectrum, the point spectrum, the continuous spectrum and residual spectrum of the operator Δ_{vuw} on the sequence space c_0. Let $v = (v_k)$ be a constant or strictly decreasing sequence of nonnegative real numbers with $\lim_{k\to\infty} v_k = V \neq 0$, $u = (u_k)$ be a sequence of nonnegative real numbers such that $u_k \neq 0$ for each $k \in \mathbb{N}_0$ with $\lim_{k\to\infty} u_k = U \neq 0$, and $w = (w_k)$

be a sequence of nonnegative real numbers such that $w_k \neq 0$ for each $k \in \mathbb{N}_0$ with $\lim_{k\to\infty} w_k = W \neq 0$. Let the linear operator $\Delta_{uvw} : c_0 \to c_0$ be defined by

$$\Delta_{vuw}x = (w_n x_{n-2} + u_n x_{n-1} + v_n x_n)_{n=0}^{\infty} \text{ with } x_{-1} = x_{-2} = 0, \tag{2.1}$$

where $x = (x_n) \in c_0$.
It is easy to see that the operator Δ_{uvw} can be represented by the matrix

$$\Delta_{vuw} = \begin{pmatrix} v_0 & 0 & 0 & 0 & 0 & 0 & \dots \\ u_1 & v_1 & 0 & 0 & 0 & 0 & \dots \\ w_2 & u_2 & v_2 & 0 & 0 & 0 & \dots \\ 0 & w_3 & u_3 & v_3 & 0 & 0 & \dots \\ 0 & 0 & w_4 & u_4 & v_4 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.2}$$

Theorem 2.1. *$\Delta_{vuw} : c_0 \longrightarrow c_0$ is a bounded linear operator and*

$$\|\Delta_{vuw}\|_{(c_0,c_0)} = \sup_k(|v_k| + |u_k| + |w_k|) \tag{2.3}$$

Proof. The proof is trivial. So we omit it. □

Before we giving the main theorem we should give the following remark. In this paper, here and in what follows, if z is a complex number then by $\sqrt{z}$ we always mean the square root of z with non-negative real part. If $\mathrm{Re}(\sqrt{z}) = 0$ then $\sqrt{z}$ represents the square root of z with $\mathrm{Im}(\sqrt{z}) \geq 0$.

Theorem 2.2. *Assume $\sqrt{U^2} = U$ and define the set S by*

$$S = \left\{ \alpha \in \mathbb{C} : \frac{2|V-\alpha|}{|-U + \sqrt{U^2 - 4W(V-\alpha)}|} \leq 1 \right\}$$

Then the spectrum of the operator Δ_{vuw} on sequence the space c_0 is given by $\sigma(\Delta_{vuw}, c_0) = S$

Proof. Our proof is divided into the following two steps.
In first step , we show that $\sigma(\Delta_{vuw}, c_0) \subseteq S$. To prove this we need to show $\alpha \in \rho(\delta_{vuw}, c_0)$ for which $\alpha \in \mathbb{C}$ with $\left|\frac{2(V-\alpha)}{-U+\sqrt{U^2-4W(V-\alpha)}}\right| > 1$. In second step, we show the reverse inclusion, i.e., $S \subseteq \sigma(\Delta_{vuw}, c_0)$.
Step 1.
Let $\alpha \in \mathbb{C}$ with $\left|\frac{2(V-\alpha)}{-U+\sqrt{U^2-4W(V-\alpha)}}\right| > 1$. Clearly, $\alpha \neq V$ and $\alpha \neq v_k$ for each $k \in \mathbb{N}_0$ as it does not satisfy the condition. So $(\Delta_{vuw} - \alpha I)$ is triangle, and hence has an inverse. Thus, $(\Delta_{vuw} - \alpha I)^{-1} = (a_{nk})$. We can calculate that

$$(a_{nk}) = \begin{pmatrix} \frac{1}{v_0-\alpha} & 0 & 0 & 0 & \dots \\ -\frac{u_1}{(v_0-\alpha)(v_1-\alpha)} & \frac{1}{v_1-\alpha} & 0 & 0 & \dots \\ \frac{u_1u_2}{(v_0-\alpha)(v_1-\alpha)(v_2-\alpha)} - \frac{w_2}{(v_0-\alpha)(v_2-\alpha)} & -\frac{u_2}{(v_1-\alpha)(v_2-\alpha)} & \frac{1}{v_2-\alpha} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.4}$$

We observe that

$$a_{nk} = -\frac{u_n}{v_n - \alpha} a_{n-1,k} - \frac{w_n}{v_n - \alpha} a_{n-2,k}, \ k = 0, 1, 2, \dots, n \tag{2.5}$$

By Lemma 1.2, the operator $(\Delta_{vuw} - \alpha I)^{-1} \in (c_0, c_0)$ if the following conditions are satisfied

(1) series $\sum_{k=0}^{\infty} |a_{nk}|$ is convergent for each $n \in \mathbb{N}_0$ and $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$.
(2) $\lim_{n\to\infty} |a_{nk}| = 0$ for each $k \in \mathbb{N}_0$.

For this consider $S_n = \sum_{k=0}^{n} |a_{nk}| = |a_{n,n}| + |a_{n,n-1}| + \cdots + |a_{n,0}|$, Clearly for n is odd, the series

$$\begin{aligned} S_n \;=\; & \left|\frac{1}{v_n - \alpha}\right| + \left|\frac{-u_n}{(v_{n-1} - \alpha)(v_n - \alpha)}\right| + \left|\frac{u_{n-1}u_n}{(v_{n-2} - \alpha)(v_{n-1} - \alpha)(v_n - \alpha)}\right. \\ & \left. - \frac{w_n}{(v_{n-2} - \alpha)(v_n - \alpha)}\right| + \cdots + \left|\frac{-u_1 u_2 \dots u_n}{(v_0 - \alpha)\dots(v_n - \alpha)}\right. \\ & \left. + \frac{u_n w_{n-1} \dots w_2}{(v_0 - \alpha)(v_2 - \alpha)\dots(v_{n-1} - \alpha)(v_n - \alpha)} + \cdots + \frac{w_n u_1 u_2 \dots u_{n-2}}{(v_0 - \alpha)\dots(v_{n-2})(v_n - \alpha)}\right| \end{aligned}$$

is convergent. Similarly for n is even, S_n is also convergent. Next we show that $\sup_n S_n$ is finite. Now by letting

$$x_1 = \frac{-U + \sqrt{U^2 - 4W(V - \alpha)}}{2(V - \alpha)} \text{ and } x_2 = \frac{-U - \sqrt{U^2 - 4W(V - \alpha)}}{2(V - \alpha)} \tag{2.6}$$

We can observe,

$$\lim_{n\to\infty} \frac{1}{(v_n - \alpha)} = \frac{1}{V - \alpha} = b_1 = \frac{1}{\sqrt{U^2 - 4W(v - \alpha)}}(x_1 - x_2)$$

$$\lim_{n\to\infty} \frac{-u_n}{(v_n - \alpha)(v_{n-1} - \alpha)} = \frac{-u}{(V - \alpha)^2} = b_2 = \frac{1}{\sqrt{U^2 - 4W(v - \alpha)}}(x_1^2 - x_2^2)$$

$$\begin{aligned} &\lim_{n\to\infty} \left(\frac{u_{n-1}u_n}{(v_n - \alpha)(v_{n-1} - \alpha)(v_{n-2} - \alpha)} - \frac{w_n}{(v_{n-2} - \alpha)(v_n - \alpha)}\right) \\ &= \frac{U^2}{(V - \alpha)^3} - \frac{W}{(V - \alpha)^2} = b_3 = \frac{1}{\sqrt{U^2 - 4W(v - \alpha)}}(x_1^3 - x_2^3). \end{aligned}$$

Hence we have,

$$b_n = \frac{1}{\sqrt{U^2 - 4W(v - \alpha)}}(x_1^n - x_2^n)$$

Suppose $U^2 = 4W(V - \alpha)$ then

$$b_n = -\left(\frac{2n}{U}\right)\left(-\frac{U}{2(V - \alpha)}\right)^n,$$

which gives,

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} |b_k| = \sum_{k=1}^{\infty} \left|\frac{2n}{U}\right| \left|\frac{U}{2(V - \alpha)}\right|^n < \infty,$$

since $\left|\frac{U}{2(V-\alpha)}\right| < 1$ and it follows from the D'alembert's ratio test. Therefore $\alpha \notin S$ implies $a_n \to 0$. So we may assume that $U^2 \neq 4W(V-\alpha)$. Since α is not in S, we have $|x_1| < 1$. Now we show that $|x_2| < 1$. Since $|x_1| < 1$, we have

$$\left|-1+\sqrt{1-4W(V-\alpha)/U^2}\right| < \left|\frac{2(V-\alpha)}{U}\right|$$

Since $|1-\sqrt{z}| \leq |1+\sqrt{z}|$ for any $z \in \mathbb{C}$, we must have

$$\left|-1-\sqrt{1-4W(V-\alpha)/U^2}\right| < \left|\frac{2(V-\alpha)}{U}\right|$$

which leads us to the fact that $|x_2| < 1$. Taking limit both sides of S_n and since $|x_1| < 1$ and $|x_2| < 1$, we get

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} |b_k| \leq \frac{1}{\sqrt{U^2-4W(V-\alpha)}} \left(\sum_{k=1}^{\infty} |x_1|^k + \sum_{k=1}^{\infty} |x_2|^k\right) < \infty$$

Since (S_n) is a sequence of positive real numbers and $\lim_{n\to\infty} S_n < \infty$. For n is even

$$\begin{aligned}
\lim_{n\to\infty} |a_{n,0}| &= \left|\frac{-u_1u_2\dots u_n}{(v_0-\alpha)\dots(v_n-\alpha)} + \frac{u_nw_{n-1}\dots w_2}{(v_0-\alpha)(v_2-\alpha)\dots(v_{n-1}-\alpha)(v_n-\alpha)} + \dots\right. \\
&\qquad \left. + \frac{w_nu_1u_2\dots u_{n-2}}{(v_0-\alpha)\dots(v_{n-2})(v_n-\alpha)}\right| \\
&\leq \frac{1}{|\sqrt{U^2-4W(V-\alpha)}|}\left(\lim_{n\to\infty}|x_1|^n + \lim_{n\to\infty}|x_2|^n\right) = 0
\end{aligned}$$

Thus, $\lim_{n\to\infty} |a_{n,0}| = 0$. Similarly, for n is odd, $\lim_{n\to\infty} |b_{n,0}| = 0$. Again, we can show that $\lim_{n\to\infty} |b_{n,k}| = 0$ for all $k = 1, 2, 3, \dots$. Thus, $(\Delta_{vuw} - \alpha I)^{-1} \in B(c_0)$ for $\alpha \in \mathbb{C}$ with

$$\left|\frac{2(V-\alpha)}{-U+\sqrt{U^2-4W(V-\alpha)}}\right| > 1.$$

Next we will show that of the operator $(\Delta_{vuw} - \alpha I)^{-1}$ is dense in c_0. This statement holds if and only if range of the operator $(\Delta_{vuw} - \alpha I)$ is dense in c_0. Since $(\Delta_{vuw} - \alpha I)^{-1} \in (c_0, c_0)$, which implies that range of the operator $(\Delta_{vuw} - \alpha I)$ is dense in c_0. Hence we have

$$\sigma(\Delta_{vuw}, c_0) \subseteq \left\{\alpha \in \mathbb{C} : \frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|} \leq 1\right\}$$

Step 2. Now we prove the reverse inclusion,i.e.,

$$\left\{\alpha \in \mathbb{C} : \frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|} \leq 1\right\} \subseteq \sigma(\Delta_{vuw}, c_0) \tag{2.7}$$

First we prove the inclusion (2.7) under the assumption that $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}_0$,i.e., we want to show that one of the conditions of Definition1.1 fails. Clearly, $(\Delta_{vuw} - \alpha I)$ is a triangle and hence $(\Delta_{vuw} - \alpha I)^{-1}$ exists. So,

condition $(R1)$ is satisfied but condition $(R2)$ fails as can be seen below:
First, let $U^2 = 4W(V-\alpha)$, then

$$b_n = -\left(\frac{2n}{U}\right)\left(-\frac{U}{2(V-\alpha)}\right)^n,$$

which gives

$$\lim_{n\to\infty} |a_{n,0}| = \left|\frac{2k}{U}\right| \left|\frac{U}{2(V-\alpha)}\right|^n = \infty$$

since $\left|\frac{U}{2(V-\alpha)}\right| \geq 1$. So, we may assume that $U^2 \neq 4W(V-\alpha)$.
Suppose $\alpha \in \mathbb{C}$ with $\left|\frac{2(V-\alpha)}{-U+\sqrt{U^2-4W(V-\alpha)}}\right| < 1$. Then $|x_1| > 1$ and $|x_1| > |x_2|$ always, consequently

$$\begin{aligned}
\lim_{n\to\infty} |a_{n,0}| &= \frac{1}{\sqrt{U^2-4W(v-\alpha)}} \lim_{n\to\infty} |x_1^n - x_2^n| \\
&\geq \frac{1}{\sqrt{U^2-4W(v-\alpha)}} \lim_{n\to\infty} (|x_1|^n - |x_2|^n) \\
&= \frac{1}{\sqrt{U^2-4W(v-\alpha)}} \lim_{n\to\infty} |x_1|^n \left(1 - \frac{|x_2|^n}{|x_1|^n}\right) \longrightarrow \infty
\end{aligned}$$

which gives $\lim_{n\to\infty} |a_{n,k}| \neq 0$ for each k.
Hence $(\Delta_{vuw} - \alpha I)^{-1} \notin B(c_0)$ for $\alpha \in \mathbb{C}$ with $\left|\frac{2(V-\alpha)}{-U+\sqrt{U^2-4W(V-\alpha)}}\right| < 1$. Next we consider $\alpha \in \mathbb{C}$ with $\left|\frac{2(V-\alpha)}{-U+\sqrt{U^2-4W(V-\alpha)}}\right| = 1$ then $|x_1| = 1$ and $|x_2| < |x_1| = 1$

$$\begin{aligned}
\lim_{n\to\infty} |a_{n,0}| &\geq \frac{1}{|\sqrt{U^2-4W(v-\alpha)}|} \lim_{n\to\infty} (|x_1|^n - |x_2|^n) \\
&= \frac{1}{|\sqrt{U^2-4W(v-\alpha)}|} \lim_{n\to\infty} (1 - |x_2|^n|) \longrightarrow \infty \\
&= \frac{1}{|\sqrt{U^2-4W(v-\alpha)}|}.
\end{aligned}$$

this tells us $\lim_{n\to\infty} |a_{n,0}| \neq 0$.
Thus

$$(\Delta_{vuw} - \alpha I)^{-1} \notin B(c_0) \ \text{ for } \alpha \in \mathbb{C} \ \text{ with } \left|\frac{2(V-\alpha)}{-U+\sqrt{U^2-4W(V-\alpha)}}\right| = 1. \quad (2.8)$$

Finally, we prove the inclusion (2.7) under the assumption that $\alpha = V$ and $\alpha = v_k$ for each $k \in \mathbb{N}_0$. We have

$$(\Delta_{vuw} - \alpha I)x = \begin{pmatrix} (v_0-\alpha)x_0 \\ u_1x_0 + (v_1-\alpha)x_1 \\ w_2x_0 + u_2x_1 + (v_2-\alpha) \\ w_3x_1 + u_3x_2 + (v_3-\alpha)x_3 \\ \vdots \end{pmatrix} \quad (2.9)$$

Case (i). If v_k is a constant sequence, say $v_k = V$ for each $k \in \mathbb{N}_0$, then $(\Delta_{vuw} - V \cdot I)x = 0 \Rightarrow x_0 = 0, x_1 = 0, x_2 = 0, \ldots$. This shows that the operator $(\Delta_{vuw} - V \cdot I)$ is one to one, but $R(\Delta_{vuw} - V \cdot I)$ is not dense in c_0. So the condition $(R3)$ fails. Hence $V \in \sigma(\Delta_{vuw}, c_0)$.
Case (ii). If (v_k) is a strictly decreasing sequence, then for fixed $k, k \geq 0$, $(\Delta_{vuw} - v_k I)x = 0 \Rightarrow x_0 = 0, x_1 = 0, \ldots, x_{k-1} = 0$,

$$\begin{aligned} x_{k+1} &= \frac{u_{k+1}}{v_k - v_{k+1}} x_k \\ x_{k+2} &= \frac{u_{k+2}x_{k+1} + w_{k+2}x_k}{v_k - v_{k+2}} \\ &= \left(\frac{u_{k+1}u_{k+2} + w_{k+2}(v_k - v_{k+1})}{(v_k - v_{k+1})(v_k - v_{k+2})} \right) x_k \end{aligned}$$

are non zero since $x_k \neq 0$ and we have chosen v_k to be a strictly decreasing sequence. Similarly it can be shown that, for $n \geq k+3, x_n$ is non-zero by using following recurrence equation

$$x_{n+1} = \frac{u_{n+1}x_n + w_{n+1}x_{n-1}}{v_{n-1} - v_{n+1}}.$$

Hence we get non-zero solution of $(\Delta_{vuw} - v_k I)x = 0$. This shows that $(\Delta_{vuw} - v_k I)$ is not injective. Hence $v_k \in \sigma(\Delta_{vuw}, c_0)$ for all $k \in \mathbb{N}$. Hence (2.7) proved. This completes the proof. □

Theorem 2.3. $\sigma_p(\Delta_{vuw}, c_0) = \begin{cases} \varnothing, & \textit{if } (v_k) \textit{ is a constant sequence} \\ \{v_0, v_1, \ldots\}, & \textit{if } (v_k) \textit{ is a strictly decreasing sequence.} \end{cases}$

Proof. The proof of this theorem is divided into the following two cases.
Case (i). Suppose $\{v_k\}_{k=0}^{\infty}$ is a constant sequence, let $v_k = V$ for each $k \in \mathbb{N}_0$. Consider $\Delta_{vuw}x = \alpha x$ for $x \in c_0$ and $x \neq \theta$, which gives following system of equation

$$\begin{aligned} v_0x_0 &= \alpha x_0 \\ u_1x_0 + v_1x_1 &= \alpha x_1 \\ w_2x_0 + u_2x_1 + v_2x_2 &= \alpha x_2 \\ w_3x_1 + u_3x_2 + v_3x_3 &= \alpha x_3 \\ &\vdots \\ w_kx_{k-2} + u_kx_{k-1}v_kx_k &= \alpha x_k \\ &\vdots \end{aligned} \tag{2.10}$$

Let x_s be the first non-zero entry of the sequence $x = \{x_n\}$. So the equation $w_sx_{s-2} + u_sx_{s-1} + Vx_s = \alpha x_s \Leftrightarrow w_sx_{s-2} + u_sx_{s-1} + Vx_s = \alpha x_s$, implies $\alpha = V$, and from the equation $w_{s+1}x_{s-1} + u_{s+1}x_s + Vx_{s+1} = \alpha x_{s+1}$, we get $x_s = 0$, which is a contradiction to our assumption. Therefore

$$\sigma_p(\Delta_{vuw}, c_0) = \varnothing.$$

Case(ii). Suppose (v_k) is a strictly decreasing sequence. Consider $\Delta_{vuw}x = \alpha x$ for $x \in c_0$ and $x \neq \theta$, which gives following system of equations (2.10). If $\alpha = v_0$

then

$$\begin{aligned} x_1 &= \frac{u_1}{(v_0 - v_1)} x_0 \\ x_2 &= \left(\frac{w_2}{v_0 - v_2} + \frac{u_1 u_2}{(v_0 - v_1)(v_0 - v_2)} \right) x_0 \end{aligned}$$

are nonzero , since v_k is a strictly decreasing sequence and by taking $x_0 \neq 0$. Similarly, it can be shown that, for $n \geq 3, x_n$ is non-zero by using following recurrence equation

$$x_{n+1} = \frac{w_{n+1}x_{n-1} + u_{n+1}x_n}{v_0 - v_{n+1}}, \quad \text{for all } n \geq 2.$$

Hence we get non-zero solution of $(\Delta_{vuw} - v_0 I)x = 0$. If $\alpha = v_k$ for all $k \geq 1$, then solving system of equations, we get $x_0 = 0, x_1 = 0, \ldots, x_{k-1} = 0$,

$$\begin{aligned} x_{k+1} &= \frac{u_{k+1}}{v_k - v_{k+1}} x_k \\ x_{k+2} &= \frac{(v_k - v_{k+1})w_{k+2} + u_{k+1}u_{k+2}}{(v_k - v_{k+1})(v_k - v_{k+2})} x_k \end{aligned}$$

are non-zero, since (v_k) is strictly decreasing sequence and by taking $x_k \neq 0$. Similarly, it can be shown that, for $n \geq k+3$, x_n is non-zero by using following expression

$$x_{n+1} = \frac{w_{n+1}x_{n-1} + u_{n+1}x_n}{v_k - v_{n+1}}, \text{ for all } n \geq k+2.$$

Hence we get non-zero solution of $(\Delta_{vuw} - v_k I)x = 0$. This shows that $(\Delta_{vuw} - v_k I)$ is not injective. So condition $(R1)$ fails. We get non-zero solution of $(\Delta_{vuw} - \alpha I)x = 0$. Thus for this case

$$\sigma_p(\Delta_{vuw}, c_0) = \{v_0, v_1, v_2, \ldots\}$$

This completes the proof. $\square$

Theorem 2.4. *The point spectrum of the adjoint operator Δ^*_{vuw} over the Banach space c^*_0 is*

$$\sigma_p(\Delta^*_{vuw}, c^*_0) = \{\alpha \in \mathbb{C} : \frac{2|V - \alpha|}{|-U + \sqrt{U^2 - 4W(V - \alpha)}|} < 1\}$$

Proof. If $\Delta^*_{vuw} f = \alpha f$ for $0 \neq f \in c^*_0 \cong \ell_1$, where

$$\Delta^*_{vuw} = \begin{pmatrix} v_0 & u_1 & w_2 & 0 & 0 & 0 & 0 & \ldots \\ 0 & v_1 & u_2 & w_3 & 0 & 0 & 0 & \ldots \\ 0 & 0 & v_2 & u_3 & w_4 & 0 & 0 & \ldots \\ 0 & 0 & 0 & v_3 & u_4 & w_5 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix} \tag{2.11}$$

and

$$f = (f_0, f_1, \ldots)^T.$$

Hence we have the following system of linear equations

$$\begin{aligned} v_0f_0+u_1f_1+w_2f_2 &= \alpha f_0 \\ v_1f_1+u_2f_2+w_3f_3 &= \alpha f_1 \\ v_2f_2+u_3f_3+w_4f_4 &= \alpha f_2 \\ &\vdots \\ v_kf_k+u_{k+1}f_{k+1}+w_{k+2}f_k &= \alpha f_k \\ &\vdots \end{aligned} \tag{2.12}$$

Solving the system of linear equation (2.12)in terms of f_0, and f_1, also using (2.5), we obtain

$$f_k=(a_{k-1,0}f_1-a_{k-1,1}f_0)\cdot\frac{(v_0-\alpha)(v_1-\alpha)\dots(v_{k-1}-\alpha)}{w_2w_3\dots w_k},$$

where $a_{k-1,0}$ and $a_{k-1,1}$ are defined as in proof of Theorem 2.2. Now let $\alpha=\alpha_1+i\alpha_2\in\mathbb{C}$ and $v=(v_k)$ is a constant or strictly decreasing positive real sequence, $w=(w_k)$ is a strictly decreasing positive real sequence, we get for $n=0,1,\dots,k-1$

$$\begin{aligned} \left|\frac{v_n-\alpha}{w_{n+2}}\right| &= \left|\frac{\alpha-v_n}{w_{n+2}}\right| \\ &= \left(\left(\frac{\alpha_1-v_n}{w_{n+2}}\right)^2+\left(\frac{\alpha_2}{w_{n+2}}\right)^2\right)^{\frac{1}{2}} \\ &\le \left(\left(\frac{\alpha_1-V}{W}\right)^2+\left(\frac{\alpha_2}{W}\right)^2\right)^{\frac{1}{2}} \\ &= \left|\frac{\alpha-V}{W}\right|. \end{aligned}$$

Then

$$|f_k|\le|a_{k-1,0}f_1-a_{k-1,1}f_0|\cdot\left|\frac{\alpha-V}{W}\right|^{k-1}|\alpha-V|.$$

Taking limit on both sides and choosing $f_0=1$ and $f_1=\frac{1}{x_1}$, we obtain

$$\begin{aligned} \lim_{k\to\infty}|f_k| &\le \lim_{k\to\infty}\frac{|(x_1^k-x_2^k)f_1-(x_1^{k-1}-x_2^{k-1})f_0|}{|\sqrt{U^2-4W(V-\alpha)}|}\cdot\left|\frac{\alpha-V}{W}\right|^{k-1}|\alpha-V| \\ &= \lim_{k\to\infty}\frac{|x_2|^{k-1}|x_1-x_2|}{|x_1||\sqrt{U^2-4W(V-\alpha)}|}\cdot\left|\frac{\alpha-V}{W}\right|^{k-1}|\alpha-V| \\ &= \lim_{k\to\infty}\frac{|x_2|^{k-1}}{|x_1|}\cdot\left|\frac{\alpha-V}{W}\right|^{k-1}. \end{aligned} \tag{2.13}$$

We have the relation

$$\frac{V-\alpha}{W} = \frac{2(V-\alpha)}{-U+\sqrt{U^2-4W(V-\alpha)}} \times \frac{2(V-\alpha)}{-U-\sqrt{U^2-4W(V-\alpha)}} = \frac{1}{x_1x_2}. \tag{2.14}$$

Then using (2.13) in (2.14), we obtain

$$\lim_{k\to\infty} |f_k| \leq \lim_{k\to\infty} \frac{|x_2|^{k-1}}{|x_1|} \cdot \frac{1}{|w_1w_2|^{k-1}} = \lim_{k\to\infty} \left|\frac{1}{x_1}\right|^k.$$

If $\alpha \in \mathbb{C}$ with $\frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|} < 1$ then $\frac{1}{|x_1|} < 1$. By Cauchy root test $\lim_{k\to\infty} |f_k|^{\frac{1}{k}} \leq \frac{1}{|x_1|} < 1$. Hence $\sum_{k=0}^{\infty} |f_k|$ converges. Conversely, we have to show that $\sum_{k=0}^{\infty} |f_k|$ converges implies $\frac{1}{|x_1|} < 1$. This is equivalent to show that $\frac{1}{|x_1|} \geq 1$ implies $\sum_{k=0}^{\infty} |f_k|$ diverges. To prove this, we write another representation of f_k in terms of f_{k-1} and f_{k-2} from the system of equation as (2.12)

$$f_k = \frac{(\alpha - v_{k-2})}{w_k} f_{k-2} - \frac{u_{k-1}}{w_k} f_{k-1}, \quad \text{for all } k \geq 2.$$

Hence dividing both sides by f_k, we obtain

$$\left(\frac{\alpha - v_{k-2}}{w_k}\right) \cdot \frac{f_{k-2}}{f_{k-1}} \cdot \frac{f_{k-1}}{f_k} - \frac{u_{k-1}}{w_k} \cdot \frac{f_{k-1}}{f_k} = 1$$

Taking limit both sides denoting $\lim_{k\to\infty} \frac{f_{k-1}}{f_k} = L$, we obtain a quadratic equation,

$$\left(\frac{\alpha - V}{W}\right) \cdot L^2 - \frac{U}{W} \cdot L - 1 = 0.$$

Solving the above equation, we get two roots as

$$L_1 = \frac{-U+\sqrt{U^2-4W(V-\alpha)}}{2(V-\alpha)} \text{ and } L_2 = \frac{-U-\sqrt{U^2-4W(V-\alpha)}}{2(V-\alpha)}.$$

Now describe some cases:

(i) If $1 < \frac{1}{|x_1|} < \frac{1}{|x_2|}$ then

$$\lim_{k\to\infty} \frac{|f_k|}{|f_{k-1}|} = \lim_{k\to\infty} \left|\frac{f_k}{f_{k-1}}\right| = \left|\frac{1}{L_1}\right| = \frac{1}{|x_1|} > 1$$

or

$$\lim_{k\to\infty} \frac{|f_k|}{|f_{k-1}|} = \lim_{k\to\infty} \left|\frac{f_k}{f_{k-1}}\right| = \left|\frac{1}{L_2}\right| = \frac{1}{|x_2|} > 1$$

Hence by the D'alembert's ratio test,series $\sum_{k=0}^{\infty} |f_k|$ diverges.

(ii) If $1 < \frac{1}{|x_1|} = \frac{1}{|x_2|}$ then

$$\lim_{k\to\infty} \frac{|f_k|}{|f_{k-1}|} = \lim_{k\to\infty} \left|\frac{f_k}{f_{k-1}}\right| = \left|\frac{1}{L_1}\right| = \frac{1}{|x_1|} > 1$$

or

$$\lim_{k\to\infty} \frac{|f_k|}{|f_{k-1}|} = \lim_{k\to\infty} \left|\frac{f_k}{f_{k-1}}\right| = \left|\frac{1}{L_2}\right| = \frac{1}{|x_2|} > 1$$

Hence by the D'alembert's ratio test,series $\sum_{k=0}^{\infty}|f_k|$ diverges.
(iii) If $1=\frac{1}{|x_1|}<\frac{1}{|x_2|}$ then

$$\lim_{k\to\infty}\frac{|f_k|}{|f_{k-1}|}=\lim_{k\to\infty}\left|\frac{f_k}{f_{k-1}}\right|=\left|\frac{1}{L_1}\right|=\frac{1}{|x_1|}=1$$

or

$$\lim_{k\to\infty}\frac{|f_k|}{|f_{k-1}|}=\lim_{k\to\infty}\left|\frac{f_k}{f_{k-1}}\right|=\left|\frac{1}{L_2}\right|=\frac{1}{|x_2|}>1$$

Hence by the D'alembert's ratio test,series $\sum_{k=0}^{\infty}|f_k|$ diverges.
Thus ratio test fails for one case. In this case, we take $f=(f_k)$ in such a way that it is an increasing sequence of positive real numbers and $\lim_{k\to\infty}\left|\frac{f_k}{f_{k-1}}\right|=1$. Clearly, $f=(f_k)$ is a divergent sequence and consequently series $\sum_{k=0}^{\infty}|f_k|$ diverges.
(iv) If $1=\frac{1}{|x_1|}=\frac{1}{|x_2|}$. Here also ratio test fails for both expression for L. Thus by choosing as same way in (iii), we able to prove that series $\sum_{k=0}^{\infty}|f_k|$ diverges. This completes the proof. □

Before we proceed to our next theorem we define following two sets S_1 and S_2 as,

$$S_1=\left\{\alpha\in\mathbb{C}:\frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|}<1\right\}$$

and

$$S_2=\left\{\alpha\in\mathbb{C}:\frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|}=1\right\}$$

Theorem 2.5. *Residual spectrum $\sigma_r(\Delta_{vuw},c_0)$ of the operator Δ_{vuw} over c_0 is*
$$\sigma_r(\Delta_{vuw},c_0)=\begin{cases} S_1, & \text{if } (v_k) \text{ is a constant sequence}\\ S_2\setminus\{v_0,v_1,\dots\}, & \text{if } (v_k) \text{ is a strictly decreasing sequence.}\end{cases}$$

Proof. The proof of theorem is divided into the following two cases.
Case(i). Let v_k is constant sequence, say $v_k=V$ for each $k\in\mathbb{N}_0$. For $\alpha\in\mathbb{C}$ with $\frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|}<1$, the operator $(\Delta_{vuw}-\alpha I)$ is a triangle except $\alpha=U$ and consequently $(\Delta_{vuw}-\alpha I)$ has an inverse. By Theorem 2.4 the operator $(\Delta_{vuw}-\alpha I)^*=\Delta^*_{vuw}-\alpha I$ is not one to one for $\alpha\in\mathbb{C}$ with $\frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|}<1$. Thus

$$\sigma_r(\Delta_{vuw},c_0)=\left\{\alpha\in\mathbb{C}:\frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|}<1\right\}$$

Case(ii) Let v_k be a strictly decreasing sequence. For $\alpha\in\mathbb{C}$ such that

$$\frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|}<1,$$

the operator $(\Delta_{vuw}-\alpha I)$ is triangle except for $\alpha=v_k$ for all $k\in\mathbb{N}_0$ and consequently the operator $(\Delta_{vuw}-\alpha I)$ is not one to one for $\alpha=v_k$ for all $k\in\mathbb{N}_0$. So,

$(\Delta_{vuw} - \alpha I)^{-1}$ does not exist. On the basis of argument as given in Case(i), it is easy to verify that the range of the operator $(\Delta_{vuw} - \alpha I)$ is not dense in c_0. Thus

$$\sigma_r(\Delta_{vuw}, c_0) = \left\{ \alpha \in \mathbb{C} : \frac{2|V - \alpha|}{|-U + \sqrt{U^2 - 4W(V - \alpha)}|} < 1 \right\} \setminus \{v_0, v_1, \dots\}.$$

□

Theorem 2.6. *Continuous spectrum $\sigma_c(\Delta_{vuw}, c_0)$ of the operator Δ_{vuw} over c_0 is* $\sigma_c(\Delta_{vuw}, c_0) = \begin{cases} S_2, & \text{if } (v_k) \text{ is a constant sequence} \\ S_2 \setminus \{v_0, v_1, \dots\}, & \text{if } (v_k) \text{ is a strictly decreasing sequence.} \end{cases}$

Proof. The proof of theorem is divided into the following two cases.
Case(i). Let v_k is constant sequence, say $v_k = V$ for each $k \in \mathbb{N}_0$. For $\alpha \in \mathbb{C}$ with $\frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|} = 1$, the operator $(\Delta_{vuw} - \alpha I)$ is a triangle because $\alpha \neq V$ and consequently $(\Delta_{vuw} - \alpha I)$ has an inverse. By Theorem 2.4 the operator $(\Delta_{vuw} - \alpha I)^{-1}$ is discontinuous by statement (2.8). Therefore, the operator $(\Delta_{vuw} - \alpha I)$ has an unbounded inverse. By Theorem(2.4), the operator $(\Delta_{vuw} - \alpha I)^*$ is not one to one for $\alpha \in \mathbb{C}$ with $\frac{2|V-\alpha|}{|-U+\sqrt{U^2-4W(V-\alpha)}|} = 1$. Hence by Lemma 1.4, range of the operator $\Delta_{vuw} - \alpha I$ is dense in c_0. Thus

$$\sigma_c(\Delta_{vuw}, c_0) = \left\{ \alpha \in \mathbb{C} : \frac{2|V - \alpha|}{|-U + \sqrt{U^2 - 4W(V - \alpha)}|} = 1 \right\}$$

Case(ii). Let v_k be a strictly decreasing sequence. For $\alpha \in \mathbb{C}$ such that

$$\frac{2|V - \alpha|}{|-U + \sqrt{U^2 - 4W(V - \alpha)}|} = 1,$$

the operator $(\Delta_{vuw} - \alpha I)$ is triangle except for $\alpha = v_k$ for all $k \in \mathbb{N}_0$ and consequently the operator $(\Delta_{vuw} - \alpha I)$ has an inverse. By Theorem 2.3 the operator $(\Delta_{vuw} - \alpha I)$ is not one to one for $\alpha = v_k$ for all $k \in \mathbb{N}_0$. So, $(\Delta_{vuw} - \alpha I)^{-1}$ does not exist. On the basis of argument as given in Case(i), it is easy to verify that the range of the operator $(\Delta_{vuw} - \alpha I)$ is not dense in c_0. Thus

$$\sigma_c(\Delta_{vuw}, c_0) = \left\{ \alpha \in \mathbb{C} : \frac{2|V - \alpha|}{|-U + \sqrt{U^2 - 4W(V - \alpha)}|} = 1 \right\} \setminus \{v_0, v_1, \dots\}.$$

□

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