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RINGS IN WHICH EVERY HOMOMORPHIC IMAGE IS A NOETHERIAN DOMAIN

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ABSTRACT. In this paper, we introduce a generalization of the well-known notion of a Noetherian ring which we call a Q-Noetherian ring. We investigate the transfer of the Q-Noetherian property to trivial ring extensions, localizations, homomorphic image of rings, direct product of rings, and amalgamated algebras along an ideal. These results provide examples of non-Noetherian Q-Noetherian rings, of non-coherent Q-Noetherian rings and examples of non-Q-Noetherian coherent rings. The article includes a brief discussion of the scope and precision of our results.

1. Introduction and preliminaries

All rings considered in this paper are commutative with identity elements and all modules are unital. We use "local" to refer to (not necessarily Noetherian) ring with a unique maximal ideal. Also, $Spec(R)$ denote the set of prime ideals of R.

A ring R is called Noetherian if every ideal of R is finitely generated. In this paper, we introduce and investigate a new class of rings called Q-Noetherian rings which generalise the Noetherian notion. A ring R is called a Q-Noetherian ring if R/P is a Noetherian domain for every prime ideal P of R. Clearly, every Noetherian ring is Q-Noetherian.

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, valuation rings, and Prüfer/semihereditary rings. See for instance [6, 7, 8, 10].

Let A be a ring, E be an A-module and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$.

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R is called the trivial ring extension of A by E. Considerable work has been concerned with trivial ring extensions. Part of it has been summarized in Glaz's book [6] and Huckaba's book (where R is called the idealization of E by A) [9]. See for instance [1, 6, 9, 10, 11].

The amalgamation algebras along an ideal, introduced and studied by D'Anna, Finocchiaro and Fontana in [4, 5] and defined as follows:

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f. In particular, they have studied amalgmations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. See for instance [4, 5].

In this paper, we examine the transfer of the Q-Noetherian property to trivial ring extensions, localizations, homomorphic image of rings, direct product of rings, and amalgamated algebras along an ideal. These results provide examples of non-Noetherian Q-Noetherian rings, of non-coherent Q-Noetherian rings and examples of non-Q-Noetherian coherent rings.

2. Main Results

This section develops results of the transfer of the Q-Noetherian property to trivial ring extensions, localizations, homomorphic image of rings, direct product of rings, and amalgamated algebras along an ideal. These results provide examples of non-Noetherian Q-Noetherian rings, of non-coherent Q-Noetherian rings and examples of non-Q-Noetherian coherent rings.

It is clear that every Noetherian ring is Q-Noetherian. Now, we give a sufficient condition to have equivalence between a Noetherian and Q-Noetherian properties.

Proposition 2.1. *Let R be an integral domain. Then R is Q-Noetherian if and only if R is Noetherian.*

Proof. Clear since (O) is a prime ideal in an integral domain R and $R/(O) \cong R$. □

First, we study the transfer of the Q-Noetherian property to trivial ring extension.

Theorem 2.2. *Let A be a ring, E an $A-$module and $R := A \propto E$ be the trivial ring extension of A by E. Then, R is a Q-Noetherian ring if and only if so is A.*

Proof. Assume that R is Q-Noetherian and let P be a prime ideal of A. Then $P \propto E$ is a prime ideal of $R := R \propto E$ and so $(A/P) \cong (A \propto E)/(P \propto E)$ is Noetherian since R is Q-Noetherian. Therefore, A is Q-Noetherian.
Conversely, assume that A is Q-Noetherian and consider a prime ideal of R which is of the form $P \propto E$, where P is a prime ideal of A. Then $(A \propto E)/(P \propto E) \cong (A/P)$ which is Noetherian since A is Q-Noetherian. Therefore, A is Q-Noetherian and this completes the proof. □

A Noetherian ring is both Q-Noetherian and coherent. We will show that the two properties Q-Noetherian and coherent are not comparable.

Now, we are able to construct examples of non-coherent (and so non-Noetherian) Q-Noetherian rings.

Example 2.3. Let D be a Noetherian domain which is not a field, $K = qf(D)$, and let $R := D \propto K$ be the trivial ring extension of D by K. Then:

(1) R is a Q-Noetherian ring by Theorem 2.2 since D is Noetherian.
(2) R is not a coherent ring by the same proof as in [10, Theorem 3.1(1)]. In particular, R is not Noetherian.

Example 2.4. Let (A, M) be a local Noetherian ring, E be an (A/M)-vector space with infinite rank, and let $R := A \propto E$ be the trivial ring extension of A by E. Then:

(1) R is a Q-Noetherian ring by Theorem 2.2 since A is Noetherian.
(2) R is not a coherent ring by [10, Theorem 2.6(2)] since E is an (A/M)-vector space with infinite rank. In particular, R is not Noetherian.

The following example shows that a coherent ring may not be a Q-Noetherian ring.

Example 2.5. Let R be a coherent non-Noetherian domain (for example, $R := K[X_1, \ldots]$ the polynomial ring with infinite indeterminates over a field K). Then:

(1) R is not Q-Noetherian ring by Proposition 2.1 since A is not Noetherian.
(2) R is coherent by hypothesis.

Now, we study the transfer of the Q-Noetherian property to localization.

Proposition 2.6. *Let R be a Q-Noetherian ring and S be a multiplicative subset of R. Then $S^{-1}R$ is a Q-Noetherian ring.*

Proof. Suppose that R is a Q-Noetherian ring and S be a multiplicative subset of R. Let's consider a prime ideal of $S^{-1}R$ which is of the form $S^{-1}P$, where P is a prime ideal of R. We have $S^{-1}R/S^{-1}P \cong S^{-1}(R/P)$ which is Noetherian. Therefore, $S^{-1}R$ is Q-Noetherian.

□

Also, we investigate the tranfert of Q-Noetherian property to the direct product of rings.

Theorem 2.7. *Let $(R_i)_{i=1,\ldots,n}$ be a family of rings. Then $\prod_{i=1}^{n} R_i$ is a Q-Noetherian ring if and only if so is R_i for each $i = 1, \ldots, n$.*

Proof. We will prove the result for $i = 1, 2$, and the Theorem will be established by induction on n.

Assume that $R_1 \times R_2$ is a Q-Noetherian ring and we must to show that R_1 is a Q-Noetherian ring (it is the same for R_2). Let P_1 be a prime ideal of R_1. Then $P_1 \times R_2$ is a prime ideal of $R_1 \times R_2$ and so $(R_1 \times R_2)/(P_1 \times R_2) \cong (R_1/P_1) \times 0$ is Noetherian. Therefore, R_1/P_1 is Noetherian as an homomorphic image. Hence, R_1 is Q-Noetherian.
Conversely, assume that R_1 and R_2 are Q-Noetherian rings and we must to show that $R_1 \times R_2$ is a Q-Noetherian ring. Let P be a prime ideal of the ring $R_1 \times R_2$. The prime ideal P is of the form $P := P_1 \times R_2$ (or $R_1 \times P_2$), where P_1 (resp., P_2) is a prime ideal of R_1 (resp., R_2). Hence, $(R_1 \times R_2)/P = (R_1 \times R_2)/(P_1 \times R_2) = (R_1/P_1) \times 0$ which is Noetherian since R_1/P_1 is it too, and this completes the proof of Theorem 2.7.

□

In the next Proposition, we establish that the homomorphic image of a Q-Noetherian ring is also a Q-Noetherian ring.

Proposition 2.8. *Let R be a Q-Noetherian ring and I be an ideal of R. Then R/I is a Q-Noetherian ring.*

Proof. Assume that R is a Q-Noetherian ring and let I be an ideal of R. Let's consider a prime ideal of R/I which is of the form P/I, where P is a prime ideal of R. We have $(R/I)/(P/I) \cong (R/P)$ which is Noetherian. Therefore, R/I is Q-Noetherian, as desired.

□

The following Theorem develops a result on the transfer of the Q-Noetherian property to amalgamation of rings $A \bowtie^f J$. Recall that if R is a ring, $Nil(R)$ denote the nilradical of R.

Theorem 2.9. *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B. Then,*

(1) *If $A \bowtie^f J$ is a Q-Noetherian ring, then so is A.*
(2) *Assume that $J \subseteq Nil(B)$. Then, $A \bowtie^f J$ is a Q-Noetherian ring if and only if so is A.*

We need the following lemma before proving this Theorem 2.9.

Lemma 2.10. *Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B such that $J \subseteq Nil(B)$. Then, $Spec(A \bowtie^f J) = \{P \bowtie^f J$, where $P \in Spec(A)\}$.*

Proof. Assume that $J \subseteq Nil(B)$. Then J is contained in P for all $P \in Spec(B)$. Consequently, the set $\{\overline{P}^f\}$ in [5, Proposition 2.6 (3)] is empty and so $Spec(A \bowtie^f J) = \{P \bowtie^f J/P \in Spec(A)\}$, as desired.

□

Proof of Theorem 2.9. **1)** By Proposition 2.8 since $(A \bowtie^f J)/(0 \bowtie^f J) \cong A$.

2) Assume that $J \subseteq Nil(B)$. If $A \bowtie^f J$ is a Q-Noetherian ring, then so is A by 1).
Conversely, assume that A is a Q-Noetherian and let's consider a prime ideal of $A \bowtie^f J$ which is of the form $P \bowtie^f J$ by Lemma 2.10, where P is a prime ideal of A. We have $(A \bowtie^f J)/(P \bowtie^f J) \cong A/P$ which is a Noetherian domain since A is a Q-Noetherian. Therefore, $A \bowtie^f J$ is Q-Noetherian, as desired.

□

Now, we are able to construct a new examples of a non-Noetherian Q-Noetherian rings.

Example 2.11. Let (A, B) be a pair of rings such that A is a non-Noetherian Q-Noetherian rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B such that $J \subseteq Nil(B)$ (for example, $J^n = 0$ for some positive integer n). Then:

(1) $A \bowtie^f J$ is a Q-Noetherian ring by Theorem 2.9.
(2) $A \bowtie^f J$ is a non-Noetherian ring by [4, Proposition 5.6].

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