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WHEN EVERY FINITELY PRESENTED PROPER IDEAL IS REGULAR

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Abstract. In this paper we introduce and investigate a class of those rings in which every finitely presented proper ideal is regular. We establish the transfer of this notion to the trivial ring extension, direct product and homomorphic image, and then generate new and original families of rings satisfying this property.

1. Introduction and preliminaries

Throughout this paper, all rings are commutative with identity element, and all modules are unitary.

For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to E \to 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules.

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, and Prüfer/semihereditary rings. For instance see [13].

Let A be a ring and E be an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R = A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A (see [1, Theorem 3.2]). Suitable background on commutative trivial ring extensions is

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[1, 3, 13, 16, 17].

In this paper, we are interested in those rings in which every finitely presented proper ideal is regular and which we called a FPR-rings. Clearly, any integral domain is an FPR-ring.

The purpose of this paper is to give some simple methods in order to construct FPR-rings. For this, we investigate the transfer of FPR property to the trivial ring extension, direct product and homomorphic image.

2. Main results

A ring is called $FPR-$ring if every finitely presented proper ideal is regular.

Proposition 2.1. (1) *Any integral domain is an FPR-ring.*
(2) *Assume that R is a Noetherian ring. Then R is an FPR-ring if and only if it is an integral domain.*

Proof. Straightforward since any ideal is finitely presented in a Noetherian ring. □

Now, we show that the direct product of rings is a never an FPR-rings.

Proposition 2.2. *Let R be a finite direct product of some family of rings $(R_i)_{i=1,\ldots,n}$, where $n \geq 2$. Then R is a never an FPR-ring.*

Proof. It suffices to check the proof for $n = 2$ and let $(R_i)_{i=1,2}$ be a two rings. The following two exact sequences of $(R_1 \times R_2)$:

$$R_1 \times 0 \to R_1 \times R_2 \to 0 \times R_2 \to 0$$

$$0 \to R_2 \times 0 \to R_1 \times R_2 \to R_1 \times 0 \to 0$$

means that $R_1 \times 0$ and $0 \times R_2$ are two finitely presented proper ideals of $R_1 \times R_2$ (since they are finitely generated). But, $R_1 \times 0$ (and also $0 \times R_2$) is not regular since $(R_1 \times 0)(0 \times R_2) = 0$, as desired. □

Recall that a ring is called a $(2,1)$-ring if every 2-presented module has projective dimension at least one. See for instance [18].

Proposition 2.3. *A local $(2,1)$-ring is an FPR-ring.*

Proof. Let I be a finitely presented proper ideal of a local $(2,1)$-ring R. Then, I is a projective ideal and so a free ideal since R is local, that is I is principal ideal generated by a regular element, as desired. □

The converse is false as the following example shows.

Example 2.4. *Let R be a Noetherian integral domain with global dimension greater than 2 (for instance, let $R := \mathbb{R}[X_1, X_2]$, where X_i are indeterminates in $\mathbb{R}$). Then,*

(1) *R is an FPR-ring.*
(2) *R is not a $(2,1)$-ring.*

Proof. Let R be a Noetherian integral domain with global dimension greater than 2 (for instance, let $R := \mathbb{R}[X_1, X_2]$, where X_i are indeterminates in $\mathbb{R}$).
1) R is an FPR-ring since R is an integral domain.
2) R is not a $(2,1)$-ring by [18] since R is a Noetherian ring with global dimension greater than 2, as desired. □

Now, to the main result which study the possible transfer of the FPR-property between a ring A and a trivial ring extension $A \ltimes E$, where E is an A-module.

Theorem 2.5. *Let A be a ring, E be an $A-$module, and let $R := A \propto E$ be the trivial ring extension of A by E. Then:*

(1) *Assume that A is a valuation domain which is not a field and $E := K := qf(A)$. Then R is an FPR-ring (which is a non-integral domain).*
(2) *Assume that (A, M) is a local ring with maximal ideal M such that $ME = 0$. Then:*
   a) *If $dim_{A/M}E = \infty$, then R is an FPR-ring.*
   b) *Assume that M is a finitely generated ideal of A. Then, R is an FPR-ring if and only if $dim_{A/M}E = \infty$.*
(3) *Assume that $E := A$. Then $R := A \propto A$ is never an FPR-ring.*

Proof. 1) Assume that A is a valuation domain which is not a field and $E := K := qf(A)$. Our aim is to show that R is an FPR-ring. Let $J := \sum_{i=1}^{n} R(b_i, e_i)$ be a nonzero finitely presented ideal of R, where $(b_i, e_i) \in J$ for each $i = 1, \ldots, n$. Set $I := \{a \in A/(a, e) \in J$ for some $e \in K\}$. We claim that $I \neq 0$.
Indeed, assume that $I = 0$. Necessarily, $J = \sum_{i=1}^{n} R(0, e_i) = 0 \propto (\sum_{i=1}^{n} Ae_i) = 0 \propto Ae = R(0, e)$ for some $e \neq 0 \in K$ since A is a valuation domain. Consider the exact sequence of R-modules:

$$0 \longrightarrow Ker(u) \longrightarrow R \overset{u}{\longrightarrow} J \longrightarrow 0$$

where $u(a, f) = (a, f)(0, e) = (0, ae)$. Hence, $Ker(u) = 0 \propto K$ is a finitely generated R-module, that is $Ker(u) = 0 \propto K = \sum_{i=1}^{n} R(0, f_i) = 0 \propto (\sum_{i=1}^{n} Af_i)$ and so $K = \sum_{i=1}^{n} Af_i$. Hence, $A = K$ is a field, a desired contradiction.

Hence, $J = I \propto K$ since $IK = K$ (since $I \neq 0$) and so J is regular (since (a, e) is regular element for any $a \in I - \{0\}$). Hence, R is an FPR-ring.

2) Now, assume that (A, M) is a local ring with maximal ideal M such that $ME = 0$.

a) Assume that $dim_{A/M}E = \infty$ and our aim is to show that R is an FPR-ring. For this, it suffices to show that there exists no proper finitely presented ideal of R. Deny. Let $J := \sum_{i=1}^{n} R(a_i, e_i)$ be a finitely presented proper ideal of R, where $(b_i, f_i)_{i=1,\ldots,n}$ is a minimal generating set of J. Clearly, $J \subseteq M \propto E$ since R is a local ring with maximal ideal $M \propto E$. Consider the exact sequence of R-modules:

$$0 \to Ker(u) \to R^n \overset{u}{\to} J \to 0$$

where $u((a_i, e_i)_{i=1,\dots,n}) = \sum_{i=1}^{n}(a_i, e_i)(b_i, f_i) = (\sum_{i=1}^{n} a_ib_i, \sum_{i=1}^{n} a_if_i)$ (since $b_i \in M$ for each $i = 1, \dots, n$). But, $Ker(u) \subseteq (M \propto E)^n$ by [8, Lemma 4.43, p. 134] since $(b_i, f_i)_{i=1,\dots,n}$ is a minimal generating set of J. Hence, $Ker(u) = V \propto E^n$, where $V = \{(a_i)_{i=1,\dots,n} \in A^n / \sum_{i=1}^{n} a_ib_i = 0\}$ since $ME = 0$. Let $(y_i, g_i)_{i=1,\dots,m}$ be a generating set of $Ker(u)$. Then, $V \propto E^n = Ker(u) = \sum_{i=1}^{m} R(y_i, g_i) = \{\sum_{i=1}^{m}(a_i, e_i)(y_i, g_i)/(a_i, e_i) \in R$ for each $i = 1, \dots, m\} = \{(\sum_{i=1}^{m} a_iy_i, \sum_{i=1}^{m} a_ig_i)/a_i \in R$ for each $i = 1, \dots, m\}$ since $y_i \in M^n$ for each $i = 1, \dots, m$ (since $Ker(u) \subseteq (M \propto E)^n$). Therefore, $E^n = \sum_{i=1}^{m}(A/M)g_i$ and so, E is an (A/M)-vector space of finite dimension. A desired contradiction. Hence, R is an FPR-ring.

b) Now, assume that M is a finitely generated ideal of A. By 2)a), it remains to show that if R is an FPR-ring, then $dim_{A/M}E = \infty$.
Assume that R is an FPR-ring, and we claim that $dim_{A/M}E = \infty$. Deny. Let $f \in E - \{0\}$ and set $J = R(0, f)$. Consider the exact sequence of R-modules:

$$0 \to Ker(v) \to R \overset{v}{\to} J \to 0$$

where $v((a, e) = (a, e)(0, f) = (0, af)$. Then, $Ker(v) = M \propto E$ (since $ME = 0$) is a finitely generated ideal of R (since M is a finitely generated ideal of A and E is a finitely generated A-module) and so J is finitely presented ideal of R. But, J is not regular since $J(0, f) = 0$, a desired contradiction since R is an FPR-ring. Hence, $dim_{A/M}E = \infty$.

3) Let A is a ring and let $R := A \propto A$. We claim that R is a never an FPR-ring. Deny. Set $J := 0 \propto A = R(0, 1)$ and consider the exact sequence of R-modules:

$$0 \to Ker(w) \to R \overset{w}{\to} J \to 0$$

where $w((a, e) = (a, e)(0, 1) = (0, a)$ shows that $Ker(w) = J$ is a finitely generated ideal of R and so J is a finitely presented ideal of R. On the other hand, $J(0, 1) = 0$, a desired contradiction. Hence, R is a never an FPR-ring and this completes the proof of Theorem 2.5. □

By the above Theorem 2.5, we have the following examples:

Example 2.6. (1) $\mathbb{Z} \propto \mathbb{R}$ *is an* FPR*-ring which is not an integral domain.*
(2) $\mathbb{R}[[X]] \propto ((\mathbb{R}[[X]])/X\mathbb{R}[[X]])^{\infty}$ *is an* FPR*-ring.*
(3) $\mathbb{R}[[X]] \propto ((\mathbb{R}[[X]])/X\mathbb{R}R[[X]])$ *is not an* FPR*-ring.*
(4) $\mathbb{Z} \propto \mathbb{Z}$ *is not an* FPR*-ring.*

The homomorphic image of an FPR-ring is not always an FPR-ring as the following two examples shows:

Example 2.7. *Let $R = \mathbb{Z}[X] = \mathbb{Z}+X\mathbb{Z}[X]$ and $I = X^2\mathbb{Z}[X]$ be an ideal of R. Then:*

(1) *R is an FPR-ring.*
(2) *R/I is not an FPR-ring.*

Proof. 1) R is an FPR-ring since it is an integral domain.

2) We have:

$$\begin{aligned} R/I =& \left\{ \left(a + X \left(\sum_{i=0}^{n} a_i X^i \right) \right) + X^2\mathbb{Z}[X] / a \in \mathbb{Z} \textit{ and } a_i \in \mathbb{Z} \right\} \\ =& \left\{ a + a_0 X + X^2\mathbb{Z}[X] / a \in \mathbb{Z} \textit{ and } a_0 \in \mathbb{Z} \right\} \\ =& \left\{ a + a_0 \overline{X} / a \in \mathbb{Z} \textit{ and } a_0 \in \mathbb{Z} \right\} \\ \cong& \mathbb{Z} \propto \mathbb{Z} \end{aligned}$$

which is not an FPR-ring by Theorem 2.5(3), as desired. □

Example 2.8. *Let $A := \mathbb{R}[[X]] \propto ((\mathbb{R}[[X]])/X\mathbb{R}[[X]])$ be a local ring with maximal ideal $M := X\mathbb{R}[[X]] \propto ((\mathbb{R}[[X]])/X\mathbb{R}[[X]])$ and set $R := A \propto (A/M)^\infty$. Then:*

(1) *R is an FPR-ring by Theorem 2.5(2.a).*
(2) *$R/(0 \propto E)(\cong A)$ is not an FPR-ring by Theorem 2.5(2.b).*

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