

APPLICATION ON ATANGANA-BALEANU FRACTIONAL DERIVATIVE VIA (τ, α) -WEAK CONTRACTIVE MAPPINGS

KOTI N. V. V. VARA PRASAD¹, VINAY MISHRA^{2*}, HAPKA SURENDRA³

ABSTRACT. This paper introduces the innovative concept of (τ, α) -weak contractive mappings. We establish several fixed point results without relying on conditions essential in prior works. Examples are presented to showcase our key findings, particularly in scenarios where earlier research outcomes are not applicable. Furthermore, we provide an application of our obtained results to fractional differential equations involving non-singular and non-local kernels.

1. INTRODUCTION AND PRELIMINARIES

The Banach contraction principle stands out as a well-established and powerful tool in the field of nonlinear analysis, renowned for its effectiveness. Several extensions and generalizations of the Banach contraction principle are documented in the literature, including one introduced by Wardowski [12]. In 2012, he innovatively presented a contraction condition known as $\mathfrak{F}$ -contraction and applied it to establish a unique fixed point theorem in a complete metric space. Wardowski [12] examined a family of auxiliary functions $\mathfrak{F} : (0, \infty) \rightarrow \mathbb{R}$ that satisfy the following conditions:

- $\mathfrak{F}_1$: $\mathfrak{F}$ is strictly increasing;
- $\mathfrak{F}_2$: For every sequence $\{\mathfrak{d}_j\}_{j \in \mathbb{N}}$ consisting of positive numbers, it holds that $\lim_{j \rightarrow \infty} \mathfrak{d}_j = 0 \iff \lim_{j \rightarrow \infty} \mathfrak{F}(\mathfrak{d}_j) = -\infty$;
- $\mathfrak{F}_3$: $\exists r \in (0, 1)$ such that $\lim_{\mathfrak{d} \rightarrow 0^+} \mathfrak{d}^r \mathfrak{F}(\mathfrak{d}) = 0$.

Let $\mathcal{F}$ denote the family of all functions satisfying conditions $\mathfrak{F}_1$, $\mathfrak{F}_2$, and $\mathfrak{F}_3$.

Then a self-map $\mathfrak{T}$ on a nonempty set X is called as an $\mathfrak{F}$ -contraction if for $\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta) > 0$, it implies that $\tau + \mathfrak{F}(\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)) \leq \mathfrak{F}(\partial(\eta, \vartheta))$, $\forall \eta, \vartheta \in X$ and some $\tau > 0$. Following the fundamental importance of this concept, numerous attempts have been made to ease the assumptions initially proposed by Wardowski (See [3, 4, 7, 8, 11, 14]). An exemplary effort in this direction was undertaken by D. Gopal et al. [5], who introduced a variant of $\mathfrak{F}$ -contraction by incorporating the concept of α -admissibility,

Date: Received: Jan 5, 2024; Accepted: Feb 26, 2024.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.

Key words and phrases. weak contraction, triangular α -orbital admissible, strictly triangular α -orbital admissible, Atangana-Baleanu fractional derivative.

Definition 1.1. [10] Consider a self map $\mathfrak{T}$ on a nonempty set X , and a function $\alpha : X \times X \rightarrow \mathbb{R}^+$, then $\mathfrak{T}$ is said to be α -admissible if,

$$\alpha(\eta, \vartheta) \geq 1 \implies \alpha(\mathfrak{T}\eta, \mathfrak{T}\vartheta) \geq 1 \text{ for all } \eta, \vartheta \in X \geq 1.$$

Definition 1.2. [5] Let $\mathfrak{T}$ be a self map defined on a metric space (X, ∂) , then $\mathfrak{T}$ is considered as, α -type $\mathfrak{F}$ -weak contraction, if $\mathfrak{T}$ satisfies

$$\tau + \alpha(\eta, \vartheta) \mathfrak{F}(\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)) \leq \mathfrak{F} \left(\max \left\{ \partial(\eta, \vartheta), \partial(\eta, \mathfrak{T}\eta), \partial(\vartheta, \mathfrak{T}\vartheta), \frac{\partial(\eta, \mathfrak{T}\vartheta) + \partial(\vartheta, \mathfrak{T}\eta)}{2} \right\} \right). \quad (1.1)$$

For all $\eta, \vartheta \in X$, satisfying $\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta) > 0$. Where $\mathfrak{F} \in \mathcal{F}$, $\tau > 0$ and, $\alpha : X \times X \rightarrow (0, +\infty)$.

By using the above definition, in 2016, Gopal et al. [5] proved the following results.

Theorem 1.3. [5] Consider the complete metric space (X, ∂) , and a self-map $\mathfrak{T}$ defined on it. Suppose $\mathfrak{T}$ is continuous α -admissible map satisfying condition (1.1), and

(Γ_1) there exists $\eta_1 \in X$ for which $\alpha(\eta_1, \mathfrak{T}\eta_1) \geq 1$.

Then $\mathfrak{T}$ possesses a fixed point in X .

Theorem 1.4. [5] Consider the complete metric space (X, ∂) , and a self-map $\mathfrak{T}$ defined on it. Suppose that $\mathfrak{F}$ is continuous, $\mathfrak{T}$ is α -admissible map satisfying condition (1.1), and (Γ_1) holds with

(Γ_2) if $\{\eta_{\mathfrak{J}}\}$ be a sequence in X for which $\alpha(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) \geq 1$, $\forall \mathfrak{J} \in \mathbb{N}$, and $\eta_{\mathfrak{J}} \rightarrow \eta$ as $\mathfrak{J} \rightarrow +\infty$, then $\alpha(\eta_{\mathfrak{J}}, \eta) \geq 1$, $\forall \mathfrak{J} \in \mathbb{N}$,

Then $\mathfrak{T}$ possesses a fixed point in X .

In a recent development from 2018, Wardowski [13] revisited the $\mathfrak{F}$ -contraction definition within a broader context. Specifically, this revision involves the elimination of the $\mathfrak{F}_3$ condition and the substitution of τ with a function, outlined as follows:

Definition 1.5. [13] Let $\mathfrak{T}$ be a self-map defined on a set X , then $\mathfrak{T}$ is characterized as a $(\varphi, \mathfrak{F})$ -contraction, also referred to as a non-linear $\mathfrak{F}$ -contraction, under the condition that there exist functions $\varphi : (0, \infty) \rightarrow (0, \infty)$ and $\mathfrak{F} : (0, \infty) \rightarrow \mathbb{R}$ such that,

(H1) Function $\mathfrak{F}$ fulfills conditions ($\mathfrak{F}_1$) and $\lim_{t \rightarrow 0^+} \mathfrak{F}(t) = -\infty$;

(H2) for all $t \geq 0$ we have $\liminf_{s \rightarrow t^+} \varphi(s) > 0$;

(H3) $\varphi(\partial(\eta, \vartheta)) + \mathfrak{F}(\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)) \leq \mathfrak{F}(\partial(\eta, \vartheta))$, $\forall \eta, \vartheta \in X$ such that $\mathfrak{T}\eta \neq \mathfrak{T}\vartheta$,

Definition 1.6. Consider a self-map $\mathfrak{T}$ on X , and a function $\alpha : X \times X \rightarrow \mathbb{R}^+$, then $\mathfrak{T}$ is considered as;

(i) α -orbital admissible [9] if,

$$\alpha(\eta, \mathfrak{T}\eta) \geq 1 \implies \alpha(\mathfrak{T}\eta, \mathfrak{T}^2\eta) \geq 1; \quad (1.2)$$

(ii) *triangular $\mathbf{a}$ -orbital admissible* [9], if $\mathfrak{T}$ complies (i) and

$$\mathbf{a}(\eta, \vartheta) \geq 1 \text{ and } \mathbf{a}(\vartheta, \mathfrak{T}\vartheta) \geq 1 \implies \mathbf{a}(\eta, \mathfrak{T}\vartheta) \geq 1. \quad (1.3)$$

In this paper, we present two distinct and independent concepts encompassing novel forms of weak $\mathfrak{F}$ -contraction conditions. Furthermore, by making slight adjustments to the definition of $\mathbf{a}$ -orbital admissible maps, we introduce the novel notion of strictly triangular $\mathbf{a}$ -orbital admissible mappings. This conceptual advancement enables us to deduce our fixed point results without the conditions $\mathfrak{F}_2$ and $\mathfrak{F}_3$, and concurrently, we relax the continuity requirements for both functions $\mathfrak{F}$ and $\mathfrak{T}$.

Definition 1.7. Let (X, ∂) be a metric space, and there exist functions $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\mathbf{a} : X \times X \rightarrow (0, \infty)$ and $\mathfrak{F} : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that for every $\eta, \vartheta \in X$ satisfying $\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta) > 0$, $\mathfrak{T}$ is said to be

(i) $(\tau, \mathbf{a})$ -weak contraction of type I, if

$$\tau(\partial(\eta, \vartheta)) + \mathbf{a}(\eta, \vartheta)\mathfrak{F}(\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)) \leq \mathfrak{F}(\mathfrak{M}(\eta, \vartheta)), \quad \mathfrak{F} \in \mathcal{F}, \quad (1.4)$$

(ii) $(\tau, \mathbf{a})$ -weak contraction of type II, if

$$\tau(\partial(\eta, \vartheta)) + \mathfrak{F}(\mathbf{a}(\eta, \vartheta)\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)) \leq \mathfrak{F}(\mathfrak{M}(\eta, \vartheta)), \quad \mathfrak{F} \in \mathfrak{F}_1, \quad (1.5)$$

$$\text{where, } \mathfrak{M}(\eta, \vartheta) = \max \left\{ \begin{array}{l} \partial(\eta, \vartheta), \partial(\eta, \mathfrak{T}\eta), \partial(\vartheta, \mathfrak{T}\vartheta), \frac{\partial(\vartheta, \mathfrak{T}\eta) + \partial(\eta, \mathfrak{T}\vartheta)}{2}, \\ \frac{\partial(\vartheta, \mathfrak{T}\vartheta)[1 + \partial(\eta, \mathfrak{T}\eta)]}{1 + \partial(\eta, \vartheta)}, \frac{\partial(\vartheta, \mathfrak{T}\eta)[1 + \partial(\eta, \mathfrak{T}\vartheta)]}{1 + \partial(\eta, \vartheta)} \end{array} \right\}.$$

In the above definitions, it can be noted that in condition (1.5), we have applied condition $\mathfrak{F}_1$ only to function $\mathfrak{F}$, showcasing its wide-ranging applicability. The following two examples show that conditions (1.4) and (1.5) are independent.

Example 1.8. Let us consider the non-empty set $X = \{0, 1, 2\}$, and define the metric $\partial : X \times X \rightarrow \mathbb{R}^+$ by

$$\partial(\eta, \vartheta) = \begin{cases} 0, & \text{if } \eta = \vartheta, \\ \frac{1}{5}, & \text{if } (\eta, \vartheta) \in \{(1, 0), (0, 1)\}, \\ \frac{1}{10}, & \text{if } (\eta, \vartheta) \in \{(1, 2), (2, 1)\}, \\ \frac{3}{10}, & \text{if } (\eta, \vartheta) \in \{(0, 2), (2, 0)\}, \end{cases}$$

additionally, let $\mathfrak{T}$ be self-map on X , such that $\mathfrak{T}0 = \mathfrak{T}1 = 0$, and $\mathfrak{T}2 = 1$.

and $\mathbf{a} : X \times X \rightarrow \mathbb{R}^+$ by $\mathbf{a}(\eta, \vartheta) = \begin{cases} 2, & \text{if } \eta \in \{(2, 0), (2, 1)\}, \\ \frac{6}{\eta+2}, & \text{otherwise.} \end{cases}$

By considering $\mathfrak{F}\eta = \ln \eta$, $\forall \eta > 0$, and $\tau(\eta) = \frac{7}{5}$ (constant), we can observe that the following cases are possible;

Case I: if $\eta = 0, \vartheta = 2$

$$\tau(\partial(0, 2)) + \mathbf{a}(0, 2)\mathfrak{F}(\partial(\mathfrak{T}0, \mathfrak{T}2)) = \frac{7}{5} + 3 \ln \left(\frac{1}{5} \right) < \mathfrak{F}(\partial(0, 2)) \leq \mathfrak{F}(\mathfrak{M}(0, 2)).$$

Case II: if $\eta = 2, \vartheta = 0$

$$\tau(\partial(2, 0)) + \mathfrak{a}(2, 0)\mathfrak{F}(\partial(\mathfrak{T}2, \mathfrak{T}0)) = \frac{7}{5} + 2\ln\left(\frac{1}{5}\right) < \mathfrak{F}(\partial(2, 0)) \leq \mathfrak{F}(\mathfrak{M}(2, 0)).$$

Case III: if $\eta = 1, \vartheta = 1$

$$\tau(\partial(1, 2)) + \mathfrak{a}(1, 2)\mathfrak{F}(\partial(\mathfrak{T}1, \mathfrak{T}2)) = \frac{7}{5} + 2\ln\left(\frac{1}{5}\right) < \mathfrak{F}(\partial(1, 2)) \leq \mathfrak{F}(\mathfrak{M}(1, 2)).$$

The case $\eta = 2, \vartheta = 1$ can be solved similarly and the remaining cases vanish trivially. Thus, $\mathfrak{T}$ satisfies (1.4).

But, for $\eta = 0, \vartheta = 2$,

$$\tau(\partial(0, 2)) + \mathfrak{F}(\mathfrak{a}(0, 2)\partial(\mathfrak{T}0, \mathfrak{T}2)) = \frac{7}{5} + \ln\left(2 \times \frac{1}{5}\right) \not\leq \mathfrak{F}(\mathfrak{M}(0, 2)).$$

Therefore, $\mathfrak{T}$ does not satisfy condition (1.5).

Example 1.9. Let X and $\mathfrak{T}$ be as defined in Example 1.8, and let $\partial : X \times X \rightarrow \mathbb{R}^+$ be a metric such that

$$\partial(\eta, \vartheta) = \begin{cases} 0, & \text{if } \eta = \vartheta, \\ 5, & \text{if } (\eta, \vartheta) \in \{(1, 0), (0, 1)\}, \\ 10, & \text{if } (\eta, \vartheta) \in \{(1, 2), (2, 1)\}, \\ 15, & \text{if } (\eta, \vartheta) \in \{(0, 2), (2, 0)\}. \end{cases}$$

Then by considering $\mathfrak{F}\eta = \ln \eta, \forall \eta > 0, \mathfrak{a}(\eta, \vartheta) = \begin{cases} |\eta - \vartheta|, & \text{if } \eta \neq \vartheta, \\ 1, & \text{if } \eta = \vartheta. \end{cases}$

And $\tau(\eta) = \frac{3}{10}$ (constant), then again we can observe that;

Case I: If $\eta = 0, \vartheta = 2$ or $\eta = 2, \vartheta = 0$.

$$\tau(\partial(0, 2)) + \mathfrak{F}(\mathfrak{a}(0, 2)\partial(\mathfrak{T}0, \mathfrak{T}2)) = \frac{3}{10} + \ln(10) < \mathfrak{F}(\partial(0, 2)) \leq \mathfrak{F}(\mathfrak{M}(0, 2)).$$

Case II: if $\eta = 1, \vartheta = 2$ or $\eta = 2, \vartheta = 1$.

$$\tau(\partial(1, 2)) + \mathfrak{F}(\mathfrak{a}(1, 2)\partial(\mathfrak{T}1, \mathfrak{T}2)) = \frac{3}{10} + \ln(10) < \mathfrak{F}(\partial(1, 2)) \leq \mathfrak{F}(\mathfrak{M}(1, 2)),$$

remaining cases can be solved similarly. Thus $\mathfrak{T}$ satisfies (1.5).

But, for $\eta = 0$, and $\vartheta = 2$,

$$\tau(\partial(0, 2)) + \mathfrak{a}(0, 2)\mathfrak{F}(\partial(\mathfrak{T}0, \mathfrak{T}2)) = \frac{3}{10} + 2\ln(5) \not\leq \mathfrak{F}(\mathfrak{M}(0, 2))$$

Therefore, $\mathfrak{T}$ does not satisfy (1.4).

The subsequent lemmas will be pivotal in deriving our main results.

Lemma 1.10. [9] Consider a triangular $\mathfrak{a}$ -orbital admissible mapping $\mathfrak{T}$. Let $\mathfrak{a}(\eta_1, \mathfrak{T}\eta_1) \geq 1$, for some $\eta_1 \in X$, also $\{\eta_{\mathfrak{J}}\}$ be a sequence defined by $\eta_{\mathfrak{J}+1} = \mathfrak{T}\eta_{\mathfrak{J}}$, then $\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_m) \geq 1 \forall m, \mathfrak{J} \in \mathbb{N}$ with $\mathfrak{J} < m$.

Lemma 1.11. [6] For every sequence $\{\eta_{\mathfrak{J}}\}$ defined on metric space (X, ∂) , if there exists $\lambda \in [0, 1)$ such that $\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2}) \leq \lambda\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \forall \mathfrak{J} \in \mathbb{N}$. Then $\{\eta_{\mathfrak{J}}\}$ is Cauchy.

In the next section, we state and prove some fixed-point theorems for our proposed contraction conditions, which are also supported by some illustrative examples. Our findings are some real generalizations and enhancements of the results of Wardowski [12, 13, 14] and Gopal et al. [5]. In the final section, we also discuss an application for our discovered fixed point result within the context of fractional differential equations, demonstrating the significance of our findings.

2. RESULTS ON $(\tau, \mathfrak{a})$ -WEAK CONTRACTION OF TYPE-I AND TYPE-II

We now demonstrate our first result.

Theorem 2.1. *Let $\mathfrak{T}$ be a self-map defined on a complete metric space (X, ∂) , such that $\mathfrak{T}$ is continuous, and satisfies the conditions (1.3), (1.4) with (F1). Then $\mathfrak{T}$ has a fixed point in X , provided $\liminf_{\eta \rightarrow t^+} \tau(\vartheta) > 0$ for all $\eta \geq 0$.*

Proof. Assuming condition (F1), let $\eta_1 \in X$ be such that $\mathfrak{a}(\eta_1, \mathfrak{T}\eta_1) \geq 1$. Define the sequence $\{\eta_{\mathfrak{J}}\}$ in X by $\eta_{\mathfrak{J}+1} = \mathfrak{T}\eta_{\mathfrak{J}}$, for all $\mathfrak{J} \in \mathbb{N}$. If there exists $\mathfrak{J}_0 \in \mathbb{N}$ such that $\eta_{\mathfrak{J}_0+1} = \eta_{\mathfrak{J}_0}$, the proof is immediately concluded. Therefore, we consider the case where $\eta_{\mathfrak{J}+1} \neq \eta_{\mathfrak{J}}$, $\forall \mathfrak{J} \in \mathbb{N}$. Utilizing Lemma 1.10, we establish $\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) \geq 1$ for all $\mathfrak{J} \geq 1$. Now, since $\mathfrak{T}$ is a $(\tau, \mathfrak{a})$ -weak contraction of type I, it follows that for every $\mathfrak{J} \in \mathbb{N}$, we have

$$\begin{aligned} \mathfrak{F}(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})) &= \mathfrak{F}(\partial(\mathfrak{T}\eta_{\mathfrak{J}-1}, \mathfrak{T}\eta_{\mathfrak{J}})) \\ &\leq \mathfrak{a}(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}) \mathfrak{F}(\mathfrak{T}\eta_{\mathfrak{J}-1}, \mathfrak{T}\eta_{\mathfrak{J}}). \end{aligned}$$

Consequently,

$$\begin{aligned} \tau(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})) + \mathfrak{F}(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})) &\leq \tau(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})) + \mathfrak{a}(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}) \mathfrak{F}(\mathfrak{T}\eta_{\mathfrak{J}-1}, \mathfrak{T}\eta_{\mathfrak{J}}) \\ &\leq \mathfrak{F}(\mathfrak{M}(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})) \\ &= \mathfrak{F} \left(\max \left\{ \begin{aligned} &\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}), \partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}), \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \frac{\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}}) + \partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}+1})}{2}, \\ &\frac{\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})[1 + \partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})]}{1 + \partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})}, \frac{\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}})[1 + \partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}+1})]}{1 + \partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})} \end{aligned} \right\} \right) \\ &\leq \mathfrak{F}(\max \{\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}), \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\}). \end{aligned} \quad (2.1)$$

Suppose, $\max \{\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}), \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\}$ is $\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})$, then

$\tau(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})) + \mathfrak{F}(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})) \leq \mathfrak{F}(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}))$, a contradiction. Therefore, the maximum of $\{\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}), \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\}$ is $\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})$. Then the equation (2.1), reformed

$$\tau(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})) + \mathfrak{F}(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})) \leq \mathfrak{F}(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})), \quad (2.2)$$

this leads to, $\mathfrak{F}(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})) < \mathfrak{F}(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}))$, by using the property $\mathfrak{F}_1$. So we can conclude that the sequence $\{\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})\}$ is a decreasing sequence. Consequently, there exists a non-negative real number r , such that $\{\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})\} \rightarrow r$ as $\mathfrak{J} \rightarrow \infty$. Also, since it is provided that $\liminf_{\eta \rightarrow t^+} \tau(\eta) > 0$ for all $\eta \geq 0$, therefore $\exists p > 0$

and $\mathfrak{J}_0 \in \mathbb{N}$ such that $\tau \partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}) > p$, $\forall \mathfrak{J} \geq \mathfrak{J}_0$. Now from (2.2) we have,

$$\begin{aligned}
 \mathfrak{F}(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})) &\leq \mathfrak{F}(\partial(\eta_{\mathfrak{J}-2}, \eta_{\mathfrak{J}-1})) - \tau(\partial(\eta_{\mathfrak{J}-2}, \eta_{\mathfrak{J}-1})) \\
 &\leq \mathfrak{F}(\partial(\eta_{\mathfrak{J}-3}, \eta_{\mathfrak{J}-2})) - [\tau(\partial(\eta_{\mathfrak{J}-2}, \eta_{\mathfrak{J}-1})) + \tau(\partial(\eta_{\mathfrak{J}-3}, \eta_{\mathfrak{J}-2}))] \\
 &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 &\leq \mathfrak{F}(\partial(\eta_0, \eta_1)) - \left[\sum_{i=1}^{\mathfrak{J}-1} \tau(\partial(\eta_{i-1}, \eta_i)) \right] \\
 &= \mathfrak{F}(\partial(\eta_0, \eta_1)) - \left[\sum_{i=1}^{\mathfrak{J}_0-1} \tau(\partial(\eta_{i-1}, \eta_i)) + \sum_{i=\mathfrak{J}_0}^{\mathfrak{J}-1} \tau(\partial(\eta_{i-1}, \eta_i)) \right] \\
 &< \mathfrak{F}(\partial(\eta_0, \eta_1)) - \mathfrak{J}p, \quad \mathfrak{J} \geq \mathfrak{J}_0
 \end{aligned} \tag{2.3}$$

by taking the limit as $\mathfrak{J} \rightarrow \infty$ on both sides, we have $\mathfrak{F}(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})) \rightarrow -\infty$, and according to property $\mathfrak{F}_2$, we deduce that $\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}) \rightarrow 0$. Utilizing property $\mathfrak{F}_3$, we can ensure the existence of $r \in (0, 1)$ such that

$$\lim_{\mathfrak{J} \rightarrow \infty} (\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}))^r F(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})) = 0.$$

It follows from (2.3) that

$$(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}))^r \left(\mathfrak{F}(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}})) - \mathfrak{F}(\partial(\eta_0, \eta_1)) \right) \leq -(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}))^r \mathfrak{J}p \leq 0,$$

and letting $\mathfrak{J} \rightarrow \infty$, we obtain

$$\lim_{\mathfrak{J} \rightarrow \infty} \mathfrak{J}(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}))^r = 0. \tag{2.4}$$

From (2.4), it can be observed that $\exists \mathfrak{J}_1 \in \mathbb{N}$, such that $\mathfrak{J}(\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}))^r \leq 1$ for all $\mathfrak{J} \geq \mathfrak{J}_1$. Therefore, we have

$$\partial(\eta_{\mathfrak{J}-1}, \eta_{\mathfrak{J}}) \leq \frac{1}{\mathfrak{J}^{\frac{1}{r}}}, \quad \forall \mathfrak{J} \geq \mathfrak{J}_1. \tag{2.5}$$

Now by using triangular inequality, for all $m > \mathfrak{J} > \mathfrak{J}_1$, we get

$$\partial(\eta_m, \eta_{\mathfrak{J}}) \leq \partial(\eta_m, \eta_{m-1}) + \dots + \partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}}) < \sum_{i=\mathfrak{J}}^{\infty} \partial(\eta_{i+1}, \eta_i) \leq \sum_{i=\mathfrak{J}}^{\infty} \frac{1}{\mathfrak{J}^{\frac{1}{r}}}, \quad (\text{By (2.5)})$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{\mathfrak{J}^{\frac{1}{r}}}$ is convergent, it implies that the sequence $\{\eta_{\mathfrak{J}}\}$ is a Cauchy sequence in the complete metric space X . Hence, there exists an element $\mathfrak{K} \in X$ such that $\lim_{\mathfrak{J} \rightarrow \infty} \eta_{\mathfrak{J}} = \mathfrak{K}$. Moreover, as $\mathfrak{T}$ is continuous, we can assert the following,

$$\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}) = \lim_{\mathfrak{J} \rightarrow \infty} \partial(\eta_{\mathfrak{J}}, \mathfrak{T}\eta_{\mathfrak{J}}) = \lim_{\mathfrak{J} \rightarrow \infty} \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) = 0.$$

This indicates that $\mathfrak{K}$ is a fixed point of $\mathfrak{T}$. □

In the next result, we impose continuity of $\mathfrak{F}$ in place of mapping $\mathfrak{T}$ in Theorem 2.1.

Theorem 2.2. Let $\mathfrak{T}$ be a self-map defined on a complete metric space (X, ∂) , satisfying the conditions (1.3), (1.4), where $\mathfrak{F}$ is continuous, and (F1), (F2) holds, then $\mathfrak{T}$ has a fixed point in X , provided $\liminf_{\eta \rightarrow t^+} \tau(\eta) > 0$ for all $\eta \geq 0$.

Proof. Assuming condition (F1), let $\eta_1 \in X$ be such that $\mathfrak{a}(\eta_1, \mathfrak{T}\eta_1) \geq 1$. Utilizing the steps outlined in the previous theorem, it is evident that the sequence $\{\eta_{\mathfrak{J}}\}$ forms a Cauchy sequence in the complete metric space X . Consequently, there exists an element $\mathfrak{K} \in X$, such that $\lim_{\mathfrak{J} \rightarrow \infty} \eta_{\mathfrak{J}} = \mathfrak{K}$.

From Lemma 1.10 we have $\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) \geq 1$, then by (F2) we get

$$\mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K}) \geq 1, \quad \forall \mathfrak{J} \in \mathbb{N}. \quad (2.6)$$

Now, we analyze two possible cases,

Case I: Suppose that for every $\mathfrak{J} \in \mathbb{N}$, $\exists i_{\mathfrak{J}} \in \mathbb{N}$ such that $\eta_{i_{\mathfrak{J}}+1} = \mathfrak{T}\mathfrak{K}$ where $i_{\mathfrak{J}} > i_{\mathfrak{J}-1}$. Then,

$$\mathfrak{K} = \lim_{\mathfrak{J} \rightarrow \infty} \eta_{i_{\mathfrak{J}}+1} = \lim_{\mathfrak{J} \rightarrow \infty} \mathfrak{T}\eta_{i_{\mathfrak{J}}} = \mathfrak{T}\mathfrak{K}, \quad (2.7)$$

i.e., $\mathfrak{K}$ is a fixed point of $\mathfrak{T}$.

Case II: Now, let's consider $\exists \mathfrak{J}_0 \in \mathbb{N}$ such that $\eta_{\mathfrak{J}+1} \neq \mathfrak{T}\mathfrak{K}$, $\forall \mathfrak{J} \geq \mathfrak{J}_0$.

Then, we have

$$\begin{aligned} \tau(\partial(\eta_{\mathfrak{J}}, \mathfrak{K})) + \mathfrak{F}(\partial(\eta_{\mathfrak{J}+1}, \mathfrak{T}\mathfrak{K})) &\leq \tau(\partial(\eta_{\mathfrak{J}}, \mathfrak{K})) + \mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K})\mathfrak{F}(\partial(\mathfrak{T}\eta_{\mathfrak{J}}, \mathfrak{T}\mathfrak{K})) \\ &\leq \max \left\{ \begin{aligned} &\partial(\eta_{\mathfrak{J}}, \mathfrak{K}), \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}), \frac{\partial(\mathfrak{K}, \eta_{\mathfrak{J}+1}) + \partial(\eta_{\mathfrak{J}}, \mathfrak{T}\mathfrak{K})}{2}, \\ &\frac{\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K})[1 + \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})]}{1 + \partial(\eta_{\mathfrak{J}}, \mathfrak{K})}, \frac{\partial(\mathfrak{K}, \eta_{\mathfrak{J}+1})[1 + \partial(\eta_{\mathfrak{J}}, \mathfrak{T}\mathfrak{K})]}{1 + \partial(\eta_{\mathfrak{J}}, \mathfrak{K})} \end{aligned} \right\}. \end{aligned} \quad (2.8)$$

If $\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}) > 0$, then by the fact that $\lim_{\mathfrak{J} \rightarrow \infty} \partial(\eta_{\mathfrak{J}}, \mathfrak{K}) = \lim_{\mathfrak{J} \rightarrow \infty} \partial(\mathfrak{K}, \eta_{\mathfrak{J}+1}) = 0$, there exists $\mathfrak{J}_1 \in \mathbb{N}$, such that

$$\max \left\{ \begin{aligned} &\partial(\eta_{\mathfrak{J}}, \mathfrak{K}), \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}), \frac{\partial(\mathfrak{K}, \eta_{\mathfrak{J}+1}) + \partial(\eta_{\mathfrak{J}}, \mathfrak{T}\mathfrak{K})}{2}, \\ &\frac{\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K})[1 + \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})]}{1 + \partial(\eta_{\mathfrak{J}}, \mathfrak{K})}, \frac{\partial(\mathfrak{K}, \eta_{\mathfrak{J}+1})[1 + \partial(\eta_{\mathfrak{J}}, \mathfrak{T}\mathfrak{K})]}{1 + \partial(\eta_{\mathfrak{J}}, \mathfrak{K})} \end{aligned} \right\} = \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}),$$

for all $\mathfrak{J} \geq \mathfrak{J}_1$. Then from (2.8), it follows that

$$\tau(\partial(\eta_{\mathfrak{J}}, \mathfrak{K}) + \mathfrak{F}(\partial(\eta_{\mathfrak{J}+1}, \mathfrak{T}\mathfrak{K}))) \leq \mathfrak{F}(\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K})), \quad \forall \mathfrak{J} \geq \max\{\mathfrak{J}_0, \mathfrak{J}_1\}.$$

Since $\mathfrak{F}$ is continuous and $\liminf_{\eta \rightarrow t^+} \tau(\eta) > 0$ for all $\eta \geq 0$,

$$\tau(\partial(\mathfrak{K}, \mathfrak{K}) + \mathfrak{F}(\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}))) \leq \mathfrak{F}(\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K})),$$

This leads to a contradiction. Therefore, $\mathfrak{T}$ must possess a fixed point in X . $\square$

Definition 2.3. Consider a self-map $\mathfrak{T}$ defined on X , and a function $\mathfrak{a} : X \times X \rightarrow \mathbb{R}^+$ such that

- (i) $\mathfrak{a}(\eta, \mathfrak{T}\eta) > 1 \implies \mathfrak{a}(\mathfrak{T}\eta, \mathfrak{T}^2\eta) > 1$;
- (ii) $\mathfrak{a}(\eta, \vartheta) > 1$ and $\mathfrak{a}(\vartheta, \mathfrak{T}\vartheta) > 1 \implies \mathfrak{a}(\eta, \mathfrak{T}\vartheta) > 1$.

If $\mathfrak{T}$ satisfies conditions (i) and (ii) then we call it strictly triangular $\mathfrak{a}$ -orbital admissible.

Remark 2.4. The notions of strict triangular $\mathfrak{a}$ -orbital admissible maps and triangular $\mathfrak{a}$ -orbital admissible maps are mutually independent.

Now, consider the following conditions;

- ($\Gamma 3$) $\mathfrak{a}(\eta_1, \mathfrak{T}\eta_1) > 1$, for some $\eta_1 \in X$,
- ($\Gamma 4$) if $\{\eta_{\mathfrak{J}}\}$ is a sequence in X for which $\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) > 1$, $\forall \mathfrak{J} \in \mathbb{N}$, and $\lim_{\mathfrak{J} \rightarrow \infty} \eta_{\mathfrak{J}} = \eta$, then $\mathfrak{a}(\eta_{\mathfrak{J}}, \eta) > 1$ for all $\mathfrak{J} \in \mathbb{N}$.

In the next results for removing conditions $\mathfrak{F}_2$ and $\mathfrak{F}_3$, we consider strict triangular $\mathfrak{a}$ -orbital admissible mapping along with $(\tau, \mathfrak{a})$ -weak contraction of type II.

Theorem 2.5. *Let $\mathfrak{T}$ be self-map defined on a complete metric space (X, ∂) , such that $\mathfrak{T}$ is continuous strictly triangular $\mathfrak{a}$ -orbital admissible mapping satisfying the condition (1.5), and ($\Gamma 3$) holds. Then $\mathfrak{T}$ has a fixed point in X .*

Proof. For any arbitrary point $\eta_0 \in X$, we can define a sequence $\{\eta_{\mathfrak{J}}\} \in X$ as $\eta_{\mathfrak{J}+1} = \mathfrak{T}\eta_{\mathfrak{J}}$, $\forall \mathfrak{J} \in \mathbb{N}$. Suppose that $\eta_{\mathfrak{J}_0} = \eta_{\mathfrak{J}_0+1}$ for some $\mathfrak{J}_0 > 0$, then $\eta_{\mathfrak{J}_0}$ is a fixed point of $\mathfrak{T}$, and then the proof is completed. We presume that for any $\mathfrak{J} \in \mathbb{N}$, $\eta_{\mathfrak{J}} \neq \eta_{\mathfrak{J}+1}$, then we get $\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) = \partial(\mathfrak{T}\eta_{\mathfrak{J}-1}, \mathfrak{T}\eta_{\mathfrak{J}}) > 0$.

Now from ($\Gamma 3$), suppose $\eta_0 \in X$ for which $\mathfrak{a}(\eta_0, \mathfrak{T}\eta_0) > 1$, and since $\mathfrak{T}$ is strictly triangular $\mathfrak{a}$ -orbital admissible, therefore

$$\mathfrak{a}(\eta_0, \eta_1) = \mathfrak{a}(\eta_0, \mathfrak{T}\eta_0) > 1 \implies \mathfrak{a}(\mathfrak{T}\eta_0, \mathfrak{T}\eta_1) = \mathfrak{a}(\eta_1, \eta_2) > 1.$$

Inductively we can obtain that, $\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) > 1$. By the fact that $\mathfrak{T}$ is $(\tau, \mathfrak{a})$ -weak contraction of type II, for every $\mathfrak{J} \in \mathbb{N}$, we get

$$\tau(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) + \mathfrak{F}(\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})) \leq \mathfrak{F}(\mathfrak{M}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})), \quad (2.9)$$

where

$$\begin{aligned} \mathfrak{M}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) &= \\ &= \max \left\{ \begin{aligned} &\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2}), \frac{1}{2}[\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+1}) + \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+2})], \\ &\frac{\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})[1 + \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})]}{1 + \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})}, \frac{\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+1})[1 + \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+2})]}{1 + \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})} \end{aligned} \right\} \\ &= \max \{ \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2}) \}. \end{aligned} \quad (2.10)$$

If $\max \{ \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2}) \} = \partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})$. Then from (2.9), we get

$$\begin{aligned} \tau(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) + \mathfrak{F}(\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})) &\leq \mathfrak{F}(\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})) \\ \text{i.e. } \mathfrak{F}(\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})) &\leq \mathfrak{F}(\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})). \end{aligned}$$

Since $\mathfrak{F}$ is increasing, therefore $\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2}) \leq \partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})$, which implies that $\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) \leq 1$, a contradiction.

Consequently, $\max \{ \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2}) \} = \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})$. This leads to the sequence $\{\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\}_{\mathfrak{J}=0}^{\infty}$ is a non-increasing, and it is bounded below by 0. Then

from (2.10), we obtain that

$$\tau(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})) + \mathfrak{F}(\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})) \leq \mathfrak{F}(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})),$$

we can write the above term in the following way

$$\mathfrak{F}(\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2})) \leq \mathfrak{F}(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})) - \tau(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})) \leq \mathfrak{F}(\partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})).$$

Since $\mathfrak{F}$ is strictly increasing, therefore $\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})\partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2}) \leq \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})$,

$$\text{which implies, } \partial(\eta_{\mathfrak{J}+1}, \eta_{\mathfrak{J}+2}) \leq \lambda \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \quad (\lambda = 1/\mathfrak{a}(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}) < 1). \quad (2.11)$$

Lemma 1.11 implies that the sequence $\{\eta_{\mathfrak{J}}\}$ is a Cauchy sequence. As X is complete, there exists an element $\mathfrak{K} \in X$ such that $\lim_{\mathfrak{J} \rightarrow \infty} \eta_{\mathfrak{J}} = \mathfrak{K}$. Additionally, since $\mathfrak{T}$ is continuous,

$$\mathfrak{K} = \lim_{\mathfrak{J} \rightarrow \infty} \eta_{\mathfrak{J}+1} = \lim_{\mathfrak{J} \rightarrow \infty} \mathfrak{T}\eta_{\mathfrak{J}} = \mathfrak{T}\mathfrak{K}.$$

□

In our next result, by incorporating an additional condition (F4), we establish a fixed-point result, without the condition of continuity for both the mappings $\mathfrak{T}$ and $\mathfrak{F}$.

Theorem 2.6. *Let $\mathfrak{T}$ be a self-map defined on a complete metric space (X, ∂) , such that $\mathfrak{T}$ is strictly triangular $\mathfrak{a}$ -orbital admissible mapping satisfying the condition (1.5), and (F3), (F4) holds. If $\liminf_{\eta \rightarrow t^+} \tau(\eta) > 0$ for all $\eta \geq 0$, then $\mathfrak{T}$ has a fixed point in X .*

Proof. Let $\{\eta_{\mathfrak{J}}\} \in X$ be a sequence in X defined as $\eta_{\mathfrak{J}+1} = \mathfrak{T}\eta_{\mathfrak{J}}$ for any $\mathfrak{J} \in \mathbb{N}$. By following the steps of Theorem 2.5 we can show that $\{\eta_{\mathfrak{J}}\}$ is a Cauchy sequence in X . Consequently, there exists a point $\mathfrak{K} \in X$, such that $\liminf_{\mathfrak{J} \rightarrow \infty} \eta_{\mathfrak{J}} = \mathfrak{K}$.

We have to show that $\mathfrak{K}$ is a fixed point of $\mathfrak{T}$. On the contrary, suppose that $\mathfrak{K}$ is not a fixed point of $\mathfrak{T}$, i.e., $\mathfrak{K} \neq \mathfrak{T}\mathfrak{K}$. Then from (1.5), we get

$$\begin{aligned} \mathfrak{F}(\mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K})\partial(\eta_{\mathfrak{J}+1}, \mathfrak{T}\mathfrak{K})) &= \mathfrak{F}(\mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K})\partial(\mathfrak{T}\eta_{\mathfrak{J}}, \mathfrak{T}\mathfrak{K})) \leq \mathfrak{F}(\mathfrak{M}(\eta_{\mathfrak{J}}, \mathfrak{K})) - \tau(\eta_{\mathfrak{J}}, \mathfrak{K}) \\ &< \mathfrak{F}(\mathfrak{M}(\eta_{\mathfrak{J}}, \mathfrak{K})). \end{aligned}$$

Since $\mathfrak{F}$ satisfies $\mathfrak{F}_1$, therefore we get

$$\mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K})\partial(\eta_{\mathfrak{J}+1}, \mathfrak{T}\mathfrak{K}) < \mathfrak{M}(\eta_{\mathfrak{J}}, \mathfrak{K}). \quad (2.12)$$

Meanwhile, on the other hand,

$$\begin{aligned} \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}) &\leq \mathfrak{M}(\eta_{\mathfrak{J}}, \mathfrak{K}) \\ &= \max \left\{ \begin{aligned} &\partial(\eta_{\mathfrak{J}}, \mathfrak{K}), \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1}), \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}), \frac{\partial(\mathfrak{K}, \eta_{\mathfrak{J}+1}) + \partial(\eta_{\mathfrak{J}}, \mathfrak{T}\mathfrak{K})}{2}, \\ &\frac{\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K})[1 + \partial(\eta_{\mathfrak{J}}, \eta_{\mathfrak{J}+1})]}{1 + \partial(\eta_{\mathfrak{J}}, \mathfrak{K})}, \frac{\partial(\mathfrak{K}, \eta_{\mathfrak{J}+1})[1 + \partial(\eta_{\mathfrak{J}}, \mathfrak{T}\mathfrak{K})]}{1 + \partial(\eta_{\mathfrak{J}}, \mathfrak{K})} \end{aligned} \right\} \end{aligned}$$

Now, taking the limit $\mathfrak{J} \rightarrow \infty$, we get $\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}) \leq \lim_{\mathfrak{J} \rightarrow \infty} \mathfrak{M}(\eta_{\mathfrak{J}}, \mathfrak{K}) \leq \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K})$.

$$\text{i.e., } \lim_{\mathfrak{J} \rightarrow \infty} \mathfrak{M}(\eta_{\mathfrak{J}}, \mathfrak{K}) = \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}). \quad (2.13)$$

It follows from (2.12) and (2.13)

$$\lim_{\mathfrak{J} \rightarrow \infty} \mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K}) \partial(\eta_{\mathfrak{J}+1}, \mathfrak{T}\mathfrak{K}) < \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}), \quad (2.14)$$

by using the triangular inequality, we get

$$\mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K}) \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}) \leq \mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K}) [\partial(\mathfrak{K}, \eta_{\mathfrak{J}+1}) + \partial(\eta_{\mathfrak{J}+1}, \mathfrak{T}\mathfrak{K})],$$

taking the limits on both sides

$$\lim_{\mathfrak{J} \rightarrow \infty} \mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K}) \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}) \leq \lim_{\mathfrak{J} \rightarrow \infty} [\mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K}) \partial(\eta_{\mathfrak{J}+1}, \mathfrak{T}\mathfrak{K})] < \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K}).$$

Let $\lim_{\mathfrak{J} \rightarrow \infty} \mathfrak{a}(\eta_{\mathfrak{J}}, \mathfrak{K}) = \beta > 1$, then $\beta(\partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K})) < \partial(\mathfrak{K}, \mathfrak{T}\mathfrak{K})$, a contradiction, thus $\mathfrak{K}$ is a fixed point of $\mathfrak{T}$. $\square$

Theorem 2.7. Let $\mathfrak{T}$ be a self-mapping defined on a complete metric space (X, ∂) , such that for any $\eta, \vartheta \in X$, $\mathfrak{T}\eta \neq \mathfrak{T}\vartheta$.

$$\tau + \mathfrak{F}(\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)) \leq \mathfrak{F}(\mathfrak{M}(\eta, \vartheta)), \quad \mathfrak{F} \in \mathcal{F}, \tau > 0.$$

Then $\mathfrak{T}$ has a unique fixed point in X if either $\mathfrak{T}$ or $\mathfrak{F}$ is continuous.

Proof. By taking $\tau(\partial(\eta, \vartheta)) = \tau(\text{constant})$ and $\mathfrak{a}(\eta, \vartheta) = 1$, the proof of the theorem follows easily from Theorem 2.1 and Theorem 2.2. $\square$

Theorem 2.8. Let $\mathfrak{T}$ be a self-mapping defined on a complete metric space (X, ∂) , such that for any $\eta, \vartheta \in X$, $\mathfrak{T}\eta \neq \mathfrak{T}\vartheta$.

$$\tau + \mathfrak{F}(\gamma \partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)) \leq \mathfrak{F}(\mathfrak{M}(\eta, \vartheta)), \quad \mathfrak{F} \in \mathfrak{F}_1, \tau > 0 \text{ and } \gamma > 1.$$

Then $\mathfrak{T}$ has a unique fixed point in X .

Proof. By taking $\tau(\partial(\eta, \vartheta)) = \tau(\text{constant})$ and $\mathfrak{a}(\eta, \vartheta) = \gamma$, the proof of the theorem follows easily from Theorem 2.6. $\square$

Remark 2.9. Let's say the set of all the fixed points in $\mathfrak{T}$ is denoted by $Fix(\mathfrak{T})$. Next, we define the following condition for this set

$$(\mathcal{V}) \quad \mathfrak{a}(\eta, \vartheta) \geq 1, \quad \forall \eta, \vartheta \in Fix(\mathfrak{T}).$$

The uniqueness of fixed points can now be attained by applying the condition $(\mathcal{V})$ to our results that have been proven.

Example 2.10. Let X , ∂ , and $\mathfrak{T}$ be the same as in Example 1.8 and,

$$\mathfrak{a}(\eta, \vartheta) = \begin{cases} 1, & \text{if } (\eta, \vartheta) \in \{(0, 0), (0, 1), (0, 2), (1, 2)\}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

By choosing $\mathfrak{F} : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $\mathfrak{F}\eta = \ln \eta, \forall \eta > 0$ and $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $\tau(\eta) = \eta + \frac{1}{10}$ for $\eta \geq 0$. Then 0 is a fixed point of $\mathfrak{T}$ in X .

Verification. It is easy to observe that (X, ∂) is a complete metric space, and $\mathfrak{F}$ satisfies conditions $\mathfrak{F}_1$, $\mathfrak{F}_2$, and $\mathfrak{F}_3$. Now, we have to show that condition (1.4) holds. By the definition of $\mathfrak{a}(\eta, \vartheta)$, it is clear that

$$\tau(\partial(\eta, \vartheta)) + \mathfrak{a}(\eta, \vartheta) \mathfrak{F}(\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)) \leq \tau(\partial(\eta, \vartheta)) + \mathfrak{F}(\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)). \quad (2.15)$$

Therefore, there are two possible cases;

Case I: If $\eta = 0, \vartheta = 2$, then $\partial(\mathfrak{T}0, \mathfrak{T}2) = \partial(0, 1) = \frac{1}{5} > 0$,

$$\text{and, } \mathfrak{M}(0, 2) = \max \left\{ \frac{3}{10}, 0, \frac{1}{10}, \frac{1}{10}, \frac{1}{13}, \frac{18}{65} \right\} = \frac{3}{10}.$$

$$\text{Now, } \tau(\partial(0, 2)) + \mathfrak{F}(\partial(\mathfrak{T}0, \mathfrak{T}2)) = \frac{3}{10} + \frac{1}{10} + \ln \left(\frac{1}{5} \right) < \ln \left(\frac{3}{10} \right) \leq \mathfrak{F}(\mathfrak{M}(0, 2)). \quad (2.16)$$

Case II: If $\eta = 1, \vartheta = 2$, then $\partial(\mathfrak{T}1, \mathfrak{T}2) = \partial(0, 1) = \frac{1}{5}$,

$$\text{and, } \mathfrak{M}(1, 2) = \max \left\{ \frac{1}{10}, \frac{1}{5}, \frac{1}{10}, \frac{3}{20}, \frac{6}{55}, \frac{3}{11} \right\} = \frac{3}{11}.$$

$$\text{Now, } \tau(\partial(1, 2)) + \mathfrak{F}(\partial(\mathfrak{T}1, \mathfrak{T}2)) = \frac{1}{10} + \frac{1}{10} + \ln \left(\frac{1}{5} \right) < \ln \left(\frac{3}{11} \right) \leq \mathfrak{F}(\mathfrak{M}(1, 2)). \quad (2.17)$$

From (2.15), (2.16), and (2.17) it can be observed that condition (1.4) holds. Consequently, from Theorem 2.2 we can conclude that $\mathfrak{T}$ has a fixed point.

Remark 2.11. If for any $\tau \in [\frac{1}{5}, \frac{3}{10}]$, then (1.1) does not hold. In this regard, Theorem 3.8 and Theorem 3.9 of Gopal, Abbas, Patel, and Vetro [5], are not applicable.

Remark 2.12. Every $\mathfrak{a}$ -type $\mathfrak{F}$ -weak contraction [5] is $(\tau, \mathfrak{a})$ -weak contraction of type I, but the converse is not necessarily true.

Example 2.13. Let $X = [0, \frac{9}{2}]$ and $\partial(\eta, \vartheta) = |\eta - \vartheta|$ for all $\eta, \vartheta \in X$. Define $\mathfrak{T}$ by

$$\mathfrak{T}(\eta) = \begin{cases} 0, & \text{if } \eta \in [0, 2], \\ \frac{9}{2}, & \text{otherwise.} \end{cases}$$

Also consider $\mathfrak{F}(\eta) = \ln \eta, \forall \eta > 0$, and $\mathfrak{a} : X \times X \rightarrow (0, \infty)$ by

$$\mathfrak{a}(\eta, \vartheta) = \begin{cases} 2, & \text{if } \eta, \vartheta \in [0, 2], \eta, \vartheta \in (2, \frac{9}{2}], \\ \frac{2}{9}, & \text{otherwise.} \end{cases}$$

It can be easily observed that $\mathfrak{T}$ is strict triangular $\mathfrak{a}$ -orbital admissible and also (T3), (T4) holds.

Now, for any $\eta \in [0, 2]$ and $\vartheta \in (2, \frac{9}{2}]$, it is easy to see that

$$\inf_{\eta \in [0, 2], \vartheta \in (2, \frac{9}{2}]} \left\{ \max \left\{ \frac{\partial(\eta, \vartheta), \partial(\eta, \mathfrak{T}\eta), \partial(\vartheta, \mathfrak{T}\vartheta), \frac{\partial(\vartheta, \mathfrak{T}\eta) + \partial(\eta, \mathfrak{T}\vartheta)}{2}}{\frac{\partial(\vartheta, \mathfrak{T}\vartheta)[1 + \partial(\eta, \mathfrak{T}\eta)]}{1 + \partial(\eta, \vartheta)}, \frac{\partial(\vartheta, \mathfrak{T}\eta)[1 + \partial(\eta, \mathfrak{T}\vartheta)]}{1 + \partial(\eta, \vartheta)}} \right\} \right\} = \frac{9}{2}.$$

And, $\tau(\partial(\eta, \vartheta)) + \mathfrak{F}(\mathfrak{a}(\eta, \vartheta)\partial(\mathfrak{T}\eta, \mathfrak{T}\vartheta)) = \tau(\partial(\eta, \vartheta)) + \ln \left(\frac{2}{9} \times \frac{9}{2} \right).$

Therefore $\mathfrak{T}$ satisfies condition (1.5) for any $\tau(\eta) \in (0, \ln \frac{9}{2}]$. Clearly, $\mathfrak{T}$ satisfies

all the hypotheses of Theorem 2.6, and consequently, $\mathfrak{T}$ has at least one fixed point.

Remark 2.14. In above example for $\eta = 2$ and $\vartheta = \frac{11}{5}$, we get

$$\tau \left(d \left(2, \frac{11}{5} \right) \right) + \mathfrak{F} \left(d \left(\mathfrak{T}2, \mathfrak{T}\frac{11}{5} \right) \right) \not\leq \ln \left(\frac{1}{5} \right) = \mathfrak{F}d \left(2, \frac{11}{5} \right)$$

for every $\tau \left(d \left(2, \frac{11}{5} \right) \right) \in \mathbb{R}^+$. Hence, condition (H3) is not satisfied. consequently, Theorem 2.1, presented by Wardowski [13] cannot be applied in this context.

Remark 2.15. Every $(\varphi, \mathfrak{F})$ -contraction (or nonlinear $\mathfrak{F}$ -contraction) [13] is $(\tau, \mathfrak{a})$ -weak contraction of type II, but the converse is not necessarily true.

3. APPLICATION ON ATANGANA BALEANU FRACTIONAL DERIVATIVE

In the last few decades, remarkable development has been accomplished in the field of fractional calculus, with widespread applications across various scientific disciplines such as Mathematics, Physics, Biology, Medicine, Fluid dynamics, Finance, Signal processing, and more. These extensive applications led the researchers to formulate various definitions for fractional derivatives, with the most commonly used being the Caputo and Riemann-Liouville derivatives. Recently, Atangana and Baleanu introduced a new fractional derivative with the generalized Mittag-Leffler function as a non-local and non-singular kernel in [2], along with some useful properties.

Let $H^1(0, \kappa) = \{g \in L^2(0, \kappa) \text{ and } g' \in L^2(0, \kappa)\}$, where $L^2(0, \kappa)$ is the space of square integrable functions on the interval $(0, \kappa)$, then the Atangana Baleanu fractional derivative is defined as

Definition 3.1. [2] Let $g \in H^1(a, b)$, $a < b$, and $0 \leq k \leq 1$. The Atangana-Baleanu fraction derivative in the Caputo sense of g of order k is defined by

$${}_a^{ABC}D^k g(\varsigma) = \frac{\mathfrak{R}(k)}{1-k} \int_a^\varsigma g'(v) E_k \left[-k \frac{(\varsigma - v)^k}{1-k} \right] dv,$$

where E_k is the Mittag-Leffler function defined by $E_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(nk+1)}$ and $\mathfrak{R}(k)$ is a normalizing positive function satisfying $\mathfrak{R}(0) = \mathfrak{R}(1) = 1$. The associated fractional integral is defined by

$${}_a^{AB}I^k g(\varsigma) = \frac{1-k}{\mathfrak{R}(k)} g(\varsigma) + \frac{k}{\mathfrak{R}(k)} {}_a I^k g(\varsigma),$$

where ${}_a I^k$ is the left Riemann-Liouville fraction integral given as

$${}_a I^k g(\varsigma) = \frac{1}{\Gamma(k)} \int_a^\varsigma (\varsigma - v)^{k-1} g(v) dv.$$

We discuss the following problem

$$({}_0^{ABC}D^k \delta)(\varsigma) = \Omega(\varsigma, \delta(\varsigma)) \quad \varsigma \in I, 0 \leq k \leq 1, \quad (3.1)$$

with the initial condition $\delta(0) = \delta_0$, where D^k denotes the Atangana - Baleanu derivative in the Caputo sense of order k and $\Omega : I \times X \rightarrow \mathbb{R}$ is a continuous function with $\Omega(0, \delta(0)) = 0$.

Proposition 3.2. [1] For $0 < k < 1$, we have $({}^{AB}I_b^{ABC} D^k \delta)(\varsigma) = \delta(\varsigma) - \delta(b)$.

Theorem 3.3. Suppose $X = \mathcal{C}(I, \mathbb{R})$, assume that there exists a function $\mathcal{J} : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\delta_1 \in X$ such that for all $\delta, \gamma, \zeta \in X$ and $\varsigma \in I$, $\mathcal{J}(\delta_1(\varsigma), \mathfrak{T}\delta_1(\varsigma)) \geq 0$, where $\mathfrak{T} : X \rightarrow X$ is defined by $\mathfrak{T}\delta(\varsigma) = \delta_0 + {}_0^{CF} I^k \Omega(\varsigma, \delta(\varsigma))$.

- (m1) $\mathcal{J}(\delta(\varsigma), \gamma(\varsigma)) \geq 0, \mathcal{J}(\gamma(\varsigma), \zeta(\varsigma)) \geq 0 \implies \mathcal{J}(\delta(\varsigma), \zeta(\varsigma)) \geq 0$;
- (m2) $\mathcal{J}(\delta(\varsigma), \gamma(\varsigma)) \geq 0 \implies \mathcal{J}(\mathfrak{T}\delta(\varsigma), \mathfrak{T}\gamma(\varsigma)) \geq 0$;
- (m3) if $\{\delta_n\} \subseteq X$ and $\lim_{n \rightarrow \infty} \delta_n = \delta$, $\implies \mathcal{J}(\delta_n, \delta) \geq 0, \forall n \in \mathbb{N}, \delta \in X$;

$$(m4) |\Omega(\varsigma, \delta(\varsigma)) - \Omega(\varsigma, \gamma(\varsigma))| \leq \frac{\Re(k)\Gamma(k) \left[|\delta(\varsigma) - \gamma(\varsigma)| + \sqrt{|\delta(\varsigma) - \gamma(\varsigma)|} \right]}{(1-k)4\Gamma(k)+4};$$

holds, then $\mathfrak{T}$ possesses at least one fixed point, which serves as the solution to the problem (3.1).

Proof. Applying the Atangana-Baleanu integral to both sides of (3.1) and using Proposition 3.2, we get, $\delta(\varsigma) = \delta_0 + {}_0^{AB} I^k \Omega(\varsigma, \delta(\varsigma)) = \mathfrak{T}\delta(\varsigma)$. In the next step, we will show that $\mathfrak{T}$ has a fixed point, let $\alpha : X \times X \rightarrow [0, \infty)$ be given by,

$$\alpha(\delta, \gamma) = \begin{cases} \lambda, & \mathcal{J}(\delta(\varsigma), \gamma(\varsigma)) \geq 0, \forall \varsigma \in I \text{ where } \lambda > 1, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \quad (3.2)$$

Then for each $\delta, \gamma \in X$,

$\alpha(\delta, \mathfrak{T}\delta) > 1 \implies \mathcal{J}(\mathfrak{T}\delta(\varsigma), \mathfrak{T}^2\delta(\varsigma)) \geq 0$, i.e. $\alpha(\mathfrak{T}\delta, \mathfrak{T}^2\delta) > 1$, and

$\alpha(\delta, \gamma) > 1, \alpha(\gamma, \mathfrak{T}\gamma(\varsigma)) > 1 \implies \mathcal{J}(\delta(\varsigma), \gamma(\varsigma)) \geq 0$, and $\mathcal{J}(\gamma(\varsigma), \mathfrak{T}\gamma(\varsigma)) \geq 0$,

which gives $\mathcal{J}(\delta(\varsigma), \mathfrak{T}\gamma(\varsigma)) \geq 0$, (using (m1)) and therefore $\alpha(\delta, \mathfrak{T}\gamma) > 1$.

which shows that $\mathfrak{T}$ is a triangular α orbital admissible map, now we will show that $\mathfrak{T}$ satisfies condition (1.5),

$$\begin{aligned} |\mathfrak{T}\delta(\varsigma) - \mathfrak{T}\gamma(\varsigma)| &= \left| {}_0^{AB} I^k (\Omega(\varsigma, \delta(\varsigma)) - \Omega(\varsigma, \gamma(\varsigma))) \right| \\ &= \left| \frac{1-k}{\Re(k)} (\Omega(\varsigma, \delta(\varsigma)) - \Omega(\varsigma, \gamma(\varsigma))) + \right. \\ &\quad \left. \frac{k}{\Re(k)} {}_0 I^k (\Omega(\varsigma, \delta(\varsigma)) - \Omega(\varsigma, \gamma(\varsigma))) \right| \\ &\leq \frac{1-k}{\Re(k)} \left| (\Omega(\varsigma, \delta(\varsigma)) - \Omega(\varsigma, \gamma(\varsigma))) \right| + \\ &\quad \frac{k}{\Re(k)} \left| {}_0 I^k (\Omega(\varsigma, \delta(\varsigma)) - \Omega(\varsigma, \gamma(\varsigma))) \right| \\ &\leq \frac{\Re(k)\Gamma(k)}{(1-k)4\Gamma(k)+4} \left(\frac{1-k}{\Re(k)} + \frac{k}{\Re(k)} {}_0 I^k(1) \right) \times \\ &\quad \left(\sup_{\varsigma \in I} |\delta(\varsigma) - \gamma(\varsigma)| + \sqrt{\sup_{\varsigma \in I} |\delta(\varsigma) - \gamma(\varsigma)|} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\Re(k)\Gamma(k)}{(1-k)4\Gamma(k)+4} \left(\frac{1-k}{\Re(k)} + \frac{k}{\Re(k)} \frac{1}{k\Gamma(k)} \right) \times \\
&\quad \left(\sup_{\varsigma \in I} |\delta(\varsigma) - \gamma(\varsigma)| + \sqrt{\sup_{\varsigma \in I} |\delta(\varsigma) - \gamma(\varsigma)|} \right) \\
&\leq \frac{1}{4} \left(\sup_{\varsigma \in I} |\delta(\varsigma) - \gamma(\varsigma)| + \sqrt{\sup_{\varsigma \in I} |\delta(\varsigma) - \gamma(\varsigma)|} \right) + \frac{1}{\lambda} \\
&= \frac{\sup_{\varsigma \in I} |\delta(\varsigma) - \gamma(\varsigma)|}{\lambda(\sqrt{\sup_{\varsigma \in I} |\delta(\varsigma) - \gamma(\varsigma)|} + 2)},
\end{aligned}$$

$$\text{this yields that, } \frac{1}{\sqrt{\lambda\partial(\mathfrak{T}\delta, \mathfrak{T}\gamma)}} \geq \frac{\sqrt{\partial(\delta, \gamma)} + 2}{\sqrt{\partial(\delta, \gamma)}},$$

$$\text{which implies, } 1 + \frac{1}{\sqrt{\partial(\delta, \gamma)}} - \frac{1}{\sqrt{\lambda\partial(\mathfrak{T}\delta, \mathfrak{T}\gamma)}} \leq -\frac{1}{\sqrt{\partial(\delta, \gamma)}}.$$

By taking $\tau(x) = 1 + \frac{1}{\sqrt{x}}$, and $F(x) = -\frac{1}{\sqrt{x}}$ we get

$$\tau(\partial(\delta, \gamma)) + F(\lambda\partial(\mathfrak{T}\delta, \mathfrak{T}\gamma)) \leq F(\partial(\delta, \gamma)).$$

Now using F_1 and from equation (3.2), it can be clearly observed that,
 $\tau(\partial(\delta, \gamma)) + F(\alpha(x, y)\partial(\mathfrak{T}\delta, \mathfrak{T}\gamma)) \leq \tau(\partial(\delta, \gamma)) + F(\lambda\partial(\mathfrak{T}\delta, \mathfrak{T}\gamma)) \leq F(\partial(\delta, \gamma)) \leq F(\mathfrak{B}(\delta, \gamma))$

also there exists $\delta_1 \in X$ such that $\mathcal{J}(\delta_1(\varsigma), \mathfrak{T}\delta_1(\varsigma)) \geq 0$, which implies $\alpha(\delta_1, \mathfrak{T}\delta_1) > 1$, and using (m3), (Γ4) can be obtained. Thus $\mathfrak{T}$ satisfies all the hypotheses of Theorem 2.6, consequently $\mathfrak{T}$ possesses a fixed point and which will be a solution of the problem (3.1). $\square$

REFERENCES

1. Abdeljawad, T., & Baleanu, D. (2018). On fractional derivatives with generalized Mittag-Leffler kernels. *Advances in Difference Equations*, 2018(468), 1-15. <https://doi.org/10.1186/s13662-018-1914-2>
2. Atangana, A., & Baleanu, D. (2016). New fractional derivatives with non-local and non-singular kernel. *Thermal Science*, 20(2), 763-769. <https://doi.org/10.2298/TSCI160111018A>
3. Chandok, S., & Radenović, S. (2022). Existence of solution for orthogonal F -contraction mappings via Picard–Jungck sequences. *The Journal of Analysis*, 30(2), 677-690. <https://doi.org/10.1007/s41478-021-00362-1>
4. Cvetkovic, M. (2023). Comparison of F -contraction and ϕ -contractions. *Filomat*, 37(12), 3951-3961. <https://doi.org/10.2298/FIL2312951C>
5. Gopal, D., Abbas, M., Patel, D.K., & Vetro, C. (2016). Fixed points of α -type F -contractive mappings with an application to nonlinear fractional differential equation. *Acta Mathematica Scientia*, 36(3), 957-970. [https://doi.org/10.1016/S0252-9602\(16\)30052-2](https://doi.org/10.1016/S0252-9602(16)30052-2)
6. Miculescu, R., & Mihail, A. (2017). New fixed point theorems for set-valued contractions in b -metric spaces. *Journal of Fixed Point Theory and Applications*, 19(3), 2153-2163. <https://doi.org/10.1007/s11784-016-0400-2>

7. Minak, G., Helvacı, A., & Altun, T. (2014). Ćirić type generalized F -contractions on complete metric spaces and fixed point results. *Filomat*, 28(6), 1143-1151. [DOI10.2298/FIL1406143M](https://doi.org/10.2298/FIL1406143M)
8. Nazam, M., Arshad, M., & Postolache, M. (2018). Coincidence and common fixed point theorems for four mappings satisfying (α_s, F) -contraction. *Nonlinear Analysis: Modelling and Control*, 23(5), 664-690. <https://doi.org/10.15388/NA.2018.5.3>
9. Popescu, O. (2014). Some new fixed point theorems for α -Geraghty contraction type maps in metric spaces. *Fixed Point Theory and Applications*, 2014(190), 1-12. <https://doi.org/10.1186/1687-1812-2014-190>
10. Samet, B., & Vetro, C., & Vetro, P. (2012). Fixed point theorems for α - ψ -contractive type mappings. *Nonlinear Analysis: Theory, Methods and Applications*, 75(4), 2154-2165. <https://doi.org/10.1016/j.na.2011.10.014>
11. Sawangsup, K., Sintunavarat, W., & Cho, Y. J. (2020). Fixed point theorems for orthogonal F -contraction mappings on O-complete metric spaces. *Journal of Fixed Point Theory and Applications*, 22, 1-14. <https://doi.org/10.1007/s11784-019-0737-4>
12. Wardowski, D. (2012). Fixed point theory of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory and Applications*, 2012(94), 1-6. <https://doi.org/10.1186/1687-1812-2012-94>
13. Wardowski, D. (2018). Solving existence problems via F -contractions. *Proceedings of the American Mathematical Society*, 146(4), 1585-1598. <http://dx.doi.org/10.1090/proc/13808>
14. Wardowski, D., & Van Dung, N. (2014). Fixed points of F -weak contractions on complete metric spaces. *Demonstratio Mathematica*, 47(1), 146-155. <https://doi.org/10.2478/dema-2014-0012>

¹ DEPARTMENT OF MATHEMATICS, GURU GHASIDAS VISHWAVIDYALAYA (A CENTRAL UNIVERSITY), KONI, BILASPUR, 495009, CHHATTISGARH, INDIA.

Email address: kvaraprasad71@gmail.com

² DEPARTMENT OF MATHEMATICS, GURU GHASIDAS VISHWAVIDYALAYA (A CENTRAL UNIVERSITY), KONI, BILASPUR, 495009, CHHATTISGARH, INDIA.

Email address: vmishra9w@gmail.com

³ DEPARTMENT OF MATHEMATICS, GURU GHASIDAS VISHWAVIDYALAYA (A CENTRAL UNIVERSITY), KONI, BILASPUR, 495009, CHHATTISGARH, INDIA.

Email address: hapkasurendra@gmail.com