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# SELF-SWITCHING OF UNION OF TWO COMPLETE GRAPHS

C. JAYASEKARAN[1] AND S. S. ATHITHIYA[2*]

Abstract. By a graph $H = (V, E)$, we mean a finite undirected graph without loops and multiple edges. Let $H$ be a graph and $\sigma \subseteq V$ be a non–empty subset of $V$. $H^\sigma$ is the graph obtained from $H$ by removing all edges between $\sigma$ and its complement $V - \sigma$ and adding as edges all non-edges between $\sigma$ and $V - \sigma$. Then $\sigma$ is said to be a self- switching of $H$ if $H \cong H^\sigma$. It can also be referred to as $k$-vertex self-switching where $k = |\sigma|$. The set of all self-switchings of the graph $H$ with cardinality $k$ is represented by $SS_k(H)$ and its cardinality by $ss_k(H)$. A graph on $m$ vertices in which each pair of distinct vertices are neighbors is called a complete graph and is denoted by $K_m$. $K_m \cup K_n$ is the union of two complete graphs and is disconnected. In this paper, we give necessary and sufficient conditions for $\sigma$ to be a self-switching for the graph $H = K_m \cup K_n$ and using this, we find the cardinality $ss_k(H)$.

## 1. Introduction and preliminaries

For a finite undirected graph $H(V, E)$ and a non-empty subset $\sigma \subseteq V$, the switching of $H$ by $\sigma$ is defined as the graph $H^\sigma(V, E')$ [2, 9] which is obtained from $H$ by removing all edges between $\sigma$ and its complement $V - \sigma$ and adding as edges all non-edges between $\sigma$ and $V - \sigma$. For $\sigma = \{v\}$, we write $H^v$ and the corresponding switching is called as vertex switching. We also call it as $|\sigma|$-vertex switching. When $|\sigma| = 2$, it is termed as 2-vertex switching. If $H$ is isomorphic to $H^\sigma$, then it is called self vertex switching [5]. In [3], switching classes are discussed. An undirected graph that has a distinct edge connecting each pair of distinct vertices is said to be complete [1, 4]. Here, we will discover how the union of two complete graphs can self-switch.

**Definition 1.1.** [8] Let $H(V, E)$ be a finite undirected graph and $\sigma \subseteq V$. The switching of $H$ by $\sigma$ is defined as the graph $H^\sigma(V, E')$, which is obtained from $H$ by removing all edges between $\sigma$ and its complement $V - \sigma$ and by adding all edges as non edges between $\sigma$ and $V - \sigma$. When $\sigma = \{v\} \subset V$, the corresponding switching $H^{\{v\}}$ is called as vertex switching and is denoted by $H^v$.

**Definition 1.2.** [6]A subset $\sigma$ of $V(H)$ is said to be a self- switching of $H$ if $H \cong H^\sigma$.

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* Corresponding author.
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The set of all self-switchings of $H$ with cardinality k is denoted by $SS_k(H)$ and its cardinality by $ss_k(H)$.

If $k = 1$, then the corresponding self-switching is called self-vertex-switching.

If $k = 2$, then the corresponding self-switching is called as 2-vertex self-switching.

**Definition 1.3.** [4] The Complement $\bar{H}$ of a graph $H$ also has $V(H)$ as its vertex set, but two vertices are adjacent in $\bar{H}$ if and only if they are not adjacent in $H$.

**Lemma 1.4.** [10] *For a graph $H(V, E)$ and $\sigma \subseteq V$, it holds that $\overline{H}^{\sigma} = \overline{H^{\sigma}}$.*

**Theorem 1.5.** [7] *If $\sigma = \{u, v\} \subseteq V$ is a 2-vertex self switching of a graph $H$, then $d_H(u) + d_H(v) = \begin{cases} p & if\ uv \in E(H) \\ p-2 & if\ uv \notin E(H) \end{cases}$*

## 2. Main results

**Theorem 2.1.** *Any self-switching of a graph $H$ is also a self-switching of the graph $\overline{H}$ and $SS_k(H) = SS_k(\overline{H})$.*

*Proof.* Let $\sigma \subseteq V$ be a self-switching of graph $H$. Let $g$ be an isomorphism between $H$ and $H^{\sigma}$. Using lemma 1.4, we get $(\overline{H^{\sigma}}) = (\overline{H})^{\sigma}$. Then $g$ is also an isomorphism between $\overline{H}$ and $(\overline{H})^{\sigma}$. This implies that $\sigma$ is a self- switching of $\overline{H}$. Hence, $SS_k(H) = SS_k(\overline{H})$. □

**Theorem 2.2.** *Let $H = K_r \cup K_n$ and $\sigma = \{d^1, d^2\} \subseteq V(H)$ be such that $d^1d^2 \notin E(H)$. Then $\sigma$ is a 2-vertex self-switching of $H$.*

*Proof.* Let $H = K_r \cup K_n$ with $V(H) = \{d_b^1, d_c^2 : 1 \leq b \leq r,\ 1 \leq c \leq n\}$. Clearly, $H$ has $p = r + n$ vertices. Let $d^1d^2 \notin E(H)$. Then $d^1$ and $d^2$ belong to different components of $H$. Without loss of generality, let $d^1 \in V(K_r)$ and $d^2 \in V(K_n)$. Clearly, $d_H(d^1) = r - 1$, $d_H(d^2) = n - 1$ and $d_H(d^1) + d_H(d^2) = r + n - 2$. Hence, by Theorem 1.5, $\sigma = \{d^1, d^2\}$ may be a 2-vertex self-switching of graph $H$. In $H^{\sigma}$, the vertex $d^1$ in $K_r$ is a neighbor of the vertices of $K_n$ other than $d^2$ and the vertex $d^2$ in $K_n$ is a neighbor of the vertices of $K_r$ other than $d^1$. Now define, $g : V(H) \to V(H^{\sigma})$ by $g(d^1) = d^2, g(d^2) = d^1$ and $g(w) = w\ \forall\ w \neq d^1, d^2$. Clearly, $g$ creates an isomorphism between $H$ and $H^{\sigma}$. Hence, $\sigma = \{d^1, d^2\}$ is a 2-vertex self-switching of the graph $H$. □

*Remark* 2.3. The converse of the Theorem 2.2 is not necessarily true. For example, let the graph $H = K_6 \cup K_4$ be the union of two complete graphs $K_6$ and $K_4$ as shown in the Figure 2.1. Let $\sigma =\{d^1, d^2\}$ be the switching of $H$. The switching graph of $H$ by $\sigma$, $H^{\sigma}$ is given in the Figure 2.2. Clearly, $H \cong H^{\sigma}$ and thereby $\sigma = \{d^1, d^2\}$ is a 2-vertex self-switching of $H$ but $d^1d^2 \in E(H)$.

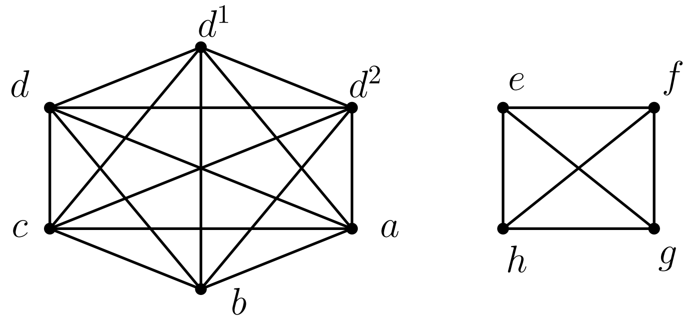


Figure 2.1. $H = K_6 \cup K_4$

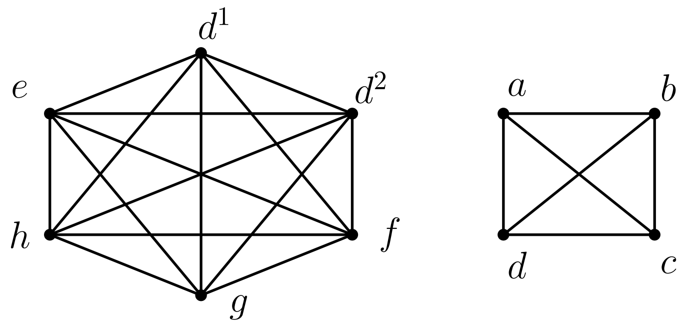


Figure 2.2. $H^\sigma$

**Theorem 2.4.** *For $r \neq s$, let $H = K_r \cup K_s$ and let $\sigma = \{d^1, d^2\} \subseteq V(H)$ be such that $d^1d^2$ is an edge in the largest order component of $H$. Then $\sigma$ is a 2-vertex self-switching of $H$ if and only if $|r - s| = 2$.*

*Proof.* Let $H = K_r \cup K_s$ be a graph with $V(H) = V_1 \cup V_2$ where $V_1 = V(K_r)$ and $V_2 = V(K_s)$. Clearly, $H$ has $r + s$ number of vertices and let $V(H) = \{d_b^1, d_c^2 : 1 \leq b \leq r; 1 \leq c \leq s\}$ where $d_b^1 \in V_1$ and $d_c^2 \in V_2$. Let $\sigma = \{d^1, d^2\} \subseteq V(H)$ be such that $d^1d^2$ lies in the largest order component of $H$.

Let $|r - s| = 2$. Then either $r - s = 2$ or $s - r = 2$. Without loss of generality, let $r > s$ and so $r - s = 2$. Since $r > s$, $d^1d^2 \in E(K_r)$. Then $d_H(d^1) + d_H(d^2) = 2r - 2 = r + r - 2 = p$. Therefore by Theorem 1.5, $\sigma$ may be a 2-vertex self-switching of $H$. In $H^\sigma$, the vertices in $\sigma$ are neighbors of the vertices of $V_2$ and the vertices in $V_1 - \sigma$ are neighbors of each other. Since $H[V_1] = K_r$, $H[V_1 - \sigma] = H^\sigma[V_1 - \sigma] = K_{r-2}$. Furthermore, $H^\sigma[V_2 \cup \sigma] = K_r$. Thus, $H^\sigma = K_r \cup K_{r-2}$ and so $H \cong H^\sigma$. Hence, $\sigma$ is a $k$-vertex self-switching of $H$.

Conversely, let $\sigma$ be a 2-vertex self-switching of $H$. If $|r - s| \neq 2$, then $d_H(d^1) + d_H(d^2)$ is either $2r - 2$ or $2s - 2$ which is not equal to $p$ or $p - 2$ and so $\sigma$ cannot be a 2-vertex self-switching of $H$ which goes against our assumption. Therefore, $|r - s| = 2$.

Hence, we obtain the desired result. $\square$

**Theorem 2.5.** *Let $H = K_r \cup K_s$ and $\sigma = \{d^1, d^2\} \subseteq V(H)$ be a 2-vertex self-switching of $H$. Then*

$$ss_2(H)= \begin{cases} \binom{r}{2}+ rs & if\ r\text{-}s=2 \\ rs & otherwise \end{cases}$$

*Proof.* Let $H = K_r \cup K_s$ be a graph with $r + s$ number of vertices and $V(H) = \{d_b^1, d_c^2 : 1 \leq b \leq r; 1 \leq c \leq s\}$ where $d_b^1 \in V(K_r)$ and $d_c^2 \in V(K_s)$. Without loss of generality, we consider that $r \geq s$. Let $\sigma = \{d^1, d^2\} \subseteq V(H)$. Then $d^1d^2 \in E(H)$ or $d^1d^2 \notin E(H)$.

**Case 1.** $d^1d^2 \in E(H)$

If $r = s$, then $d_H(d^1) + d_H(d^2) = 2s - 2$, which is not the same as $p$. Therefore, by Theorem 1.5, $\sigma$ cannot be a 2-vertex self-switching of $H$. So, let $r > s$. We have either $d^1d^2 \in E(K_r)$ or $d^1d^2 \in E(K_s)$. However, if $d^1d^2 \in E(K_s)$, then $d_H(d^1) + d_H(d^2) = 2s - 2 \neq p$ or $p - 2$. Hence by Theorem 1.5, $\sigma$ cannot be a 2-vertex self-switching of $H$. If $d^1d^2 \in E(K_r)$, then by Theorem 2.4, $\sigma$ is a 2-vertex self-switching of $H$ if and only if $r - s = 2$. As there are $\binom{r}{2}$ possible pairs of vertices in $K_r$, $ss_2(H) \geq \binom{r}{2}$.

**Case 2.** $d^1d^2 \notin E(H)$

By Theorem 2.2, $\sigma$ is a 2-vertex self-switching of $H$. Thus for $r \geq s$ in $H$, any non-adjacent pair of vertices in $H$ is a 2-vertex self-switching of $H$. Since there are $rs$ such pairs in $H$, the number of 2-vertex self-switchings in $H$ is $rs$.

From the above two cases, we get, $ss_2(H)= \begin{cases} \binom{r}{2}+rs & if\ r - s = 2 \\ rs & otherwise \end{cases}$ □

**Theorem 2.6.** *Let $H = K_r \cup K_{r-k}$ and $\sigma = \{v_1, v_2, ..., v_k\} \subseteq V(H)$. Then*

$$ss_k(H) = \begin{cases} \binom{r}{k} + \binom{r}{\frac{k}{2}}\binom{r-k}{\frac{k}{2}} & if\ k\ is\ even \\ \binom{r}{k} & if\ k\ is\ odd \end{cases}$$

*Proof.* Let $H = K_r \cup K_{r-k}$ be a graph with $V(H) = V_1 \cup V_2$ where $V_1 = V(K_r)$ and $V_2 = V(K_{r-k})$. Clearly, $H$ has $2r - k$ vertices. Let $\sigma = \{v_1, v_2, ..., v_k\} \subseteq V(H)$. Then we have, $\sigma \subseteq V_1$ or $\sigma \subseteq V_2$ or $\sigma \subseteq V_1 \cup V_2$.

**Case 1.** $\sigma \subseteq V_1$

Let $V_1 = \{d_b^1 : 1 \leq b \leq r\}$ and $V_2 = \{d_c^2 : 1 \leq c \leq r - k\}$. Then in $H^\sigma$, the vertices in $\sigma$ are neighbors of the vertices of $V_2$ and not neighbors of the vertices of $V_1 - \sigma$. Since $H[V_1] = K_r$, $H[V_1 - \sigma] = H^\sigma[V_1 - \sigma] = K_{r-k}$. Moreover, $H^\sigma[V_2 \cup \sigma] = K_r$. Thus, $H^\sigma = K_r \cup K_{r-k}$ and so $H \cong H^\sigma$. Hence, $\sigma$ is a $k$-vertex self-switching of $H$.

Since there are $\binom{r}{k}$ possible combinations of selecting $k$ vertices from $r$ vertices, $ss_k(H) \geq \binom{r}{k}$.

**Case 2.** $\sigma \subseteq V_2$

Then in $H^\sigma$, the vertices in $\sigma$ are neighbors of the vertices of $V_1$ and not neighbors of the vertices of $V_2 - \sigma$. Since $H[V_2] = K_{r-k}$, $H[V_2 - \sigma] = H^\sigma[V_2 - \sigma] = K_{r-2k}$. Moreover, $H^\sigma[V_1 \cup \sigma] = K_{r+k}$. Thus, $H^\sigma = K_{r+k} \cup K_{r-2k} \ncong H$. Hence, $\sigma$ cannot be a $k$-vertex self-switching of $H$.

**Case 3.** $\sigma \subseteq V_1 \cup V_2$

Let $r_1$ vertices of $\sigma$ be in $V_1$ and the remaining $k - r_1$ vertices of $\sigma$ be in $V_2$. Then we have either $r_1 = k - r_1$ or $r_1 \neq k - r_1$.

**Subcase 3.1.** $r_1 = k - r_1$

Then $2r_1 = k$ and so $k$ is even. Without loss of generality, let $\sigma = \{d_b^1, d_b^2 : 1 \leq b \leq \frac{k}{2}\}$ where $d_b^1 \in V_1$ and $d_b^2 \in V_2$. Then in $H^\sigma$, the $\frac{k}{2}$ vertices $d_b^1$'s are neighbors of the vertices of $V_2 - \sigma$ and the $\frac{k}{2}$ vertices $d_b^{2\prime}s$ are neighbors of the vertices of $V_1 - \sigma$. Define $g : V(H) \to V(H^\sigma)$ by $g(d_b^1) = d_b^2; g(d_b^2) = d_b^1; g(w) = w \ \forall \ w \neq d_b^1, d_b^2$, $1 \leq b \leq \frac{k}{2}$. Clearly, $g$ establishes an isomorphism between $H$ and $H^\sigma$ and so $\sigma$ is a $k$-vertex self-switching of $H$. As there are $\binom{r}{\frac{k}{2}}$ possibilities of selecting $\frac{k}{2}$ vertices from $r$ vertices of $K_r$ and $\binom{r-k}{\frac{k}{2}}$ possib0lities of selecting $\frac{k}{2}$ vertices from $r - k$ vertices of $K_{r-k}$, there are $\binom{r}{\frac{k}{2}}\binom{r-k}{\frac{k}{2}}$ $k$-vertex self-switchings in $H$.

**Subcase 3.2.** $r_1 \neq k - r_1$

Let a vertex of $\sigma$ be in $V_1$ and the remaining $k - 1$ vertices of $\sigma$ be in $V_2$. Then in $H^\sigma$, the vertex of $\sigma$ in $V_1$ is a neighbor of the vertices of $V_2 - \sigma$ and the $k - 1$ vertices of $\sigma$ in $V_2$ are neighbors of the vertices of $V_1 - \sigma$. Thus we have, $H^\sigma = K_{r+k-2} \cup K_{r-2k+2}$. Also, if 2 vertices of $\sigma$ are in $V_1$ and $k - 2$ vertices of $\sigma$ are in $V_2$, then in $H^\sigma$, the 2 vertices of $\sigma$ in $V_1$ are neighbors of the vertices of $V_2 - \sigma$ and the $k - 2$ vertices of $\sigma$ in $V_2$ are neighbors of the vertices of $V_1 - \sigma$. Thus we have, $H^\sigma = K_{r+k-4} \cup K_{r-2k+4}$. Similarly, if 3 vertices of $\sigma$ are in $V_1$ and $k - 3$ vertices of $\sigma$ are in $V_2$. Then $H^\sigma = K_{r+k-6} \cup K_{r-2k+6}$ and so on.

Therefore in general, if $r_1$ vertices of $\sigma$ are in $V_1$ and $k - r_1$ vertices of $\sigma$ are in $V_2$, then $H^\sigma = K_{r+k-2r_1} \cup K_{r-2k+2r_1}$. This implies that $H^\sigma \ncong H$ and hence $\sigma$ cannot be a self-switching of $H$ .

Thus from the above cases, we get $ss_k(H) = \binom{r}{k} + \binom{r}{\frac{k}{2}}\binom{r-k}{\frac{k}{2}}$, if $k$ is even and $ss_k(H) = \binom{r}{k}$, if $k$ is odd. □

**Theorem 2.7.** *Let $H = K_r \cup K_s$ and the vertices of $\sigma = \{u_1, u_2, u_3, ..., u_k\} \subseteq V(H)$ be contained only in $K_r$. Then $\sigma$ is a $k$-vertex self-switching of $H$ if and only if $s = r - k$.*

*Proof.* Let $H = K_r \cup K_s$ with $V(H) = V_1 \cup V_2$ where $V_1 = V(K_r)$ and $V_2 = V(K_s)$ and $\sigma = \{u_1, u_2, u_3, ..., u_k\} \subseteq V(K_r)$.

Let $s = r - k$. Then $H = K_r \cup K_{r-k}$. Since $\sigma \subseteq V(K_r)$, we have in $H^\sigma$, the vertices of $\sigma$ are neighbors of $V_2$ and the vertices of $V_1 - \sigma$ are neighbors of each other. Thus we have, $H^\sigma = K_{r-k} \cup K_r \cong H$. Hence, $\sigma$ is a $k$-vertex self-switching of $H$.

Conversely, let $\sigma = \{u_1, u_2, u_3, ..., u_k\} \subseteq V(K_r)$ be a $k$-vertex self-switching of $H$. Then, $H \cong H^\sigma$. Since in $H^\sigma$, the vertices of $\sigma$ are neighbors of the vertices of $V_2$ and the vertices of $V_1 - \sigma$ are neighbors of each other, $H^\sigma = K_{r-k} \cup K_{s+k}$. Hence $K_r \cup K_s \cong K_{r-k} \cup K_{s+k}$. Thus either $r - k = r$ and $s + k = s$ or $r - k = s$ and $s + k = r$. Since $k = 0$ is not possible, we have $s = r - k$. Hence, we obtain the desired result. □

**Result 2.8.** Let $H = K_r \cup K_s$ and $\sigma = \{u_1, u_2, u_3, ..., u_k\} \subseteq V(H)$. If $|V_1 \cap \sigma| = r_1$ and $|V_2 \cap \sigma| = k - r_1$, then $H^\sigma = K_{r+k-2r_1} \cup K_{s-k+2r_1}$.

*Proof.* Let $H = K_r \cup K_s$ and $\sigma = \{u_1, u_2, u_3, ..., u_k\} \subseteq V(G) \subseteq V_1 \cup V_2$ where $V_1 = V(K_r)$ and $V_2 = V(K_s)$ such that $|V_1 \cap \sigma| = r_1$ and $|V_2 \cap \sigma| = k - r_1$. In $H^\sigma$, the $r_1$ vertices of $\sigma$ in $K_r$ are neighbors of the $s - k + r_1$ vertices of $V_2 - \sigma$ and the $k - r_1$ vertices of $\sigma$ in $K_s$ are neighbors of the $r - r_1$ vertices of $V_1 - \sigma$. Hence, $H^\sigma = K_{r+k-2r_1} \cup K_{s-k+2r_1}$. $\square$

**Example 2.9.** Consider the graph $H = K_8 \cup K_5$ which is the union of two complete graphs $K_8$ and $K_5$ as shown in the Figure 2.3 where $V(K_8) = \{a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1\}$ and $V(K_5) = \{i_1, j_1, k_1, l_1, m_1\}$. Here $k = 5$, $r_1 = 3$, $k - r_1 = 2$ and $\sigma = \{a_1, b_1, c_1, i_1, j_1\}$. Then the switching graph of $H$ by $\sigma$, $H^\sigma$ will be as in the Figure 2.4. i.e., $H^\sigma = K_{8+5-6} \cup K_{5-5+6} = K_7 \cup K_6$.

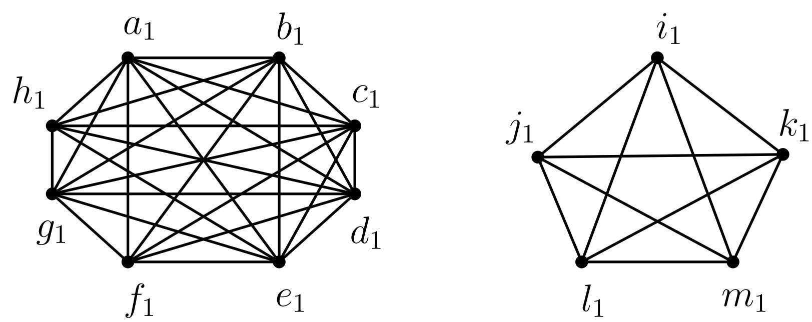


*Figure* 2.3. $H = K_8 \cup K_5$

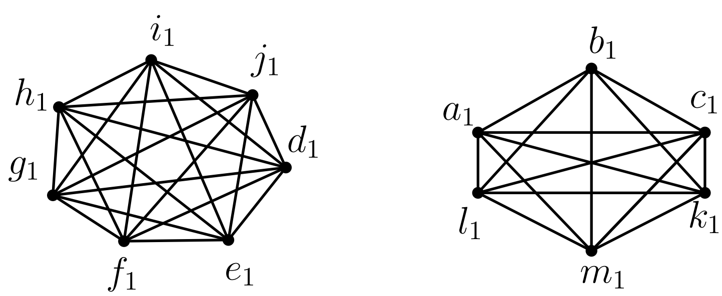


*Figure* 2.4. $H^\sigma = K_7 \cup K_6$

**Theorem 2.10.** *Let $H = K_r \cup K_s$ and the vertices of $\sigma = \{u_1, u_2, u_3, ..., u_k\} \subseteq V(H)$ lie in both the components of $H$. Then $\sigma$ is a $k$-vertex self-switching of $H$ if and only if one of the following conditions holds:*

*(i) $\frac{k+t}{2}$ vertices of $\sigma$ lie in $K_r$ and $\frac{k-t}{2}$ vertices of $\sigma$ lie in $K_s$ where $s = r - t$ and either both $k$ and $t$ are even or both $k$ and $t$ are odd.*

*(ii)Number of vertices of $\sigma$ in $K_r$ is equal to the number of vertices of $\sigma$ in $K_s$ only if $k$ is even.*

*Proof.* Let $H = K_r \cup K_s$ with $V(H) = V_1 \cup V_2$ where $V_1 = V(K_r)$ and $V_2 = V(K_s)$. Given $\sigma = \{u_1, u_2, u_3, ..., u_k\} \subseteq V(H)$ such that $V_1 \cap \sigma \neq \varphi$ and $V_2 \cap \sigma \neq \varphi$.

Let $\sigma = \{u_1, u_2, u_3, ..., u_k\} \subseteq V(H)$ be a $k$-vertex self-switching of $H$ and $|V_1 \cap \sigma| = r_1$ and $|V_2 \cap \sigma| = k - r_1$. Then by Result 2.8, we have $H^\sigma = K_{r+k-2r_1} \cup K_{s-k+2r_1}$. Since $\sigma$ is a $k$-vertex self-switching of $H$, $H \cong H^\sigma$ and hence $K_r \cup K_s \cong K_{r+k-2r_1} \cup K_{s-k+2r_1}$. Thus, either $r = s - k + 2r_1$ and $s = r + k - 2r_1$ or $r = r + k - 2r_1$ and $s = s - k + 2r_1$. That is, the chances are either $k = s - r + 2r_1$ or $k = 2r_1$.

**Case 1.** $k = s - r + 2r_1$

Without loss of generality, let $r \geq s$ which means $s = r - t$ where $t \geq 0$. If $r > s$, then $s = r - t$ and $t > 0$. Hence, $k = 2r_1 - t$ that implies $r_1 = \frac{k+t}{2}$, $k - r_1 = \frac{k-t}{2}$ and since $r_1$ and $k - r_1$ indicates the number of vertices, either both $k$ and $t$ are even or both $k$ and $t$ are odd. Moreover, if $r = s$ then $t = 0$, thus $k = 2r_1$ which implies $r_1 = \frac{k}{2}$, $k - r_1 = \frac{k}{2}$.

**Case 2.** $k = 2r_1$

It follows that $r_1 = k - r_1$ and $k$ is even. Moreover $k = 2r_1$ implies $r_1 = \frac{k}{2}$, $k - r_1 = \frac{k}{2}$ for all $r, s$.

Therefore, if $\sigma$ is a $k$-vertex self-switching of $H$, then the result follows from the above two cases. That is, if $\sigma$ is a $k$-vertex self-switching of $H$, then either $|V_1 \cap \sigma| = \frac{k+t}{2}$ and $|V_2 \cap \sigma| = \frac{k-t}{2}$ where $s = r - t$ or $|V_1 \cap \sigma| = |V_2 \cap \sigma| = \frac{k}{2}$ only if $k$ is even.

Conversely, let $\frac{k+t}{2}$ vertices of $\sigma$ lie in $K_r$ and $\frac{k-t}{2}$ vertices of $\sigma$ lie in $K_s$ when $s = r - t$. That is, $|V_1 \cap \sigma| = \frac{k+t}{2}$ and $|V_2 \cap \sigma| = \frac{k-t}{2}$. Then by Result 2.8, we get, $H^\sigma = K_{r-t} \cup K_r \cong H = K_{r-t} \cup K_r$. Hence, $\sigma$ is a $k$-vertex self-switching of $H$.

Also, let $|V_1 \cap \sigma| = |V_2 \cap \sigma|$. Then $r_1 = k - r_1$ follows $k = 2r_1$ which is even. Without loss of generality, let $\sigma = \{d_b^1, d_b^2 : 1 \leq b \leq \frac{k}{2}\}$ where $d_b^1 \in V_1$ and $d_b^2 \in V_2$. Then in $H^\sigma$, the $\frac{k}{2}$ vertices $d_b^1$'s are neighbors of the vertices of $V_2 - \sigma$ and the $\frac{k}{2}$ vertices $d_b^2{}'s$ are neighbors of the vertices of $V_1 - \sigma$. Define $g : V(H) \to V(H^\sigma)$ by $g(d_b^1) = d_b^2; g(d_b^2) = d_b^1; g(w) = w \ \forall \ w \neq d_b^1, d_b^2,\ 1 \leq b \leq \frac{k}{2}$. Clearly, $g$ creates an isomorphism between $H$ and $H^\sigma$ and so $\sigma$ is a $k$-vertex self-switching of $H$. Hence, we obtain the desired result.

□

**Conclusion**. In this paper, we gave necessary and sufficient conditions for a subset $\sigma$ of $V(H)$ to be a self-switching for the union of two complete graphs and we used this to determine the cardinality of the set of all self-switchings of $H$. That is, $ss_k(H)$.

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[1] Department of Mathematics,, Pioneer Kumaraswamy College, Nagercoil - 629003, Tamil Nadu, India

*Email address*: jayacpkc@gmail.com

[2] Research Scholar, Reg. No: 23113132092001, Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil - 629003, Tamil Nadu, India. Affliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627012, Tamil Nadu, India

*Email address*: ssathithiya@gmail.com