SELF-SWITCHING OF UNION OF TWO COMPLETE GRAPHS

C. JAYASEKARAN¹ AND S. S. ATHITHIYA²*

ABSTRACT. By a graph \( H = (V, E) \), we mean a finite undirected graph without loops and multiple edges. Let \( H \) be a graph and \( \sigma \subseteq V \) be a non-empty subset of \( V \). \( H^\sigma \) is the graph obtained from \( H \) by removing all edges between \( \sigma \) and its complement \( V - \sigma \) and adding as edges all non-edges between \( \sigma \) and \( V - \sigma \). Then \( \sigma \) is said to be a self-switching of \( H \) if \( H \cong H^\sigma \). It can also be referred to as \( k \)-vertex self-switching where \( k = |\sigma| \). The set of all self-switchings of the graph \( H \) with cardinality \( k \) is represented by \( SS_k(H) \) and its cardinality by \( ss_k(H) \). A graph on \( m \) vertices in which each pair of distinct vertices are neighbors is called a complete graph and is denoted by \( K_m \). \( K_m \cup K_n \) is the union of two complete graphs and is disconnected. In this paper, we give necessary and sufficient conditions for \( \sigma \) to be a self-switching for the graph \( H = K_m \cup K_n \) and using this, we find the cardinality \( ss_k(H) \).

1. INTRODUCTION AND PRELIMINARIES

For a finite undirected graph \( H(V, E) \) and a non-empty subset \( \sigma \subseteq V \), the switching of \( H \) by \( \sigma \) is defined as the graph \( H^\sigma(V, E') \) which is obtained from \( H \) by removing all edges between \( \sigma \) and its complement \( V - \sigma \) and adding as edges all non-edges between \( \sigma \) and \( V - \sigma \). For \( \sigma = \{v\} \), we write \( H^v \) and the corresponding switching is called as vertex switching. We also call it as \( |\sigma| \)-vertex switching. When \( |\sigma| = 2 \), it is termed as 2-vertex switching. If \( H \) is isomorphic to \( H^\sigma \), then it is called self vertex switching [5]. In [3], switching classes are discussed. An undirected graph that has a distinct edge connecting each pair of distinct vertices is said to be complete [1, 4]. Here, we will discover how the union of two complete graphs can self-switch.

Definition 1.1. [8] Let \( H(V, E) \) be a finite undirected graph and \( \sigma \subseteq V \). The switching of \( H \) by \( \sigma \) is defined as the graph \( H^\sigma(V, E') \), which is obtained from \( H \) by removing all edges between \( \sigma \) and its complement \( V - \sigma \) and by adding all edges as non-edges between \( \sigma \) and \( V - \sigma \). When \( \sigma = \{v\} \subset V \), the corresponding switching \( H^{(v)} \) is called as vertex switching and is denoted by \( H^v \).

Definition 1.2. [6] A subset \( \sigma \) of \( V(H) \) is said to be a self-switching of \( H \) if \( H \cong H^\sigma \).

Date: Received: Feb 1, 2024; Accepted: Mar 29, 2024.
* Corresponding author.

2020 Mathematics Subject Classification. 05C60.

Key words and phrases. switching, complete graphs, union of two complete graphs.
The set of all self-switchings of $H$ with cardinality $k$ is denoted by $SS_k(H)$ and its cardinality by $ss_k(H)$.

If $k = 1$, then the corresponding self-switching is called self-vertex-switching.

If $k = 2$, then the corresponding self-switching is called as 2-vertex self-switching.

**Definition 1.3.** [4] The Complement $\bar{H}$ of a graph $H$ also has $V(H)$ as its vertex set, but two vertices are adjacent in $\bar{H}$ if and only if they are not adjacent in $H$.

**Lemma 1.4.** [10] For a graph $H(V, E)$ and $\sigma \subseteq V$, it holds that $\bar{H}^\sigma = \bar{H}^\sigma$.

**Theorem 1.5.** [7] If $\sigma = \{u, v\} \subseteq V$ is a 2-vertex self switching of a graph $H$, then $d_H(u) + d_H(v) = \begin{cases} p & \text{if } uv \in E(H) \\ p - 2 & \text{if } uv \notin E(H) \end{cases}$

2. **Main results**

**Theorem 2.1.** Any self-switching of a graph $H$ is also a self-switching of the graph $\bar{H}$ and $SS_k(H) = SS_k(\bar{H})$.

**Proof.** Let $\sigma \subseteq V$ be a self-switching of graph $H$. Let $g$ be an isomorphism between $H$ and $\bar{H}^\sigma$. Using lemma 1.4, we get $(\bar{H}^\sigma) = (\bar{H})^\sigma$. Then $g$ is also an isomorphism between $\bar{H}$ and $(\bar{H})^\sigma$. This implies that $\sigma$ is a self-switching of $\bar{H}$. Hence, $SS_k(H) = SS_k(\bar{H})$. \hfill $\Box$

**Theorem 2.2.** Let $H = K_r \cup K_n$ and $\sigma = \{d^1, d^2\} \subseteq V(H)$ be such that $d^1d^2 \notin E(H)$. Then $\sigma$ is a 2-vertex self-switching of $H$.

**Proof.** Let $H = K_r \cup K_n$ with $V(H) = \{d^1, d^2 : 1 \leq b \leq r, 1 \leq c \leq n\}$. Clearly, $H$ has $p = r + n$ vertices. Let $d^1d^2 \notin E(H)$. Then $d^1$ and $d^2$ belong to different components of $H$. Without loss of generality, let $d^1 \in V(K_r)$ and $d^2 \in V(K_n)$. Clearly, $d_H(d^1) = r - 1$, $d_H(d^2) = n - 1$ and $d_H(d^1) + d_H(d^2) = r + n - 2$. Hence, by Theorem 1.5, $\sigma = \{d^1, d^2\}$ may be a 2-vertex self-switching of graph $H$. In $H^\sigma$, the vertex $d^1$ in $K_r$ is a neighbor of the vertices of $K_n$ other than $d^2$ and the vertex $d^2$ in $K_n$ is a neighbor of the vertices of $K_r$ other than $d^1$. Now define, $g : V(H) \to V(H^\sigma)$ by $g(d^1) = d^2$, $g(d^2) = d^1$ and $g(w) = w$ for $w \neq d^1, d^2$. Clearly, $g$ creates an isomorphism between $H$ and $H^\sigma$. Hence, $\sigma = \{d^1, d^2\}$ is a 2-vertex self-switching of the graph $H$. \hfill $\Box$

**Remark 2.3.** The converse of the Theorem 2.2 is not necessarily true. For example, let the graph $H = K_6 \cup K_4$ be the union of two complete graphs $K_6$ and $K_4$ as shown in the Figure 2.1. Let $\sigma = \{d^1, d^2\}$ be the switching of $H$. The switching graph of $H$ by $\sigma$, $H^\sigma$ is given in the Figure 2.2. Clearly, $H \cong H^\sigma$ and thereby $\sigma = \{d^1, d^2\}$ is a 2-vertex self-switching of $H$ but $d^1d^2 \in E(H)$. 


Theorem 2.4. For \( r \neq s \), let \( H = K_r \cup K_s \) and let \( \sigma = \{d^1, d^2\} \subseteq V(H) \) be such that \( d^1d^2 \) is an edge in the largest order component of \( H \). Then \( \sigma \) is a 2-vertex self-switching of \( H \) if and only if \( |r - s| = 2 \).

Proof. Let \( H = K_r \cup K_s \) be a graph with \( V(H) = V_1 \cup V_2 \) where \( V_1 = V(K_r) \) and \( V_2 = V(K_s) \). Clearly, \( H \) has \( r + s \) number of vertices and let \( V(H) = \{d^1_b, d^2_c : 1 \leq b \leq r; 1 \leq c \leq s\} \) where \( d^1_b \in V_1 \) and \( d^2_c \in V_2 \). Let \( \sigma = \{d^1, d^2\} \subseteq V(H) \) be such that \( d^1d^2 \) lies in the largest order component of \( H \).

Let \( |r - s| = 2 \). Then either \( r - s = 2 \) or \( s - r = 2 \). Without loss of generality, let \( r > s \) and so \( r - s = 2 \). Since \( r > s \), \( d^1d^2 \in E(K_r) \). Then \( d_H(d^1) + d_H(d^2) = 2r - 2 = r + r - 2 = p \). Therefore by Theorem 1.5, \( \sigma \) may be a 2-vertex self-switching of \( H \). In \( H^\sigma \), the vertices in \( \sigma \) are neighbors of the vertices of \( V_2 \) and the vertices in \( V_1 - \sigma \) are neighbors of each other. Since \( H[V_1] = K_r \), \( H[V_1 - \sigma] = H^\sigma[V_1 - \sigma] = K_{r-2} \). Furthermore, \( H^\sigma[V_2 \cup \sigma] = K_r \). Thus, \( H^\sigma = K_r \cup K_{r-2} \) and so \( H \cong H^\sigma \). Hence, \( \sigma \) is a \( k \)-vertex self-switching of \( H \).

Conversely, let \( \sigma \) be a 2-vertex self-switching of \( H \). If \( |r - s| \neq 2 \), then \( d_H(d^1) + d_H(d^2) \) is either \( 2r - 2 \) or \( 2s - 2 \) which is not equal to \( p \) or \( p - 2 \) and so \( \sigma \) cannot be a 2-vertex self-switching of \( H \) which goes against our assumption. Therefore, \( |r - s| = 2 \).

Hence, we obtain the desired result. \( \square \)
Theorem 2.5. Let \( H = K_r \cup K_s \) and \( \sigma = \{d^1, d^2\} \subseteq V(H) \) be a 2-vertex self-switching of \( H \). Then

\[
ss_2(H) = \begin{cases} \binom{r}{2} + rs & \text{if } r - s = 2 \\ rs & \text{otherwise} \end{cases}
\]

Proof. Let \( H = K_r \cup K_s \) be a graph with \( r + s \) number of vertices and \( V(H) = \{d^1_b, d^2_c : 1 \leq b \leq r; 1 \leq c \leq s\} \) where \( d^1_b \in V(K_r) \) and \( d^2_c \in V(K_s) \). Without loss of generality, we consider that \( r \geq s \). Let \( \sigma = \{d^1, d^2\} \subseteq V(H) \). Then \( d^1d^2 \in E(H) \) or \( d^1d^2 \not\in E(H) \).

Case 1. \( d^1d^2 \in E(H) \)

If \( r = s \), then \( d_H(d^1) + d_H(d^2) = 2s - 2 \), which is not the same as \( p \). Therefore, by Theorem 1.5, \( \sigma \) cannot be a 2-vertex self-switching of \( H \). So, let \( r > s \). We have either \( d^1d^2 \in E(K_r) \) or \( d^1d^2 \in E(K_s) \). However, if \( d^1d^2 \in E(K_s) \), then \( d_H(d^1) + d_H(d^2) = 2s - 2 \neq p \) or \( p - 2 \). Hence by Theorem 1.5, \( \sigma \) cannot be a 2-vertex self-switching of \( H \). If \( d^1d^2 \in E(K_r) \), then by Theorem 2.4, \( \sigma \) is a 2-vertex self-switching of \( H \) if and only if \( r - s = 2 \). As there are \( \binom{r}{2} \) possible pairs of vertices in \( K_r \), \( ss_2(H) \geq \binom{r}{2} \).

Case 2. \( d^1d^2 \not\in E(H) \)

By Theorem 2.2, \( \sigma \) is a 2-vertex self-switching of \( H \). Thus for \( r \geq s \) in \( H \), any non-adjacent pair of vertices in \( H \) is a 2-vertex self-switching of \( H \). Since there are \( rs \) such pairs in \( H \), the number of 2-vertex self-switchings in \( H \) is \( rs \).

From the above two cases, we get,

\[
ss_2(H) = \begin{cases} \binom{r}{2} + rs & \text{if } r - s = 2 \\ rs & \text{otherwise} \end{cases}
\]

\( \square \)

Theorem 2.6. Let \( H = K_r \cup K_{r-k} \) and \( \sigma = \{v_1, v_2, ..., v_k\} \subseteq V(H) \). Then

\[
ss_k(H) = \begin{cases} \binom{r-1}{k} + \frac{1}{2}\binom{r-k}{\frac{k}{2}} & \text{if } k \text{ is even} \\ \binom{r}{k} & \text{if } k \text{ is odd} \end{cases}
\]

Proof. Let \( H = K_r \cup K_{r-k} \) be a graph with \( V(H) = V_1 \cup V_2 \) where \( V_1 = V(K_r) \) and \( V_2 = V(K_{r-k}) \). Clearly, \( H \) has \( 2r - k \) vertices. Let \( \sigma = \{v_1, v_2, ..., v_k\} \subseteq V(H) \).

Then we have, \( \sigma \subseteq V_1 \) or \( \sigma \subseteq V_2 \) or \( \sigma \subseteq V_1 \cup V_2 \).

Case 1. \( \sigma \subseteq V_1 \)

Let \( V_1 = \{d^1_b : 1 \leq b \leq r\} \) and \( V_2 = \{d^2_c : 1 \leq c \leq r-k\} \). Then in \( H^\sigma \), the vertices in \( \sigma \) are neighbors of the vertices of \( V_2 \) and not neighbors of the vertices of \( V_1 - \sigma \). Since \( H[V_1] = K_r \), \( H[V_1 - \sigma] = H^\sigma[V_1 - \sigma] = K_{r-k} \). Moreover, \( H^\sigma[V_2 \cup \sigma] = K_{r-k} \). Thus, \( H^\sigma = K_r \cup K_{r-k} \) and so \( H \cong H^\sigma \). Hence, \( \sigma \) is a \( k \)-vertex self-switching of \( H \).

Since there are \( \binom{r}{k} \) possible combinations of selecting \( k \) vertices from \( r \) vertices, \( ss_k(H) \geq \binom{r}{k} \).

Case 2. \( \sigma \subseteq V_2 \)

Then in \( H^\sigma \), the vertices in \( \sigma \) are neighbors of the vertices of \( V_1 \) and not neighbors of the vertices of \( V_2 - \sigma \). Since \( H[V_2] = K_{r-k} \), \( H[V_2 - \sigma] = H^\sigma[V_2 - \sigma] = K_{r-2k} \). Moreover, \( H^\sigma[V_1 \cup \sigma] = K_{r+k} \). Thus, \( H^\sigma = K_{r+k} \cup K_{r-2k} \not\cong H \). Hence, \( \sigma \) cannot be a \( k \)-vertex self-switching of \( H \).
Case 3. \( \sigma \subseteq V_1 \cup V_2 \)

Let \( r_1 \) vertices of \( \sigma \) be in \( V_1 \) and the remaining \( k - r_1 \) vertices of \( \sigma \) be in \( V_2 \). Then we have either \( r_1 = k - r_1 \) or \( r_1 \neq k - r_1 \).

Subcase 3.1. \( r_1 = k - r_1 \)

Then \( 2r_1 = k \) and so \( k \) is even. Without loss of generality, let \( \sigma = \{ d_1^r, d_2^r : 1 \leq b \leq \frac{k}{2} \} \) where \( d_1^r \in V_1 \) and \( d_2^r \in V_2 \). Then in \( H^\sigma \), the \( \frac{k}{2} \) vertices \( d_1^r \)'s are neighbors of the vertices of \( V_2 - \sigma \) and the \( \frac{k}{2} \) vertices \( d_2^r \)'s are neighbors of the vertices of \( V_1 - \sigma \).

Define \( g : V(H) \to V(H^\sigma) \) by \( g(d_1^r) = d_2^r; g(d_2^r) = d_1^r; g(w) = w \forall w \neq d_1^r, d_2^r, 1 \leq b \leq \frac{k}{2} \). Clearly, \( g \) establishes an isomorphism between \( H \) and \( H^\sigma \) and so \( \sigma \) is a \( k \)-vertex self-switching of \( H \). As there are \( \binom{k}{\frac{k}{2}} \) possibilities of selecting \( \frac{k}{2} \) vertices from \( r \) vertices of \( K_r \) and \( \binom{r}{k} \) possibilities of selecting \( \frac{k}{2} \) vertices from \( r - k \) vertices of \( K_{r-k} \), there are \( \binom{k}{\frac{k}{2}} \binom{r}{k} \) \( k \)-vertex self-switchings in \( H \).

Subcase 3.2. \( r_1 \neq k - r_1 \)

Let a vertex of \( \sigma \) be in \( V_1 \) and the remaining \( k - 1 \) vertices of \( \sigma \) be in \( V_2 \). Then in \( H^\sigma \), the vertex of \( \sigma \) in \( V_1 \) is a neighbor of the vertices of \( V_2 - \sigma \) and the \( k - 1 \) vertices of \( \sigma \) in \( V_2 \) are neighbors of the vertices of \( V_1 - \sigma \). Thus we have, \( H^\sigma = K_{r+k-2} \cup K_{r-2k+2} \).

Also, if \( 2\) vertices of \( \sigma \) are in \( V_1 \) and \( k - 2 \) vertices of \( \sigma \) are in \( V_2 \), then in \( H^\sigma \), the \( 2 \) vertices of \( \sigma \) in \( V_1 \) are neighbors of the vertices of \( V_2 - \sigma \) and the \( k - 2 \) vertices of \( \sigma \) in \( V_2 \) are neighbors of the vertices of \( V_1 - \sigma \). Thus we have, \( H^\sigma = K_{r+k-4} \cup K_{r-2k+4} \).

Similarly, if \( 3 \) vertices of \( \sigma \) are in \( V_1 \) and \( k - 3 \) vertices of \( \sigma \) are in \( V_2 \), then \( H^\sigma = K_{r+k-6} \cup K_{r-2k+6} \)

Therefore in general, if \( r_1 \) vertices of \( \sigma \) are in \( V_1 \) and \( k - r_1 \) vertices of \( \sigma \) are in \( V_2 \), then \( H^\sigma = K_{r+k-2r_1} \cup K_{r-2k+2r_1} \). This implies that \( H^\sigma \not\cong H \) and hence \( \sigma \) cannot be a self-switching of \( H \).

Thus from the above cases, we get \( ss_k(H) = \binom{r}{k} + \binom{r}{\frac{k}{2}} \binom{r}{\frac{k}{2}} \), if \( k \) is even and \( ss_k(H) = \binom{r}{k} \), if \( k \) is odd. \( \square \)

**Theorem 2.7.** Let \( H = K_r \cup K_s \) and the vertices of \( \sigma = \{ u_1, u_2, u_3, \ldots, u_k \} \subseteq V(H) \) be contained only in \( K_r \). Then \( \sigma \) is a \( k \)-vertex self-switching of \( H \) if and only if \( s = r - k \).

**Proof.** Let \( H = K_r \cup K_s \) with \( V(H) = V_1 \cup V_2 \) where \( V_1 = V(K_r) \) and \( V_2 = V(K_s) \) and \( \sigma = \{ u_1, u_2, u_3, \ldots, u_k \} \subseteq V(K_r) \).

Let \( s = r - k \). Then \( H = K_r \cup K_{r-k} \). Since \( \sigma \subseteq V(K_r) \), we have in \( H^\sigma \), the vertices of \( \sigma \) are neighbors of \( V_2 \) and the vertices of \( V_1 - \sigma \) are neighbors of each other. Thus we have, \( H^\sigma = K_{r-k} \cup K_r \cong H \). Hence, \( \sigma \) is a \( k \)-vertex self-switching of \( H \).

Conversely, let \( \sigma = \{ u_1, u_2, u_3, \ldots, u_k \} \subseteq V(K_r) \) be a \( k \)-vertex self-switching of \( H \). Then, \( H \cong H^\sigma \). Since in \( H^\sigma \), the vertices of \( \sigma \) are neighbors of the vertices of \( V_2 \) and the vertices of \( V_1 - \sigma \) are neighbors of each other, \( H^\sigma = K_{r-k} \cup K_{s+k} \).

Hence \( K_r \cup K_s \cong K_{r-k} \cup K_{s+k} \). Thus either \( r - k = r \) and \( s + k = s \) or \( r - k = s \) and \( s + k = r \). Since \( k = 0 \) is not possible, we have \( s = r - k \). Hence, we obtain the desired result. \( \square \)

**Result 2.8.** Let \( H = K_r \cup K_s \) and \( \sigma = \{ u_1, u_2, u_3, \ldots, u_k \} \subseteq V(H) \). If \( |V_1 \cap \sigma| = r_1 \) and \( |V_2 \cap \sigma| = k - r_1 \), then \( H^\sigma = K_{r+k-2r_1} \cup K_{s-k+2r_1} \).
Proof. Let \( H = K_r \cup K_s \) and \( \sigma = \{u_1, u_2, u_3, \ldots, u_k\} \subseteq V(G) \subseteq V_1 \cup V_2 \) where \( V_1 = V(K_r) \) and \( V_2 = V(K_s) \) such that \(|V_1 \cap \sigma| = r_1\) and \(|V_2 \cap \sigma| = k - r_1\). In \( H^\sigma \), the \( r_1 \) vertices of \( \sigma \) in \( K_r \) are neighbors of the \( s - k + r_1 \) vertices of \( V_2 - \sigma \) and the \( k - r_1 \) vertices of \( \sigma \) in \( K_s \) are neighbors of the \( r - r_1 \) vertices of \( V_1 - \sigma \). Hence, \( H^\sigma = K_{r+k-2r_1} \cup K_{s-k+2r_1} \). □

Example 2.9. Consider the graph \( H = K_8 \cup K_5 \) which is the union of two complete graphs \( K_8 \) and \( K_5 \) as shown in the Figure 2.3 where \( V(K_8) = \{a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1\} \) and \( V(K_5) = \{i_1, j_1, k_1, l_1, m_1\} \). Here \( k = 5, r_1 = 3, k - r_1 = 2 \) and \( \sigma = \{a_1, b_1, c_1, i_1, j_1\} \). Then the switching graph of \( H \) by \( \sigma \), \( H^\sigma \) will be as in the Figure 2.4. i.e., \( H^\sigma = K_{8+5-6} \cup K_{5-5+6} = K_7 \cup K_6 \).

![Figure 2.3](image1.png)

**Figure 2.3.** \( H = K_8 \cup K_5 \)

![Figure 2.4](image2.png)

**Figure 2.4.** \( H^\sigma = K_7 \cup K_6 \)

Theorem 2.10. Let \( H = K_r \cup K_s \) and the vertices of \( \sigma = \{u_1, u_2, u_3, \ldots, u_k\} \subseteq V(H) \) lie in both the components of \( H \). Then \( \sigma \) is a \( k \)-vertex self-switching of \( H \) if and only if one of the following conditions holds:

(i) \( \frac{k+t}{2} \) vertices of \( \sigma \) lie in \( K_r \) and \( \frac{k-t}{2} \) vertices of \( \sigma \) lie in \( K_s \) where \( s = r - t \) and either both \( k \) and \( t \) are even or both \( k \) and \( t \) are odd.

(ii)\( \)Number of vertices of \( \sigma \) in \( K_r \) is equal to the number of vertices of \( \sigma \) in \( K_s \) only if \( k \) is even.
Proof. Let $H = K_r \cup K_s$ with $V(H) = V_1 \cup V_2$ where $V_1 = V(K_r)$ and $V_2 = V(K_s)$. Given $\sigma = \{u_1, u_2, u_3, \ldots, u_k\} \subseteq V(H)$ such that $V_1 \cap \sigma \neq \emptyset$ and $V_2 \cap \sigma \neq \emptyset$.

Let $\sigma = \{u_1, u_2, u_3, \ldots, u_k\} \subseteq V(H)$ be a $k$-vertex self-switching of $H$ and $|V_1 \cap \sigma| = r$ and $|V_2 \cap \sigma| = k - r$. Then by Result 2.8, we have $H^\sigma = K_{r+k-2r_1} \cup K_{s-k+2r_1}$. Since $\sigma$ is a $k$-vertex self-switching of $H$, $H \cong H^\sigma$ and hence $K_r \cup K_s \cong K_{r+k-2r_1} \cup K_{s-k+2r_1}$. Thus, either $r = s - k + 2r_1$ and $s = r + k - 2r_1$ or $r = r + k - 2r_1$ and $s = s - k + 2r_1$. That is, the chances are either $k = s - r + 2r_1$ or $k = 2r_1$.

Case 1. $k = s - r + 2r_1$

Without loss of generality, let $r \geq s$ which means $s = r - t$ where $t \geq 0$. If $r > s$, then $s = r - t$ and $t > 0$. Hence, $k = 2r_1 - t$ that implies $r_1 = \frac{k + t}{2}$, $k - r_1 = \frac{k - t}{2}$ and since $r_1$ and $k - r_1$ indicates the number of vertices, either both $k$ and $t$ are even or both $k$ and $t$ are odd. Moreover, if $r = s$ then $t = 0$, thus $k = 2r_1$ which implies $r_1 = \frac{k}{2}$, $k - r_1 = \frac{k}{2}$.

Case 2. $k = 2r_1$

It follows that $r_1 = k - r_1$ and $k$ is even. Moreover $k = 2r_1$ implies $r_1 = \frac{k}{2}$, $k - r_1 = \frac{k}{2}$ for all $r, s$.

Therefore, if $\sigma$ is a $k$-vertex self-switching of $H$, then the result follows from the above two cases. That is, if $\sigma$ is a $k$-vertex self-switching of $H$, then either $|V_1 \cap \sigma| = \frac{k + t}{2}$ and $|V_2 \cap \sigma| = \frac{k - t}{2}$ where $s = r - t$ or $|V_1 \cap \sigma| = |V_2 \cap \sigma| = \frac{k}{2}$ only if $k$ is even.

Conversely, let $\frac{k + t}{2}$ vertices of $\sigma$ lie in $K_r$ and $\frac{k - t}{2}$ vertices of $\sigma$ lie in $K_s$ when $s = r - t$. That is, $|V_1 \cap \sigma| = \frac{k + t}{2}$ and $|V_2 \cap \sigma| = \frac{k - t}{2}$. Then by Result 2.8, we get, $H^\sigma = K_{r-t} \cup K_r \cong H = K_{r-t} \cup K_r$. Hence, $\sigma$ is a $k$-vertex self-switching of $H$.

Also, let $|V_1 \cap \sigma| = |V_2 \cap \sigma|$. Then $r_1 = k - r_1$ follows $k = 2r_1$ which is even. Without loss of generality, let $\sigma = \{d_b^1, d_b^2 : 1 \leq b \leq \frac{k}{2}\}$ where $d_b^1 \in V_1$ and $d_b^2 \in V_2$. Then in $H^\sigma$, the $\frac{k}{2}$ vertices $d_b^1$'s are neighbors of the vertices of $V_2 - \sigma$ and the $\frac{k}{2}$ vertices $d_b^2$'s are neighbors of the vertices of $V_1 - \sigma$. Define $g : V(H) \rightarrow V(H^\sigma)$ by $g(d_b^1) = d_b^2$; $g(d_b^2) = d_b^1$, $g(w) = w \forall w \neq d_b^1, d_b^2$, $1 \leq b \leq \frac{k}{2}$. Clearly, $g$ creates an isomorphism between $H$ and $H^\sigma$ and so $\sigma$ is a $k$-vertex self-switching of $H$.

Hence, we obtain the desired result.

\[\square\]

Conclusion. In this paper, we gave necessary and sufficient conditions for a subset $\sigma$ of $V(H)$ to be a self-switching for the union of two complete graphs and we used this to determine the cardinality of the set of all self-switchings of $H$. That is, $ss_k(H)$.

References


1 Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil - 629003, Tamil Nadu, India
Email address: jayacpkc@gmail.com

2 Research Scholar, Reg. No: 23113132092001, Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil - 629003, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekappati, Tirunelveli - 627012, Tamil Nadu, India
Email address: ssathithiya@gmail.com