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DISCONTINUOUS GALERKIN METHOD FOR LINEAR PARABOLIC EQUATIONS WITH L^1-DATA

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Abstract. In this work, we examine the discontinuous Galerkin method for parabolic linear problem with data in $L^1(\Omega\times(0,T))$. On one hand, using a Euler time advancing scheme that goes backwards, we can discretize a time interval. Furthermore, the discretization of space is based on Symmetric Weighted Interior Penalty (SWIPG) method. We use the technique of construction of the renormalized solution to obtain existence of the discrete solution. Then, our research demonstrates that the discrete solution converges in $L^1(Q)$ to the unique renormalized solution of the problem, where $Q=\Omega\times(0,T)$.
In the case where the coefficients are smooth, we offer an estimate of the error in $L^1(Q)$, when the side on the right is assigned to Marcinkiewicz space $L^{s,\infty}(Q)$ where $1<s<2$.

1. Introduction

Parabolic problems with L^1 data are encountered in numerous physical and engineering applications, for example in continuous casting in the metallurgical industry, in fluid flow modeling and image restoration. In this paper, our approach is to consider numerical analysis of linear parabolic problems with data belongs L^1, namely the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(A(x)\nabla u) & = f \quad \text{in } \Omega\times(0,T), \\ \qquad u & = 0 \quad \text{on } \partial\Omega\times(0,T), \\ \qquad u(\cdot,0) & = u_0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded polygonal subset and Lipschitz of $\mathbb{R}^d$ with $d=2$ or $d=3$, $T>0$ and $Q=\Omega\times(0,T)$. The matrix $A\in L^\infty(\Omega)$ is the diffusion cœfficient and the data f and u_0 belongs respectively to $L^1(Q)$ and $L^1(\Omega)$. The use and analysis of discontinuous Galerkin methods has increased in recent years, and they have been utilized in different types of applications. Generally, the problems addressed for this method are right-hand side $f\in L^2(\Omega)$, which allows to use the variational formulation of the problem, see [1, 2, 8]. In [12], the authors

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address the case where the data is a Dirac measure. For our study, the data u_0 and f belong respectively to $L^1(\Omega)$ and $L^1(Q)$. This is the main contribution of this paper. Thus, the analysis of the problem (1.1) is difficult and requires news tools because discontinuous Galerkin finite element method schemes are based on variational formulations which do not give estimates in the L^1 norm. Here, we introduce the framework of renormalized solution. Employing this concept, we show that every sequence of a approximate solutions, obtained by a backward-in-time and SWIP discontinuous Galerkin finite element method, converges to the unique renormalized solution of the problem (1.1).

The rest of the paper is organized in the following manner. In Section 2, we introduce the notion of renormalized solution and we give functionals settings of the discontinuous Galerkin framework. Moreover, we give some primary estimates. In Section 3, we prove, in the one part, that the unique discrete solution converges to the unique renormalized solution of (1.1). Secondly, we have completed the main outcome of the current paper, which is the convergence of discrete solution to the renormalized solution, by the error estimate when $d = 2$ or $d = 3$ and $f \in L^{s,\infty}(Q)$.

2. Preliminaries

2.1. Renormalized solution.

We consider the linear parabolic problem defined in (1.1) under the following assumptions : the matrix A is coercive. It implies that A is symmetric matrix such that :

$$A \in L^\infty(\Omega)^{d\times d} \tag{2.1}$$

and satisfying the ellipticity assumption

$$\exists \alpha > 0, \text{ a.e. } x \text{ in } \Omega, \ \forall \xi \in \mathbb{R}^d, \ A(x)\xi \cdot \xi \geq \alpha|\xi|^2. \tag{2.2}$$

The data of this problem are such that

$$f \in L^1(\Omega \times (0,T)), \qquad u_0 \in L^1(\Omega). \tag{2.3}$$

∇ is the gradient operator and the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is opened, bounded and Lipschitz. We assume that the space-time cylinder $\overline{Q} = \overline{\Omega} \times [0,T]$. We consider the Marcinkiewicz space or weak-Lebesgue space $L^{s,\infty}(Q)$ with the norm

$$\|f\|_{L^{s,\infty}(Q)} := \sup_{\lambda>0} \lambda \Big|\{z \in Q = \Omega \times (0,T) : |f(z)| > \lambda\}\Big|^{1/s} < \infty. \tag{2.4}$$

T_k is what we refer to as the truncation function at height k $(k \geq 0)$ i.e.

$$T_k(r) = \min\Big(k, \ \max(r, -k)\Big)$$

Now, we are able to define the renormalized solution of our problem.

Definition 2.1. A measurable function u defined from Q to $\mathbb{R}$ is a renormalized solution of (1.1) if for any $k > 0$,

$$u \in L^\infty(0,T; L^1(\Omega)), \tag{2.5}$$

$$T_k(u) \in L^2(0,T; H_0^1(\Omega)), \tag{2.6}$$

$$\lim_{k\longrightarrow+\infty}\frac{1}{k}\int_0^T\int_\Omega |\nabla T_k(u)|^2dxdt=0 \tag{2.7}$$

and

$$\begin{aligned}&-\int_Q S(u)\frac{\partial\phi}{\partial t}dxdt+\int_Q S'(u)A\nabla u\nabla\phi dxdt\\&\quad+\int_Q S''(u)\,A\nabla u\,\nabla u\,\phi\,dxdt=\int_Q fS'(u)\phi dxdt,\end{aligned} \tag{2.8}$$

for every function $\phi\in C_0^\infty(Q)$, for all functions $S\in C^\infty(\mathbb{R})$ such that S' has a compact support in $\mathbb{R}$ (i.e. $S'\in C_0^\infty(\mathbb{R})$) and $S'(u)\phi$ belongs to $L^2(0,T;H_0^1(\Omega))$.

2.2. **The discrete setting.** Here, we briefly present a discontinuous Galerkin finite element approximation of problem (1.1). For more details of this method, see [8]. The initial introduction of the discontinuous Galerkin method was meant to model and simulate neutron transport by Lasaint and Raviart in [10].
Consider $\mathcal{T}_h$ to be uniform partition of Ω into disjoint open triangles K such that $\overline{\Omega}=\bigcup_{K\in\mathcal{T}_h}\overline{K}$. We set $h=\max\{\mathrm{diam}(K),\ K\in\mathcal{T}_h\}$.

For every integer $k\geq 0$, we define the broken polynomial space :

$$\mathbb{P}_d^k(\mathcal{T}_h):=\{w_h\in L^2(\Omega)|\ \forall K\in\mathcal{T}_h,\ w_h|_K\in\mathbb{P}_d^k(K)\},$$

with polynomial degree $k\geq 1$. Note that $\mathbb{P}_d^k$ is spanned by the restriction to K of polynomial functions in d variables and of total degree at most k. We say that a (closed) subset $F\subset\overline{\Omega}$ is a *mesh face* if F has positive $(d-1)$-dimensional measure and if one of the two following mutually exclusive conditions is satisfied :

- (i) There are separate mesh component $K_1,\ K_2\in\mathcal{T}_h$ where $F=\partial K_1\cap\partial K_2$; in such case, F is called an *interface*. Moreover, $n_F:=n_{K_1}$ designate the unit normal vector to F pointing from K_1 to K_2.
- (ii) There is $K\in\mathcal{T}_h$ such that $F=\partial K\cap\partial\Omega$; in such circumstances, F is called a *boundary face* and we indicate by $n_F:=n$ the outward unit normal to $\partial\Omega$.

Interfaces are collected in the set $\mathcal{F}_h^i$, boundary faces in $\mathcal{F}_h^b$, and mesh faces in $\mathcal{F}_h:=\mathcal{F}_h^i\cup\mathcal{F}_h^b$. Moreover, for every mesh element $K\in\mathcal{T}_h$, the set

$$\mathcal{F}_K:=\{F\in\mathcal{F}_h\ |F\subset\partial K\}$$

contains the mesh faces forming the boundary of K. We suppose that $\mathcal{T}_h$ is shape regular and contact-regular, that is to say there exists C independent of the mesh-size h such that, for all $K\in\mathcal{T}_h$ and all $F\in\mathcal{F}_K$, we have

$$h_K\leq Ch_F,$$

where h_F designates the diameter of F. Letting

$$N_\partial:=\max_{Q,\ K\in\mathcal{T}_h}\mathrm{card}(\mathcal{F}_K)$$

contact regularity implies that N_∂ is bounded.

Definition 2.2. (Jumps) Let ψ be a function defined on Ω. We suppose that ψ is smooth enough to admit on all $F \in \mathcal{F}_h$ a trace. Then, if $F \in \mathcal{F}_h^i$ with $F = \partial K_1 \cap \partial K_2$, the jump of ψ at F is defined for a.e. $x \in F$ as :

$$[\![\psi]\!]_F(x) := \psi_{K_1}(x) - \psi_{K_2}(x),$$

while if $F \in \mathcal{F}_h^b$ with $F = \partial K \cap \partial\Omega$, we set $[\![\psi]\!]_F(x) := \psi_K(x)$.

Definition 2.3. (Weighted averages) Either ψ a function defined as above. To any interface $F \in \mathcal{F}_h^i$ with $F = \partial K_1 \cap \partial K_2$, we attribute two non-negative real numbers $\omega_{K_1,F}$ and $\omega_{K_2,F}$ where

$$\omega_{K_1,F} + \omega_{K_2,F} = 1.$$

Then, the weighted average of ψ at $F \in \mathcal{F}_h^i$ is defined for a.e. $x \in F$ as

$$\{\psi\}_{\omega,F}(x) := \omega_{K_1,F}\psi_{|K_1}(x) + \omega_{K_2,F}\psi_{|K_2}(x).$$

while on boundary faces $F \in \mathcal{F}_h^b$ with $F = \partial K \cap \partial\Omega$, we set $\{\psi\}_{\omega,F}(x) := \psi_{|K}(x)$.

Afterward, we consider the diffusion-dependent factor for the weights, in particular for all $F \in \mathcal{F}_h^i,\ F = \partial K_1 \cap \partial K_2$,

$$\begin{aligned} \omega_{K_1,F} &:= \frac{A_2}{A_1 + A_2}, \\ \omega_{K_2,F} &:= \frac{A_1}{A_1 + A_2}, \end{aligned} \tag{2.9}$$

where $A_i = A_{|K_i},\ i \in \{1,2\}$. The following embedding $\mathbb{P}_d^1(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)$, allows to define the *broken gradient* $\nabla_h : H^1(\mathcal{T}_h) \longrightarrow \Big[L^2(\Omega)\Big]^d$ such that, $\forall w \in H^1(\mathcal{T}_h)$,

$$\forall K \in \mathcal{T}_h :\ (\nabla_h w)|_K := \nabla(w|_K).$$

Now, we give the diffusion-dependent penalty parameter as

$$\gamma_{A,F} := \begin{cases} \dfrac{2A_1A_2}{A_1 + A_2}, & \text{if} \quad F \subset \partial K_1 \cap \partial K_2, \\ \Big(A_{|K} n\Big) \cdot n, & \text{if} \quad F \subset \partial K \cap \partial\Omega, \end{cases} \tag{2.10}$$

In the following, we will use a Symmetric Weighted Interior Penalty (SWIP) method (see [8]) to discretize (1.1). The SWIP discrete bilinear form is determined by :

$$\begin{aligned} a_h^{swip}(v,w) \ &:= \ \int_\Omega A\nabla_h v \nabla_h w dx + \sum_{F\in\mathcal{F}_h} \eta \frac{\gamma_{A,F}}{h_F} \int_F [\![v,w]\!] \\ &- \sum_{F\in\mathcal{F}_h} \int_F \Big(\{A\nabla_h v\}_w \cdot n_F [\![w]\!] + \{A\nabla_h w\}_w \cdot n_F [\![v]\!] \Big). \end{aligned} \tag{2.11}$$

The quantity $\eta > 0$ denotes a user-dependent penalty parameter which is independent of the diffusion coefficient. Therefore, the SWIP norms are defined by :

$$\|v\|_{swip} := \Big(\|A^{1/2}\nabla_h v\|^2_{[L^2(\Omega)]^d} + |v|^2_{A,J} \Big)^{1/2}.$$

For all $q \geq 1$, we have :

$$\|v\|_{swip,q} := \left(\|A^{1/2}\nabla_h v\|^q_{[L^q(\Omega)]^d} + |v|^q_{J,A,q}\right)^{1/q} \tag{2.12}$$

when the propagation jump semi-norms

$$|v|^2_{J,A} := \sum_{F\in\mathcal{F}_h} \frac{\gamma_{A,F}}{h_F}\|[\![v]\!]\|^2_{L^2(F)}.$$

For all $q \geq 1$, we designate :

$$|v|^q_{J,A,q} := \sum_{F\in\mathcal{F}_h} \frac{\gamma_{A,F}}{h_F^{q-1}}\|[\![v]\!]\|^q_{L^q(F)}. \tag{2.13}$$

For every d-simplex K in $\mathbb{R}^d$, we introduce the following notations :

(1) $s_K^{(i)},\ i = 0,\dots,d$, designate the vertices of K.

(2) $c_K^{(i)},\ i = 0,\dots,d$, designate the centers of the faces $F_i \in K$.

(3) $\Lambda_{i,K},\ i = 0,\dots,d$, nominate the barycentric coordinates regarding to the $s_K^{(i)}$.

(4) for every $x \in \mathbb{R}^d$, we set

$$\varphi_{i,K}(x) := 1 - d\Lambda_{i,K}(x) \text{ for } i = 0,\dots,d,$$

where $(\varphi_{i,K})_{0\leq i\leq d}$ denote the $\mathbb{P}_1$ configuration functions connected to K. To simplify the notation, we consider respectively

$$s_K^{(i)} = s_i \text{ and } c_K^{(i)} = c_i,\ \forall\ i = 0,\dots,d,$$

the vertices of K and the centers of the face F_i. Therefore, we have

$$\varphi_i \in L^2(\Omega_h),$$

$$\varphi_i|_K \in \mathbb{P}_1[K],\ \forall i \in \{0,1,\dots,d\},\ \forall K \in \mathcal{T}_h,$$

$$\varphi_i|_{F_i} = 1,$$

$$\varphi_i(s_i) = 1,\ \text{ and } \varphi_i(s_j) = 1 - d \text{ such that } j \neq i,$$

$$\frac{1}{|F_j|}\int_{F_j}\varphi_i = \varphi_i(c_j) = \delta_{i,j} \text{ and } \sum_{i=0}^{d}\varphi_{i,K}(x) = 1,\ \text{for every } x \in \mathbb{R}^d,$$

with

$$\Omega_h := \bigcup\left\{K,\ K \in \mathcal{T}_h\right\} \subset \overline{\Omega}.$$

(5) If N designates the number of all interior centers c_i of faces F in $\mathcal{T}_h$, we define the interpolation operator Π_h and the truncated interpolation operator I_h^k by

$$\Pi_h(v) \in V_h, \ \forall v \in V_{dG},$$

$$\Pi_h(v) := \sum_{i=1}^{N} \alpha_i^v \varphi_i,$$

$$I_h^k(v) := \sum_{i=1}^{N} T_k(\alpha_i^v)\varphi_i,$$

with

$$\alpha_i^v = \Pi_h(v)(c_i) := (1/|F_i|)\int_{F_i} \Pi_h(v) := (1/|F_i|)\int_{F_i} v.$$

2.3. **The discrete problem.** We use respectively, the SWIP bilinear formulation in space and Euler backward method in time to obtain the fully discrete scheme. The space semi-discretization of problem (1.1) becomes

$$\begin{cases} \text{Find } u_h \in V_{dG}(\mathcal{T}_h) \ \text{ such that, for any } t \in [0,T], \\ \displaystyle\int_\Omega \frac{\partial u_h}{\partial t} \cdot v_h dx + a_h^{swip}(u_h, v_h) = \int_\Omega f v_h dx \text{ for any } v_h \in V_{dG}(\mathcal{T}_h) \\ u_{h|t=0} = u_{h0} \end{cases} \tag{2.14}$$

where

$$V_{dG}(\mathcal{T}_h) := \mathbb{P}_d^k(\mathcal{T}_h), \ k \geq 1.$$

Now, we study the entirely discretization of problem (2.14) by using to the finite difference scheme with time-step $\triangle t > 0$. We first subdivide the time gap $(0,T)$ into N uniformly-spaced subintervals by the following points :

$$0 = t_0 < t_1 < \ldots < t_N = T$$

with $t_n = n\triangle t$ and $\triangle t = T/N$ the time step size. Let $I_n = (t_{n-1}, tn]$ be the nth sub-interval. For any sequence $\{\psi^n\}_{n=0}^N \subset L^2(\mathcal{T}_h)$, we consider the backward difference quotient

$$\widetilde{\nabla}\psi^n = \frac{\psi^n - \psi^{n-1}}{\triangle t}.$$

We use the finite-dimensional space

$$V_{hk} = \{\psi : [0,T] \longrightarrow V_{dG}(\mathcal{T}_h) : \psi|_{I_n} \in V_{dG}(\mathcal{T}_h) \text{ is constant in time}\}.$$

For $\psi \in V_{dG}(\mathcal{T}_h)$, we let ψ^n be the value of ψ at t_n. Furthermore, we write V_{hk}^n for the restriction to I_n of the functions in V_{hk}. The fully discrete discontinuous Galerkin finite element approximation to (2.14) is defined as follows. For each $n = 1, 2, \ldots, N$, find $u_h^n \in V_{hk}^n$ such that :

$$\int_\Omega \widetilde{\nabla} u_h^n v_h dx + a_h^{swip}(u_h^n, v_h) = \int_\Omega f^n v_h, \ \forall v_h \in V_{hk}^n. \tag{2.15}$$

We define the $N \times N$ stifness matrices $P = (P_{ij})$ and $Q = (Q_{ij})$, namely

$$P_{ij} = \int_\Omega \widetilde{\nabla}\varphi_i \cdot \varphi_j dx, \qquad Q_{ij} = a_h^{swip}(\varphi_i, \varphi_j), \qquad \text{for } i,j \in \{1,2,\dots,N\}.$$

The matrices P and Q satisfy, $\forall i,j \in \{1,2,\dots,N\}$ the following main assumptions :

$$\begin{cases} P_{ii} - \sum\limits_{\substack{1\le j\le N \\ j\neq i}} |P_{ij}| \ge 0, \\ \\ Q_{ii} - \sum\limits_{\substack{1\le j\le N \\ j\neq i}} |Q_{ij}| \ge 0. \end{cases} \tag{2.16}$$

Proposition 2.4. *Under the assumptions* (2.1)-(2.3)*, the fully discrete discontinuous Galerkin scheme defined by*

$$\begin{cases} \text{Find } u_h^n \in V_{hk}^n \ \text{ such that,} \\ \\ \displaystyle\int_\Omega \widetilde{\nabla} u_h^n \cdot v_h dx + \int_0^T a_h^{swip}(u_h^n, v_h)dt = \int_Q f v_h dxdt \text{ for all } v_h \in V_{hk}^n, \end{cases} \tag{2.17}$$

admits a unique solution.

Proof. The term of right side in (2.17) has meaning since f belongs to $L^1(Q)$ and $V_{hk}^n \subset L^\infty(Q)$. So, by a direct repercussion of the Lax-Milgram Theorem, we deduce that the solution u_h^n of (2.17) exists and is unique, for $n = 1,\dots,N$. □

We gives two stability result of the discrete solution of (2.17).

Theorem 2.5. *Suppose that $v_h \in V_{hk}^n$ satisfies*

$$\forall k > 0, \ \sup_{t\in(0,T)} \int_\Omega |I_h^k(v_h)|^2 dx + \int_Q |\nabla_h I_h^k(v_h)|^2 dxdt \le kM \tag{2.18}$$

for some $M > 0$. Then, for every q with $1 \le q < \dfrac{d}{d-1}$

$$\int_0^T \|v_h\|_{swip,q} dt \le C(d, |Q|, q, \|A\|_{L^\infty(\Omega)^{d\times d}})M. \tag{2.19}$$

We demonstrate it by using the techniques developed in Lemma 2.4 of [3] and Theorem 5 of [11].

Lemma 2.6. *Let u_h^n be the solution of* (2.17)*. Then we have, for $n = 1,\dots,N$*

$$\|u_h^N\|_{L^2(\Omega)} + \sum_{n=1}^N \triangle t \|u_h^n\|_{swip} \le C(Q, u_0, h)\|f\|_{L^1(Q)}. \tag{2.20}$$

Proof. We take $v_h = u_h^n$ in (2.17) and then, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|u_h^n\|_{L^2(\Omega)}^2 - \|u_h^{n-1}\|_{L^2(\Omega)}^2 + \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2 + 2\triangle t \|u_h^n\|_{swip}^2 &\le \triangle t \int_\Omega f^n u_h^n dx \\ &\le \triangle t \|f^n\|_{L^1(\Omega)} \|u_h^n\|_{L^1(\Omega)} \end{aligned}$$

Summing over n from $n = 1$ to $n = N$, using the discrete Gronwall's inequality and the inequality (see [13], Lemma 5.2)

$$\sum_{n=1}^{N} \triangle t \|f^n\|_{L^1(\Omega)} \leq C\|f\|_{L^1(Q)},$$

we get the estimate (2.20). □

3. Results and Findings

Here, we prove the convergence of the discrete solution to the unique renormalized solution of (1.1). Initially, we give an a priori estimation of the solution u_h^n of (2.17).

Proposition 3.1. *Under assumption* (2.16)*, one has for every* $v_h \in V_{hk}^n$ *and every* $k > 0$

$$B_1 = \int_\Omega \widetilde{\nabla}\Big(u_h - I_h^k(u_h)\Big) \cdot I_h^k(u_h)dx \geq 0, \tag{3.1}$$

$$B_2 = a_h^{swip}\Big(u_h - I_h^k(u_h), I_h^k(u_h)\Big) \geq 0. \tag{3.2}$$

Proof. We use the principle of the discrete maximum by considering the relationship given in (2.16). □

Proposition 3.2. *Assume that* A*,* f *satisfy* (2.1)-(2.3) *and that the family of subdivisions* $\mathcal{T}_h$ *satisfies the shape regular hypothesis. Then the solution* u_h *of* (2.17) *satisfies for every* $h > 0$ *and every* $k > 0$

$$\int_\Omega \widetilde{\nabla} I_h^k(u_h) \cdot I_h^k(u_h)dx + \int_0^T a_h^{swip}(I_h^k(u_h), I_h^k(u_h))dt \leq \int_0^T \int_\Omega f I_h^k(u_h)dxdt, \tag{3.3}$$

and, in particular, u_h *satisfies*

$$\|I_h^k(u_h)\|_{L^2(\Omega)} + \int_Q \Big|\nabla_h I_h^k(u_h)\Big|^2 dxdt \leq C\|f\|_{L^1(Q)}, \tag{3.4}$$

where the constant C *depends only on* $\Omega, k, \alpha, C_\eta$.

Proof. Using $I_h^k(u_h)$ as a test function in (2.17), one has

$$\int_0^T \int_\Omega \widetilde{\nabla} u_h \cdot I_h^k(u_h)dxdt + \int_0^T a_h^{swip}(u_h, I_h^k(u_h))dt = \int_0^T \int_\Omega f I_h^k(u_h)dxdt.$$

From Proposition (3.1), we obtain (3.3).
On the other hand, by combining (3.3) with coercivity (2.2) of A, we obtain

$$\begin{aligned}
\|I_h^k(u_h)\|_{L^2(\Omega)} &+ \alpha C_\eta \int_Q |\nabla_h I_h^k(u_h)|^2 dxdt \\
&\leq \|I_h^k(u_h)\|^2_{L^2(\Omega)} + C_\eta \int_0^T \|I_h^k(u_h)\|^2_{swip} dt \\
&\leq \int_\Omega \widetilde{\nabla} I_h^k(u_h) \cdot I_h^k(u_h) + \int_0^T a_h^{swip}(I_h^k(u_h), I_h^k(u_h))dt \\
&\leq \int_0^T \int_\Omega f I_h^k(u_h) dxdt \leq \max_Q \left| I_h^k(u_h) \right| \int_Q |f| dxdt \\
&\leq C_1 \|f\|_{L^1(Q)}.
\end{aligned}$$

□

Our principal result is the following :

Theorem 3.3. *Suppose that A, f satisfy (2.1)-(2.3) and that the family of subdivisions $\mathcal{T}_h$ satisfies the shape regular hypothesis. Then the unique solution u_h^n of (2.17) satisfies for every $h > 0$*

$$u_h^n \longrightarrow u \text{ strongly in } L^1(Q), \tag{3.5}$$

$$\nabla u_h^n \longrightarrow u \text{ strongly in } \left[L^1(Q)\right]^d, \tag{3.6}$$

$$I_h^k(u_h^n) \longrightarrow T_k(u) \text{ strongly in } L^2(Q), \tag{3.7}$$

while h decreases to zero, for $n = 1, \ldots, N$, and where u is the unique renormalized solution of (1.1).

Proof.
We introduce a sequence f^ε in $L^2(Q)$, converging strongly in $L^1(Q)$ to f (we can take, for example $f^\varepsilon = T_{1/\varepsilon}(f)$). Let u_h^ε to be the unique solution of (2.17) for the data f^ε. Then $u_h - u_h^\varepsilon$ satisfies :

$$\begin{cases}
u_h - u_h^\varepsilon \in V_{hk}^n, \\
\forall v_h \in V_{hk}^n, \ \displaystyle\int_\Omega \widetilde{\nabla}(u_h - u_h^\varepsilon) \cdot v_h dx + \int_0^T a_h^{swip}(u_h - u_h^\varepsilon, v_h) dt = \int_Q (f - f^\varepsilon) v_h dxdt.
\end{cases}$$

Applying estimate (3.4) to this problem, we get for every $h > 0$, every $k > 0$ and every $\varepsilon > 0$

$$\|I_h^k(u_h - u_h^\varepsilon)\|_{L^2(\Omega)} + \int_Q |\nabla_h I_h^k(u_h - u_h^\varepsilon)|^2 dxdt \leq C(k, d, \alpha, |Q|) \|f - f^\varepsilon\|_{L^1(Q)}. \tag{3.8}$$

Therefore, we use the Theorem (2.5) and we acquire that for each q such that $1 \leq q < \dfrac{d}{d-1}$, every $h > 0$, and every $\varepsilon > 0$

$$\int_0^T \|u_h - u_h^\varepsilon\|_{swip,q} dt \leq C(k, d, \alpha, |Q|, q, \|A\|_{L^\infty(\Omega)^{d\times d}}) \|f - f^\varepsilon\|_{L^1(Q)}. \tag{3.9}$$

Also, as f^ε is square integrable and the shape-regularity of the family of triangulations, we obtain (see [8]) for each ε

$$u_h^\varepsilon \longrightarrow u^\varepsilon \text{ strongly in } L^2(Q),$$

$$\nabla u_h^\varepsilon \longrightarrow \nabla u^\varepsilon \text{ strongly in } \left[L^2(Q)\right]^d, \tag{3.10}$$

where h decreases to zero, where u^ε is the only solution of

$$\begin{cases} u^\varepsilon \in L^2(0,T; H_0^1(\Omega)) \\ \dfrac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(A\nabla u^\varepsilon) = f^\varepsilon \text{ in } \mathcal{D}'(Q). \end{cases} \tag{3.11}$$

The following problem has a unique renormalized solution, which is the function u^ε

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(A\nabla u^\varepsilon) & = f^\varepsilon \text{ in } \Omega \times (0,T), \\ & u^\varepsilon = 0 \text{ on } \partial\Omega \times (0,T), \\ & u^\varepsilon = u_0^\varepsilon \text{ on } \partial\Omega \times \{0\}. \end{cases} \tag{3.12}$$

The fact that

$$\int_0^T \|u^\varepsilon - u\|_{swip,q} dt \leq \|A\|_{L^\infty(\Omega)^{d\times d}} \|u^\varepsilon - u\|_{L^1(0,T;W_0^{1,q}(\Omega))} \tag{3.13}$$

alllows one to have

$$\int_0^T \|u^\varepsilon - u\|_{swip,q} dt \leq \|A\|_{L^\infty(\Omega)^{d\times d}} C(d, q, |Q|, \alpha) \|f - f^\varepsilon\|_{L^1(Q)} \tag{3.14}$$

for every q with $1 \leq q < \dfrac{d}{d-1}$, where u is the unique renormalized solution of (1.1).

We use the triangle inequality by considering (3.9), (3.10), (3.14), one has for every $\varepsilon > 0$ and every q with $1 \leq q < \dfrac{d}{d-1}$

$$\limsup_{h \longrightarrow 0} \int_0^T \|u^\varepsilon - u\|_{swip,q} dt \leq C\Big(d, q, |Q|, \alpha, \|A\|_{L^\infty(\Omega)^{d\times d}}\Big) \|f - f^\varepsilon\|_{L^1(Q)}. \tag{3.15}$$

Passing to the limit when ε tends to zero proves (3.5) and (3.6).

To complete this proof, we show (3.7). For this, we consider (see Proposition 15 in [7]) that

$$I_k^h(u_h^n) - T_k(u_h^n) \longrightarrow 0 \text{ in measure.}$$

Moreover, we have

$$I_h^k(u_h^n) \rightharpoonup T_k(u) \text{ weakly in } L^2(0,T;H_0^1(\Omega)) \tag{3.16}$$

and

$$|fI_h^k(u_h^n)| \leq C(k,d)|f| \in L^1(Q).$$

Then, by the compact injection theorem, we have

$$\int_Q fI_h^k(u_h^n)dxdt \longrightarrow \int_Q fT_k(u)dxdt.$$

When h tends towards zero in (3.3), we obtain

$$\limsup_{h\longrightarrow 0}\Big(\int_\Omega \widetilde{\nabla} I_h^k(u_h)\cdot I_h^k(u_h)dx + \int_0^T a_h^{swip}(I_h^k(u_h), I_h^k(u_h^n))dt\Big) \leq \int_Q fT_k(u)dxdt \tag{3.17}$$

Also, since u is the renormalized solution of (1.1), we have

$$\int_\Omega \widetilde{\nabla} T_k(u)\cdot T_k(u)dx + \int_0^T a_h^{swip}(T_k(u), T_k(u))dt = \int_Q fT_k(u)dxdt. \tag{3.18}$$

From (3.17) and (3.18), we deduce that

$$\limsup_{h\longrightarrow 0}\Big[\int_\Omega \widetilde{\nabla} I_h^k(u_h)\cdot I_h^k(u_h)dx + \int_0^T a_h^{swip}(I_h^k(u_h), I_h^k(u_h^n))dt\Big] \leq \int_\Omega \widetilde{\nabla} T_k(u)\cdot T_k(u)dx + \int_0^T a_h^{swip}(T_k(u), T_k(u))dt,$$

which associated with the weak convergence (3.16) gives the strong convergence (3.7). $\square$

Now, we give an estimation error of the strong convergence established in (3.3). For this, we assume that Ω and the elements of A are regular, and f belong to the Marcinkiewicz space $L^{s,\infty}(Q)$ with $s > 1$. The next result is another major finding of this manuscript.

Theorem 3.4. *According the hypotheses of Theorem (3.3) ($d = 2$ or $d = 3$), if f belongs to $L^{s,\infty}(\Omega\times(0,T))$ for all s with $1 < s < 2$, Ω a convex polyhedron and $A \in [W^{1,\infty}(\Omega)]^{d\times d}$, then we have the following error estimate :*

$$\|u_h^n - u\|_{L^1(Q)} \leq C\Big(d, |Q|, \alpha, s, \|A\|_{W^{1,\infty}(\Omega)^{d\times d}}\Big)h^{2(1-\frac{1}{s})}\|f\|_{L^{s,\infty}(Q)}. \tag{3.19}$$

Proof. The unique solution $u_h^{n,\varepsilon}$ of (2.17) when the data $f^\varepsilon = T_{1/\varepsilon}(f) \in L^\infty(Q) \subset L^2(Q)$ satisfies

$$\int_0^T \|u_h^{n,\varepsilon} - u^\varepsilon\|_{swip}dt \leq C\Big(\|A\|_{W^{1,\infty}(\Omega)^{d\times d}}\Big)h\|f^\varepsilon\|_{L^2(Q)}$$

so

$$\int_0^T \|u_h^{n,\varepsilon} - u^\varepsilon\|_{swip,q}dt \leq C\Big(q, d, \eta, |Q|, \|A\|_{W^{1,\infty}(\Omega)^{d\times d}}\Big)h\|f^\varepsilon\|_{L^2(Q)} \tag{3.20}$$

Consider (3.9), (3.14) and (3.20). By the triangle inequality, we obtain :

$$\int_0^T \|u_h^n - u\|_{swip,q} dt \le C\Big(q, d, \eta, |Q|, \|A\|_{W^{1,\infty}(\Omega)^{d\times d}}\Big)\Big(\|f - f^\varepsilon\|_{L^1(Q)} + h\|f^\varepsilon\|_{L^2(Q)}\Big). \tag{3.21}$$

By considering the norm (2.4), we obtain :

$$\|f - f^\varepsilon\|_{L^1(Q)} \le \frac{1}{s-1}\varepsilon^{s-1}\|f\|^s_{L^{s,\infty}(Q)}, \tag{3.22}$$

and

$$\|f^\varepsilon\|^2_{L^2(Q)} \le \frac{2}{s-2}\frac{1}{\varepsilon^{2-s}}\|f\|^s_{L^{s,\infty}(Q)}. \tag{3.23}$$

Therefore, (3.21) gives :

$$\int_0^T \|u_h^n - u\|_{swip,q} dt \le C\Big(\frac{1}{s-1}\varepsilon^{s-1}\|f\|^s_{L^{s,\infty}(Q)} + \sqrt{\frac{2}{2-s}}\frac{h}{\varepsilon^{1-\frac{s}{2}}}\|f\|^{\frac{s}{2}}_{L^{s,\infty}(Q)}\Big). \tag{3.24}$$

By the Sobolev inequality, we obtain for every $q \ge 1$:

$$\|u_h^n - u\|_{L^q(Q)} \le \int_0^T \|u_h^n - u\|_{swip,q} dt.$$

We consider this inequality and we take $\varepsilon = \dfrac{h^{\frac{s}{2}}}{\|f\|_{L^{s,\infty}(Q)}}$ in (3.24). Finally, we have

$$\|u_h^n - u\|_{L^q(Q)} \le C\Big(d, q, s, |Q|, \|A\|_{W^{1,\infty}(\Omega)^{d\times d}}, \alpha\Big) h^{2(1-\frac{1}{s})}\|f\|_{L^{s,\infty}(Q)}.$$

□

4. Conclusion and Future scope

In the present paper, we developed and analyzed a discontinuous Galerkin method to approximation parabolic diffusion with L^1 data. Since such problems do not admit weak solutions, we use the renormalized solution framework. We establish the convergence of the discrete solution to the unique renormalized solution. We consider a priori estimates on $L^1(Q)$, which is not an ideal space for this numerical method.

We plan to adapt this numerical approximation to an elliptical problem involving variable exponent and measure data. Such problems appear for example in elasticity and image processing.

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