

MULTILINGUAL PRE-SERVICE TEACHER'S KNOWLEDGE AND UNDERSTANDING OF ANALYTICAL GEOMETRY CONTENT

SIBONGILE ZULU^{1*} AND DEONARAIN BRIJLALL²

ABSTRACT. This South African study explored how pre-service teachers understand concepts in analytical geometry. Extending the work of Skemp and Carrillo-Yañez, et al., we captured data and carried out this qualitative study. We noted that the participants displayed complete understanding for some geometrical concepts but lacked relational understanding. Their learning process was interrupted during the covid epidemic, and this had a negative impact on their relational understanding of geometric concepts.

Keywords: relational understanding, content knowledge, geometry education.

1. INTRODUCTION AND PRELIMINARIES

Training Mathematics Preservice Teachers (PTs) in multilingual teacher training classrooms presents the challenge of having to teach the Mathematics language and Mathematics content in English, which is the language of instruction for teaching Mathematics. Focus on the Mathematics language is crucial because the PTs' understanding of it has a direct impact on their ability to effectively impart their knowledge in their future practice. Most importantly, their understanding of the Mathematics language affects their understanding of the Mathematics content, which is the basis for pedagogical content knowledge[11]. For this reason, various researchers have called for more research on multilingual Mathematics PTs' learning of Mathematics [14], which has not received as much attention as the teaching of Mathematics in multilingual classrooms in primary and secondary schools.

This study examines multilingual PTs' knowledge and understanding of analytical geometry, also known as coordinate geometry, which is part of the special domain of Mathematics, geometry [9]. To fulfil its purpose, the study is guided by the following questions:

- (1) *How do multilingual Mathematics pre-service teachers understand analytical geometry content?*

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* Corresponding author.

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Key words and phrases. Instrumental, relational understanding, content knowledge.

- (2) *What knowledge of analytical geometry content do multilingual Mathematics pre-service teachers have?*

The response to Question 1 is guided by the framework of understanding in Mathematics [12],[13]. Question 2 is guided by the Mathematical knowledge domain of the Mathematics Teacher's Specialised Knowledge (MTSK) model [4]. The first author captured the data and formulated the draft paper. The second author was involved in the analysis of the data and the discussion thereof.

2. METHODOLOGY

This qualitative interpretative study seeks to understand and describe the experiences of the participants involved [6], [7] by analysing and interpreting the multilingual PTs' knowledge [2], [3] and understanding of analytical geometry. It is conducted on Mathematics PTs from one of the South African universities in the KwaZulu Natal (KZN) province. KZN is a province that is predominantly populated by Africans whose home language is not the language of instruction in the university of study. The sampled PTs are in their final year of taking Mathematics as a specialisation in the Bachelor of Education in senior phase and further education and training. They were sampled using non-probabilistic sampling method, which is based on the judgement of the researchers, and not randomly [5]. This method involves the selection of individuals that are proficient and well-informed about the concept of study [7].

Data collection was in the form of an analytical geometry test and individual interviews. The test covered all aspects of analytical geometry as prescribed in the South African CAPS documents for Mathematics in the senior phase (Grades 7 – 9) and further education and training phase (Grades 10 – 12). The test allowed the study to probe into the PTs' knowledge of Mathematical concepts that are interdependent in solving analytical geometry problems. The purpose of the interviews was to gain in-depth insight from the PTs about their experiences and understanding of analytical geometry concepts.

3. MAIN RESULTS

The collected data was analysed using the Mathematical knowledge and understanding framework. From the list of all participating PTs, twenty were sampled based on their test scores and ordered based on the following categories with five from each.

$$\begin{aligned} 0 &< mark \leq 40 \\ 40 &< mark < 50 \\ 50 &\leq mark < 70 \\ mark &\geq 70 \end{aligned}$$

The analysis process began with an analysis of the average performance of the 20 PTs on each sub question on the test, and then looked at the types of questions

on the test and how the PTs performed on them. Table 1 details the descriptors that were used to analyse the type of questions on the test.

<u>Types of questions</u>	<u>Descriptor</u>
1. Calculations	
1.1 Substitutions and algebraic manipulations	<i>Questions that require the basic use of formulae and equations. For example, finding the distance between two points.</i>
1.2 Application of geometric properties.	<i>Knowing and applying geometric properties in calculations.</i>
1.3 Integration of geometric concepts with other mathematical concepts.	<i>Being able to integrate different mathematical concepts to solve problems. For example, finding the angle of inclination of a line, which is also used to find the gradient of the line.</i>
2. Explanation/understanding of complex concepts and language involved.	<i>Questions that require an explanation or understanding a scenario for application in complex problems.</i>

Table 1: Descriptors for the types of questions on the test.

Furthermore, a diagnostic analysis of the PTs responses to the test questions is done to examine their knowledge and understanding by exploring some of their common misconceptions and errors. This is then supplemented by an analysis of their responses to the follow-up interviews, which were intended to further probe into the PTs' key problems and thinking in the content area.

The analysis process employs descriptors that were derived from the definitions of the subdomains of Mathematical knowledge, and those of instrumental and relational understanding. These are outlined in Table 2.

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4. DATA ANALYSIS AND DISCUSSION

This section provides a breakdown of the type of questions in the test, which had three major questions, each one with its sub questions. The section also provides the analysis and discussion of the collected data. Table 3(a) and Table 3(b) capture different concepts that were important for each sub question on the test, and their cognitive demands. The analysis explores the average performance of the PTs on each of the sub questions on the test, and thereafter, explores the misconceptions, errors, and the impact of language on the PTs' knowledge and understanding of analytical geometry.

Question	Concepts	Type of question	Cognitive demand
1.1	Gradient of a straight line.	Calculation: substitution	Lower-level demand: Procedures without connections.
1.2	Equation of a straight line. $y_2(x - x_1)$	Calculation: algebraic manipulation	Lower-level demand: Procedures without connections
1.3	Angle of inclination, CAST diagram and angle of a straight line, and working with trigonometric ratios.	Calculation: integration of concepts.	Higher-level demand: Procedures with connections.
1.4	Concept understanding of perpendicular bisector, relationship between the gradients of perpendicular lines, and manipulating the gradient formula and equation of a straight line.	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: procedures with connection
1.5	Properties of a kite, distance formula, and midpoint formula.	Calculation and explanation: application of geometric properties and integration of concepts.	Higher-level demand: Procedures with connections.
1.6	Area of a triangle, properties of triangles, and distance formula.	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: Procedures with connections.
2.1.1	Equation of a circle – substituting the center and radius.	No calculation: substitution	Lower-level demand: memorization
2.1.2	Properties of a circle and understanding the cartesian plane.	No calculation: substitution	Lower-level demand: memorization
2.1.3	Distance formula and understanding the equation of a circle.	Calculation: substitution	Lower-level demand: Procedures without connections.
2.1.4	Finding intercepts using equation of a circle or manipulating distance formula.	Calculation: algebraic manipulation	Lower-level demand: Procedures without connections.
2.1.5	Relationship between a tangent line and radius of a circle and finding gradient and equation of a line.	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: Procedures with connections.
2.2.1	Manipulating the equation of a circle.	Calculation: algebraic manipulation	Lower-level demand: Procedures without connections.
2.2.2	Calculating distance.	Calculation: substitution	Lower-level demand: Procedures without connections.
2.2.3	Using equations to draw circles on the cartesian plane and understanding the relationship between circles.	Explanation	Higher-level demand: Procedures with connections.
3.1	Equation of a circle, midpoint formula, and distance formula.	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: Procedures with connections.
3.2	Gradient of a line, relationship between perpendicular lines, and relationship between tangent to a circle and radius.	Calculation: application of geometric properties.	Higher-level demand: Procedures with connections.
3.3	The y-intercepts of a circle.	Calculation: algebraic manipulation and application of geometric properties	Higher-level demand: Procedures with connections.
3.4	Using angle of inclination to find angle between two lines. Angles of a straight line and a triangle. Using trig ratios.	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: Procedures with connections and doing mathematics.

Table 3: Breakdown of concepts in each question on the test.

4.1. Test sub-question average performance. The graph on Figure 1 indicates that most of the PTs performed better in calculation questions that simply require substitution and algebraic manipulations.

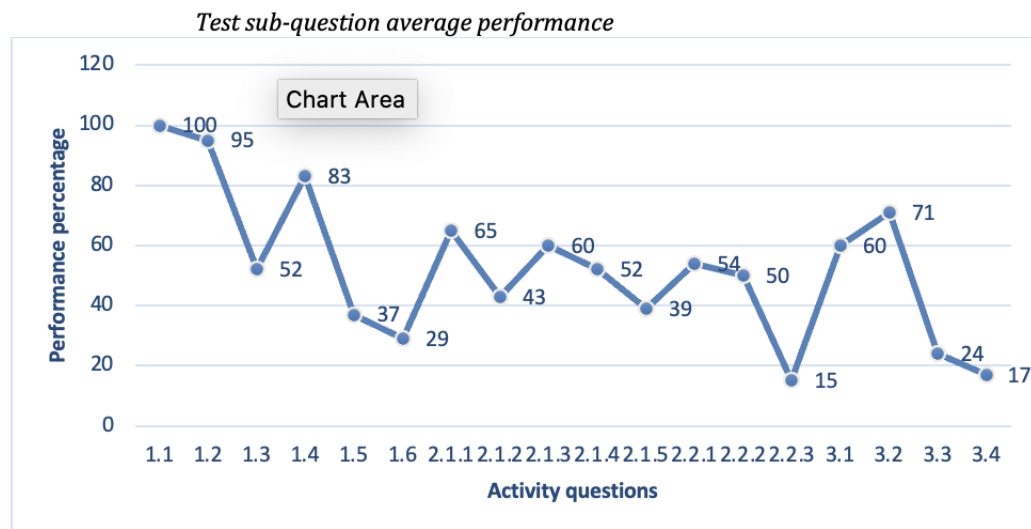


Figure 1: Sub-question average performance percentage of the PSTs.

These are questions of the lower-level demand where procedures are done without connections. This is evident in the high-performance percentages in sub questions 1.1 and 1.2, which were the most basic. The PTs had difficulties in solving problems of higher-level demand where they had to use procedures with connections. One such sub question is 3.3 in which they were supposed to determine the coordinates of V, the y-intercept of the circle as shown in Figure 2.

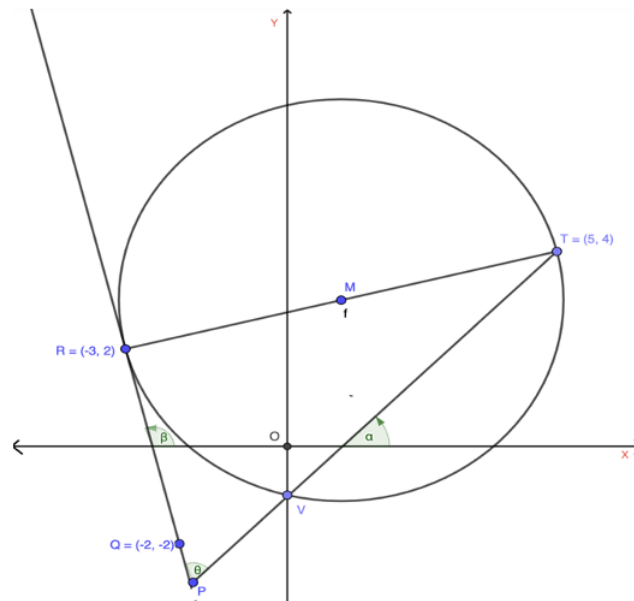
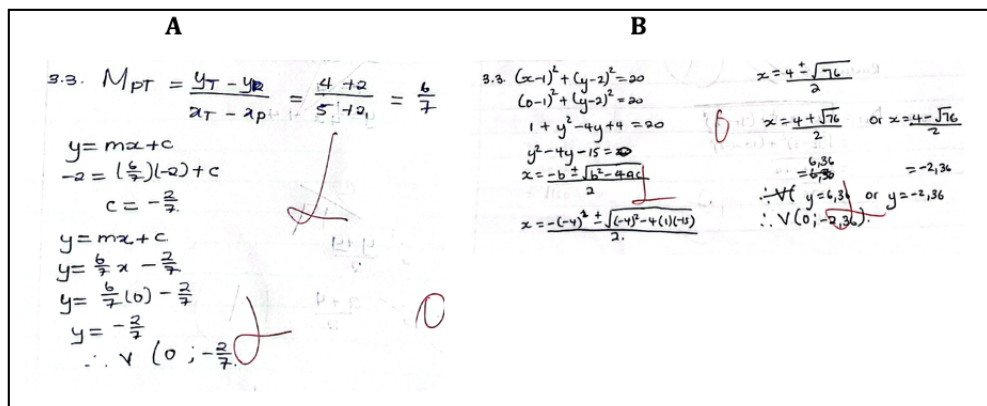


Figure 2: Diagram for Question 3

This sub question was a build up from 3.1 where they were supposed to find the equation of the circle, and for which the performance percentage was 71%. Yet, the average performance in sub question 3.3 was 24%. From the 20 sampled PTs 7 ran out of time and could not respond or finish the test, 8 scored zero, some of which had the correct equation of the circle but did not use it on this sub question. Only 3 got the total marks for the sub question, and the others got less than the total marks of 4. Box 1 shows some of the PTs' attempts to sub question 3.3.



Box 1: Examples A and B of attempts to sub question 3.3.

One common observation about the attempts in Box 1 is that they both expose a lack of relational understanding because the presented coordinates of V do not match the point V on the diagram. For example, the final answer in Attempt B is $v(0;-2.36)$, and yet point Q is below point V in position and its y value is 2. This shows that the PTs were simply using procedures without comprehension and could not make connections between their solutions and the presented problem. They had instrumental understanding, which is a disadvantage for them because of its rigid and context-dependent nature [8]. Attempt A, further confirms this by using the coordinated of Q in the place of P.

Sub questions 2.2.3 and 3.4 had the lowest average performance percentages of 15% and 17%, respectively. These are both higher level demand questions that required the use of procedures and ability to connect concepts, that is both instrumental and relational understanding. For sub question 2.2.3 which is shown in Figure 3, the PTs needed to have an understanding and a visual idea of properties of a circle. There was no need for the use of procedures as the response relied on answers to the preceding questions.

2.2 Two other circles are given:

- One has centre K, and equation $x^2 - 6x + y^2 + 4y - 12 = 0$.
- The other has centre T, and equation $(x - 12)^2 + (y - 10)^2 = 100$

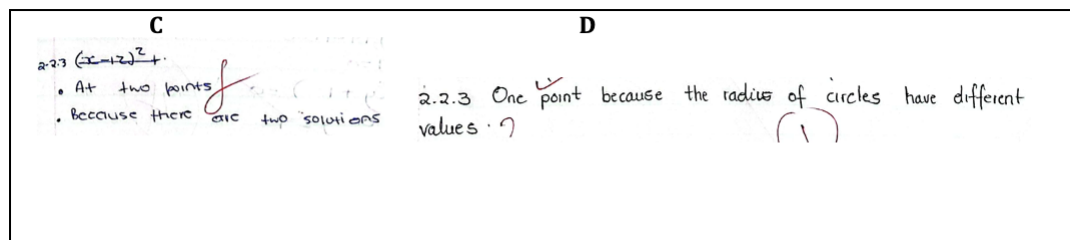
2.2.1 Determine the centre and radius of the circle with centre K. (4)

2.2.2 Calculate the distance between the centres, K and T. (2)

2.2.3 At how many points do the two circles intersect? Motivate your answer. (2)

Figure 3: sub questions leading to question 2.2.3.

Some of the PSTs got correct answers for sub questions 2.2.1 and 2.2.2 but could not connect the distance between the centers and the sum of the radii lengths in their response to sub question 2.2.3. This mishap is also related to their inability to visually imagine the two circles which indicates a lack of relational understanding as shown in Attempt D in Box 2.



Box 2: Examples C and D of attempts to sub question 2.2.3.

Attempt C is not clear in terms of what the PT was thinking, and they were not available for an interview where a probe into what the “two solutions” in their response meant. Looking at sub question 3.4, the PTs’ inability to link concepts is also observed on the attempts. The sub question is captured in Figure 4.

3.4 If $R\hat{P}T = \theta$, calculate θ to ONE decimal place. (7)

Figure 4: Sub question 3.4

This sub question was the least attempted with 11 of the 20 PTs not attempting it, and 5 of the remaining 9 scoring zero on their attempts. Only 3 PTs got the full 7 marks, and 1 got 3 out of 7 marks in this sub question. Box 3 shows an attempt from one of the PTs.

E

$$34. \text{ MPT} + \text{MTP} + \text{MPR} = 90 \text{ [Sum of L's in } \Delta \text{]}$$

$$\theta + 14.04 + 59.03 + 104.04 = 90$$

$$\theta + 177.11 = 90$$

$$\theta = 87.11$$

Box 3: Example E of attempts to sub question 3.4.

This attempt highlights some of the misconceptions and factors that contribute to the PTs' understanding of concepts. These are discussed in the next subsection of the data analysis and discussion.

4.2. Identified errors and misconceptions. The attempt in Box 3 indicates that the PT did not understand the difference between angles and gradients because they add gradients to get an answer in degrees. During the interview, the PT was asked about their reason on the first equation, and this is captured on Except 1.

Interviewer: What did you mean by "sum of angles in triangle" on your attempt, which angles were you referring to?

Attempt E PT: I think because I was talking about the right triangle.

Interviewer: Why is it a right triangle?

Attempt E PT: Eish I remember... (looks at diagram, and points at R) I know this is 90 degrees.

Except 1: Engagement about sub question 3.4

$\hat{R}$

is indeed 90° because of the tangent radius theorem, and in sub-question 3.2, the PTs had to prove that PR is a tangent at R by using analytical methods showing that RT is perpendicular to RQ, but this PT still could not give a reason for the angle being 90 degrees. In this case, we could argue that the PT lacks both instrumental and relational understanding. This can be observed where the PT uses incorrect mathematical notations, and from equation 3 where $\theta + 177.11^\circ = 90^\circ$ leads to $\theta = 87^\circ.11^\circ$, which also indicates that the PT does not have adaptive reasoning[10]. Attempt E also shows that the PT does not have sufficient KoT and KSM [4]. The limited KoT is seen in the way they could not use analytical geometry concepts such as the angle of inclination to find the angle between the two intersecting lines. The PT could not use the inter and intra

relationships between Mathematics concepts, which also exposes limited KSM. Another sub question with the most observed errors and misconception was sub question 1.6 where the PTs had to identify the height of triangle ABD in Figure 5. The sub question had 29% average performance with most of the PTs only scoring a mark for the area formula and finding the length of BD.

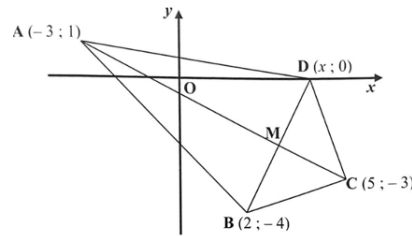
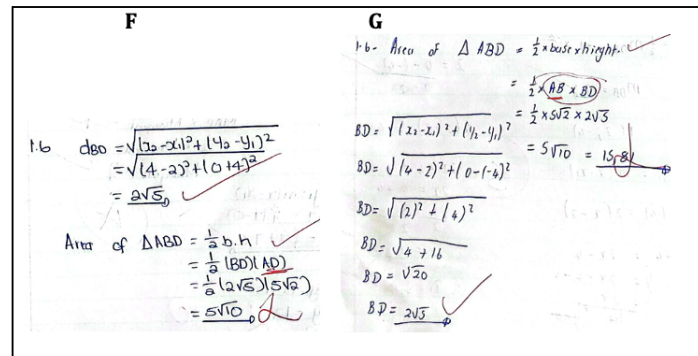


Figure 5: Question 1 diagram

In sub question 1.6 the PTs needed the coordinates of D which were calculated in sub question 1.4. Some of the attempts confirmed that some PTs rely on procedural or instrumental understanding [3] when solving problems. This can be observed in Box 4 Attempts F and G.

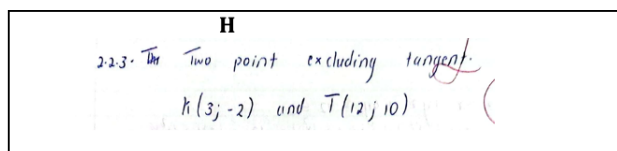


Box 4: Examples F and G of attempts to sub question 1.6.

The PTs failed to identify the height of the triangle and used AD and AB as the height. The challenge seems to be in understanding what the height of a triangle is. The PTs did not associate the height of a triangle with the altitude which coincides with the misconception that was noted in the NSC Diagnostic Report [5], which is the same year on which this group of PTs were in Grade 12. The diagnostic report states that the candidates substituted the length of one of the sides of a triangle as the perpendicular height. This misconception may still be there because, for this group of PTs, the first two years of their higher education was mostly online learning due to the Corona virus pandemic regulations. Unfortunately, the analytical geometry topic was covered in the first and second year of training in this institution.

4.3. The impact of language in the PTs' content knowledge and understanding. The PTs lack of understanding the Mathematics language and the language of instruction can be observed in Box 3 Attempt E and Except 1. This shows how the PT misuses the notation and finds it difficult to explain their reasoning mathematically. The impact of the language of instruction can also be

observed in Box 5 Attempt H and Except 2, where an attempt to sub question 2.2.3 and an engagement about it are captured.



Box 5: Examples F and G of attempts to sub question 1.6.

Sub question 2.2.3 can be seen in Figure 3. Attempt H highlights the impact of language in the PTs' understanding and use of mathematical language. The PT explained that the circles intersect at two points excluding the tangent. However, this response suggests that the PT is of the view that a tangent also intersects the circle. The PT was engaged about this attempt, and their response which confirms the observation in the attempt is captured in Except 2.

Interviewer: Why did you mention the tangent here?

PST₉: Because a tangent can also intersect a circle.

Interviewer: But the question is just about the two circles...

PST₉: I was making sure.

Except 2: Engagement about sub question 2.2.3

The PT may be of the view that the words 'intersect' and 'touch' mean the same thing. This confusion which is caused by language affected the PT's relational understanding. Noting K and T, the centers of the two circles also highlights that the PT might not have understood the question well. When they were asked about this, the PT could not justify this.

As already indicated, this group of PSTs had their learning interrupted from high school and into university. The missed content could not be covered extensively in their teacher training because of time and curriculum constraints. This has affected the development and improvement of their content knowledge and understanding in analytical geometry.

Yet, [8] emphasize that the knowledge that is needed by Mathematics teachers is inclusive of deep content knowledge and knowledge for teaching.

5. CONCLUSION

When it comes to the mathematical knowledge, we found that the PTs had limited KoT and KSM as their test responses show that they mostly rely on instrumental understanding only. They performed well in questions that required procedures without connections. Questions that required the application of concepts from other topics did not receive the same performance. This finding suggests that the PTs may not be able to demonstrate clear understanding of how analytical geometry concepts are used in practical applications, which implies lack of KPM. The KPM subdomain was difficult to observe directly from the PTs' attempts because the test did not have application questions.

Having established that most of these PTs were in the critical grades or starting teacher training when the Coronavirus pandemic struck, we also noted that their performance in the test corresponds with the performance of the 2020 matriculants in analytical geometry. The Diagnostic Report [5] shows that many of these matriculants had difficulty with substituting correct values to the formula $\tan \theta = m$, which we also found in the PTs' responses to sub question 3.4. We also noted that the background of the PTs affects their development of content knowledge because some of their misconceptions seem to stem from their command and understanding of the language of instruction, which plays a major role in the Mathematics register [14].

6. LIMITATIONS OF THE STUDY

This qualitative study came with the limitation of having findings that are context based [15] because it was conducted in a university that is based in the KZN province of South Africa where the population is mostly African. Therefore, the findings are highly contextual, and applicable to specific institutions and PTs. Furthermore, we noted that the group of PTs who participated in this study were amongst those who had their education disturbed and affected by the emergency online teaching and learning that was introduced in the year 2020 and carried on for more than a year [1]. Therefore, the findings of this study may also be applicable only to the PTs in this cohort.

7. RECOMMENDATIONS

To close the gap that exists in Mathematics PTs' content knowledge and to improve the quality of teacher training and education in Mathematics, we must consult national diagnostic reports of each year. In this way, we are informed about the misunderstandings most PTs have and we could treat the issues seriously starting from the first year of each cohort. This would allow us to keep track of the quality of education in high school Mathematics, and also its subsequent progression to Mathematics education in tertiary level.

REFERENCES

1. Batmang, B., Sultan, M., Azis, A., and Gunawan, F. (2021). Perceptions of pre-service teachers on online learning during the COVID-19 pandemic. *International Journal of Education in Mathematics, Science and Technology*, 9(3), 449-461. <https://doi.org/10.46328/ijemst.1595>
2. Bjerke, A. H., and Eriksen, E. (2016). Measuring pre-service teachers' self-efficacy in tutoring children in primary mathematics: an instrument. *Research in Mathematics Education*, 18(1), 61-79. <https://doi.org/10.1080/14794802.2016.1141312>
3. Bjerke, A. H., and Solomon, Y. (2020). Developing self-efficacy in teaching mathematics: Pre-service teachers' perceptions of the role of subject knowledge. *Scandinavian Journal of Educational Research*, 64(5), 692-705. <https://www.tandfonline.com/doi/full/10.1080/00313831.2019.1595720>
4. Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., ... and Muñoz-Catalán, M. C. (2018). The mathematics teacher's specialised knowledge (MTSK) model. *Research in Mathematics Education*, 20(3), 236-253. <http://dx.doi.org/10.1080/14794802.2018.1479981>

5. Department of Basic Education (DBE). (2020). NSC Diagnostic Report. South Africa: Seriti Printing.
6. Etikan, I., and Bala, K. (2017). Sampling and sampling methods. *Biometrics and Biostatistics International Journal*, 5(6), 00149. DOI:10.15406/bbij.2017.05.00149
7. Etikan, I., Musa, S. A., and Alkassim, R. S. (2016). Comparison of convenience sampling and purposive sampling. *American journal of theoretical and applied statistics*, 5(1), 1-4. doi:10.11648/j.ajtas.20160501.11
8. Herheim, R. (2023). On the origin, characteristics, and usefulness of instrumental and relational understanding. *Educational Studies in Mathematics*, 113(3), 389-404. <https://doi.org/10.1007/s10649-023-10225-0>
9. Kholid, M. N., Pradana, L. N., Maharani, S., and Swastika, A. (2022). GeoGebra in Project-Based Learning (Geo-PJBL): A dynamic tool for analytical geometry course. *JOTSE*, 12(1), 112-120. <https://doi.org/10.3926/jotse.1267>
10. Findell, B., Swafford, J., and Kilpatrick, J. (Eds.). (2001). Adding it up: Helping children learn mathematics. National Academies Press. https://books.google.co.za/books?id=pvI7uDPo0-YC&lpg=PA1&ots=vD_391xrfk&dq=Kilpatrick%2C%20Swafford%2C%20and%20Findell%2C%202001&lr&pg=PA1#v=onepage&q=Kilpatrick,%20Swafford,%20and%20Findell,%202001&f=false
11. Rosli, R., Goldsby, D., Onwuegbuzie, A. J., Capraro, M. M., Capraro, R. M., and Gonzalez, E. G. Y. (2020). Elementary Preservice Teachers' Knowledge, Perceptions and Attitudes towards Fractions: A Mixed-Analysis. *Journal on Mathematics Education*, 11(1), 59-76. <http://ejournal.unsri.ac.id/index.php/jme/issue/view/408>
12. Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics teaching*, 77(1), 20-26. <http://www.davidtall.com/skemp/pdfs/instrumental-relational.pdf>
13. Skemp, R. R. (2006). Relational understanding and instrumental understanding. *Mathematics teaching in the middle school*, 12(2), 88-95. <https://gargicollge.in/wp-content/uploads/2020/03/Skemp-article.pdf>
14. Thompson, D. R., Kersaint, G., Vorster, H., Webb, L., and Van der Walt, M. S. (2016). Addressing multi-language diversity in mathematics teacher education programs. *Mathematics education and language diversity: The 21st ICMI Study*, 121-139. https://doi.org/10.1007/978-3-319-14511-2_7
15. Queirós, A., Faria, D., and Almeida, F. (2017). Strengths and limitations of qualitative and quantitative research methods. *European journal of education studies*. www.oapub.org/edu

¹ SCHOOL OF EDUCATION, DURBAN UNIVERSITY OF TECHNOLOGY, SOUTH AFRICA.
Email address: sibongilez2@dut.ac.za

² DEPARTMENT OF MATHEMATICS, DURBAN UNIVERSITY OF TECHNOLOGY, SOUTH AFRICA
Email address: deonarainb@dut.ac.za