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NIHO BENT FUNCTIONS AND OVALS IN SMALL DIMENSIONS

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Abstract. We consider Niho bent functions in small dimensions. They are in one-to-one correspondence with ovals in the projective plane $PG(2,q)$, $q=2^m$. Niho bent functions and ovals are listed up to equivalency for dimensions $m \leq 6$. They are described by means of g-functions. We also give characterization of supports of Niho bent functions and their duals in terms of associated ovals.

1. Introduction

Bent functions were introduced by Rothaus [31] and then they were studied by Dillon [20]. A bent function is a Boolean function with an even number of variables that achieves the maximum possible distance from the set of affine functions [17, 27]. Bent functions have relations to coding theory, cryptography, sequences, combinatorics, and design theory [1, 2, 17, 18, 27, 21].

Dillon [20] introduced bent functions related to partial spreads of $F \times F$, $F = \mathbb{F}_{2^m}$. He constructed bent functions that are constant on each element of a spread. Additionally, Dillon [20] studied a class of bent functions that are linear on the elements of a Desarguesian spread. These functions were thoroughly studied in [7, 8, 13, 14, 15, 16, 22, 23, 26] as Niho bent functions.

Denote by $PG(2,q)$ a finite projective plane of order $q=2^m$. An *oval* [24] is a set of $q+1$ points, no three of which are collinear; and a *hyperoval* is a set of $q+2$ points, no three of which are collinear. Each oval can be uniquely extended to a hyperoval [24], contrariwise, taking a point out from a hyperoval will produce an oval. A list of known hyperovals can be seen, for example, in [10]. Hyperovals are classified up to dimension $m=6$ [28, 30, 32].

In [3, 4] it was shown that the equivalence classes of Niho bent functions are in one-to-one correspondence with the equivalence classes of ovals in the projective plane $PG(2,q)$. Therefore, each hyperoval determines several equivalence classes of ovals. Consequently, each hyperoval defines several equivalence classes of Niho bent functions.

In [3, 4] special g-functions were introduced as a new analog of o-polynomials in case of univariate presentation of functions and hyperovals. In our paper, we

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describe Niho bent functions and ovals in the language of g-functions. We also describe supports of those functions and their duals in terms of associated ovals.

This paper finds all ovals and equivalence classes of Niho bent functions in small dimensions $m \leq 6$. We describe corresponding g-functions in explicit forms. The main motivation to find all g-functions for $m \leq 6$ is the following: Assume that there is a new series of g-functions, then for $m \leq 6$, these g-functions should be equivalent to some functions from our list. Therefore, one could guess possible new g-functions for $m > 6$ (see Remark 5.3 in Section 5 for details). In other words, finding the explicit list of all g-functions in dimensions $m \leq 6$ is the essential step toward a full classification of hyperovals in $PG(2, q)$.

The outline of the paper is as follows: First, in Section 2 we recall definitions and notation regarding projective planes, ovals, and bent functions. In Section 3 we provide characterization of supports of Niho bent functions and their duals in terms of associated ovals. In Section 4 we present general formulas for Niho bent functions and ovals corresponding to a fixed hyperoval (Theorems 4.1 and 4.2). In Section 5 (Theorem 5.1) we describe all Niho bent functions and ovals in small dimensions up to $m = 6$.

2. Preliminary considerations and notation

In this section, some definitions and notations are recalled.

2.1. **Polar representations.** Denote by $F = \mathbb{F}_{2^m}$ a finite field of order $q = 2^m$. F is a subfield of $K = \mathbb{F}_{2^n}$, where $n = 2m$, thus K can be considered as a two-dimensional vector space over F.

For a finite field extension M/L, we denote by $Tr_{M/L}(x)$ the corresponding trace function. Furthermore, we denote $Tr(x) = Tr_{K/\mathbb{F}_2}(x)$, $T(x) = Tr_{K/F}(x)$, $tr(x) = Tr_{F/\mathbb{F}_2}(x)$. The *conjugate* of $x \in K$ over F is $\bar{x} = x^q$. Then the *trace* and the *norm* maps from K to F are

$$T(x) = Tr_{K/F}(x) = x + \bar{x} = x + x^q,$$

$$N(x) = N_{K/F}(x) = x\bar{x} = x^{1+q}.$$

The elements of norm 1 form the *unit circle* of K:

$$S = \{u \in K : u\bar{u} = 1\}.$$

Hence, S is the multiplicative group of $(q+1)$st roots of unity in K, $|S| = q+1$, and each non-zero element of K has a unique polar coordinate representation $x = \lambda u$ with $\lambda \in F^*$ and $u \in S$. For any $x \in K^*$ we have $\lambda = \sqrt{x\bar{x}}$ and $u = \sqrt{x/\bar{x}}$.

A nondegenerate bilinear form $\langle \cdot, \cdot \rangle : K \times K \to F$ is defined [4] by

$$\langle x, y \rangle = T(x\bar{y}) = x\bar{y} + \bar{x}y.$$

Then the form $\langle \cdot, \cdot \rangle$ is alternating and symmetric: $\langle a, a \rangle = 0$, $\langle a, b \rangle = \langle b, a \rangle$.

2.2. **Projective and affine planes.** Consider points of a projective plane $PG(2,q)$ in homogeneous coordinates [24] as triples $(x:y:z)$, where $x,y,z \in F$, $(x,y,z) \neq (0,0,0)$, and we identify $(x:y:z)$ with $(\lambda x:\lambda y:\lambda z)$, $\lambda \in F^*$. Then points of $PG(2,q)$ are

$$\{(x:y:1) \mid x \in F,\ y \in F\} \cup \{(x:1:0) \mid x \in F\} \cup \{(1:0:0)\},$$

For $a,b,c \in F$, $(a,b,c) \neq (0,0,0)$, the line $[a:b:c]$ in $PG(2,q)$ is defined as

$$[a:b:c] = \{(x:y:z) \in PG(2,q) \mid ax+by+cz = 0\}.$$

The line $[a:b:c]$ is incident with the point $(x:y:z)$ if and only if $ax+by+cz=0$.

Points of the form $(x:y:0)$ are called the points at infinity. Then $[0:0:1]$ refers to the line at infinity (it consists of all points at infinity). An affine plane $AG(2,q)$ is defined as $AG(2,q) = PG(2,q) \setminus [0:0:1]$, thus points of this affine plane $AG(2,q)$ are

$$\{(x:y:1) \mid x,y \in F\}.$$

Associating $(x:y:1)$ with (x,y). one can identify [4] points of $AG(2,q)$ with elements of the vector space

$$V(2,q) = \{(x,y) \mid x,y \in F\},$$

thus we can write $AG(2,q) = V(2,q)$.

Lines in $AG(2,q) = V(2,q)$ are $\{(c,y) \mid y \in F\}$ and $\{(x,xb+a) \mid x \in F\}$, $a,b,c \in F$. We can define these lines by equations $x=c$ and $y=xb+a$.

Now, another representation of $PG(2,q)$ is introduced using the field K. Consider pairs $(x:z)$, such that $x \in K$, $z \in F$, $x \neq 0$ or $z \neq 0$, and identify $(x:z)$ with $(\lambda x:\lambda z)$, $\lambda \in F^*$. Then points of $PG(2,q)$ are

$$\{(x:1) \mid\ x \in K\} \cup \{(u:0) \mid u \in S\}.$$

For $\alpha \in K$ and $\beta \in F$, lines $[\alpha:\beta]$ in $PG(2,q)$ are defined as

$$[\alpha:\beta] = \{(x:z) \in PG(2,q) \mid \langle \alpha, x\rangle + \beta z = 0\}.$$

Pairs $[\alpha:\beta]$ and $[\lambda\alpha:\lambda\beta]$ with $\lambda \in F^*$ represent same lines. The point $(x:z)$ is incident with the line $[\alpha:\beta]$ if and only if $\langle \alpha, x\rangle + \beta z = 0$.

Element $u_\infty = (u:0)$, $u \in S$ is called the point at infinity in the direction of u. Thus $[0:1]$ represents the line at infinity.

We define an affine plane $AG(2,q) = PG(2,q) \setminus [0:1]$, therefore points of this affine plane $AG(2,q)$ are

$$\{(x:1) \mid x \in K\}.$$

Associating $(x:1)$ with $x \in K$ one can identify [4] points of $AG(2,q)$ with elements of the field K, and we will write $AG(2,q) = K$.

2.3. **Ovals and hyperovals.** Let $PG(2,q)$ be a finite projective plane of order $q = 2^m$. An *oval* is a set of $q+1$ points, no three of which are collinear [24]. Any line of the plane meets the oval $\mathcal{O}$ at either 0, 1 or 2 points and is called exterior, tangent or secant, respectively. All the tangent lines to the oval $\mathcal{O}$ concur [24] at the same point N, called the *nucleus* of $\mathcal{O}$. The set $\mathcal{H} = \mathcal{O} \cup \{N\}$ becomes a *hyperoval*, that is a set of $q+2$ points, no three of which are collinear [24]. Conversely, taking any point out from a hyperoval gives an oval.

2.4. **Bent functions.** Consider the finite fields $K = \mathbb{F}_{2^n}$ and $\mathbb{F}_2$ of orders 2^n and 2 respectively. The field $\mathbb{F}_{2^n}$ is an $\mathbb{F}_2$-vector space of dimension n. A Boolean function on $\mathbb{F}_{2^n}$ is defined to be a mapping from $\mathbb{F}_{2^n}$ to the prime field $\mathbb{F}_2$.

If f is a Boolean function on $\mathbb{F}_{2^n}$, then the Walsh transform of f is defined [27] as follows:

$$W_f(b) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+b\cdot x}, \tag{2.1}$$

where $b \cdot x$ is an inner product from $\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$ to $\mathbb{F}_2$. A Boolean function f on $\mathbb{F}_{2^n}$ is said to be *bent* [27] if its Walsh transform satisfies $W_f(b) = \pm 2^{n/2}$ for all $b \in \mathbb{F}_{2^n}$. Then n is an even integer.

Let f be a bent function over $\mathbb{F}_{2^n}$, we can define its *dual function* [27], denoted by $\tilde{f}$, by considering the signs of the values of the Walsh transform $W_f(b)$ of f. More precisely, $\tilde{f}$ is defined by the equation:

$$(-1)^{\tilde{f}(x)} 2^{n/2} = W_f(x).$$

It is known that the dual of a bent function is bent, and $\tilde{\tilde{f}} = f$.

2.5. **Niho bent functions.** Desarguesian spread on $V(2,q)$ defined as a collection of $q+1$ one-dimensional subspaces $\{(x, xt) \mid x \in F\}$, $t \in F$, and $\{(0, y) \mid y \in F\}$ [19]. Thus every nonzero element of $V(2,q)$ lies in a unique subspace of those subspaces.

If we consider the field K as a two-dimensional vector space over F, then the set

$$\{uF : u \in S\}$$

is a spread on this vector space. Let us consider Boolean functions $f : K \to \mathbb{F}_2$, which are $\mathbb{F}_2$-linear on each element of the spread $\{uF : u \in S\}$. Then [4] for any $x = \lambda u \in K$, where $\lambda \in F$, $u \in S$, the function f can be defined by

$$f(x) = tr(\lambda g(u)) \tag{2.2}$$

for some function $g : S \to F$. If such a function is bent then it is called *Niho bent function* [22].

Let Boolean function $f : K \to \mathbb{F}_2$ be linear on elements of the spread $\{uF : u \in S\}$. Now we consider bent functions, and we use the scalar product $a \cdot b = tr(\langle a, b\rangle)$ in the formula (2.1) for the Walsh transform (as it was introduced in [4]).

The following result explains the connection between Niho bent functions and ovals.

Theorem 2.1 ([4])**.** *Let function $f : K \to \mathbb{F}_2$ be defined by $f(\lambda u) = tr(\lambda g(u))$, for some function $g : S \to F$. Then the following statements are equivalent:*

(1) *The function f is bent;*
(2) *The set $\{\frac{u}{g(u)}) \mid u \in S\}$ is an oval in $PG(2,q)$ whose nucleus is the origin.*

Here we assume that if $g(u) = 0$ then $\frac{u}{g(u)} = u_\infty$ is an element at infinity in the direction u.

Therefore, Niho bent functions are in one-to-one correspondence with the ovals in $PG(2,q)$ with nucleus at the origin. A function $g : S \to F$ is said to be

a g-function if it satisfies conditions of Theorem 2.1. Necessary and sufficient conditions for the existence of g-functions are given in [4].

Boolean functions $f, f' : \mathbb{F}_{2^n} \to \mathbb{F}_2$ are *extended-affine equivalent* [27] if there exist an affine permutation L of $\mathbb{F}_{2^n}$ and an affine function $\ell : \mathbb{F}_{2^n} \to \mathbb{F}_2$ such that $f'(x) = (f \circ L)(x) + \ell(x)$. We call such functions just *equivalent*.

Recall [24] that the automorphism (collineation) group of $PG(2, q)$ is the group $P\Gamma L(3, q) = PGL(3, q)\langle\sigma\rangle$ and the automorphism group of $AG(2, q)$ is $A\Gamma L(2, q) = AGL(2, q)\langle\sigma\rangle$, where $\langle\sigma\rangle$ is the Galois group of F, and $AGL(2, q) = F^2 \cdot GL(2, q)$ is the affine group. Hyperovals (and ovals) are called (projectively) equivalent if they are equivalent under the action of the group $P\Gamma L(3, q)$.

Niho bent functions are equivalent if and only if the corresponding ovals (with nucleus at the origin) are projectively equivalent [3]. Note also that functions $g(u)$ and $g(u) + \langle c, u\rangle$, where $c \in K$, lead to equivalent ovals [4].

3. Characterization of Niho bent functions and duals in terms of ovals

Theorem 2.1 describes Niho bent functions and ovals in terms of g-functions. This section describes the supports of Niho bent functions and their duals in terms of associated ovals. Recall that the support [27] of a Boolean function $f(x)$ is defined by $\mathrm{Supp}(f) = \{x \in K \mid f(x) = 1\}$.

We define the following sets

$$V_1 = \{x \in F \mid tr(x) = 1\},\ G_x = \left\{ \frac{g(x^{(q-1)/2}) g(wx^{(q-1)/2})}{T(w)} \mid w \in S \setminus \{1\} \right\} \subset F.$$

The next theorem provides a characterization of supports of Niho bent functions and their duals in terms of associated ovals.

Theorem 3.1. *Let the set $\mathcal{O} = \{\frac{u}{g(u)} \mid u \in S\}$ be an oval in $AG(2, q) = K$ with nucleus at 0. Let Niho bent function $f : K \to \mathbb{F}_2$ be defined by $f(\lambda u) = tr(\lambda g(u))$. Then*

$$\mathrm{Supp}(f) = \bigcup_{x \in \mathcal{O}} xV_1,\ \mathrm{Supp}(\tilde{f} + 1) = \bigcup_{x \in \mathcal{O}} xG_x.$$

Proof. We can write $x \in \mathcal{O}$ as $x = \frac{u}{g(u)}$ for some $u \in S$. Then for $\nu \in F$ we have that

$$f(\frac{u}{g(u)}\nu) = 1 \Leftrightarrow f(\frac{\nu}{g(u)}u) = 1 \Leftrightarrow tr(\frac{\nu}{g(u)}g(u)) = 1 \Leftrightarrow r(\nu) = 1.$$

Further, by [4, Theorem 4.3], we have $\tilde{f}(y) = 0 \Leftrightarrow \langle v, y\rangle = g(v)$ for some $v \in S$. Therefore,

$$\begin{aligned}
\tilde{f}(\frac{u}{g(u)}\nu) = 0 \quad &\Leftrightarrow \quad \langle v, \frac{u}{g(u)}\nu\rangle = g(v) \text{ for some } v \in S \\
&\Leftrightarrow \quad \nu\langle v, u\rangle = g(u)g(v) \text{ for some } v \in S \\
&\Leftrightarrow \quad \nu = \frac{g(u)g(v)}{\langle v, u\rangle} \text{ for some } v \in S \setminus \{u\} \\
&\Leftrightarrow \quad \nu = \frac{g(u)g(v)}{\langle v\bar{u}, 1\rangle} \text{ for some } v \in S \setminus \{u\} \\
&\Leftrightarrow \quad \nu = \frac{g(u)g(uw)}{\langle w, 1\rangle} \text{ for some } w \in S \setminus \{1\} \\
&\Leftrightarrow \quad \nu = \frac{g(u)g(uw)}{T(w)} \text{ for some } w \in S \setminus \{1\}.
\end{aligned}$$

If $x = \frac{u}{g(u)} \in \mathcal{O}$ then $x^{(1-q)/2} = u^{(1-q)/2} = u^{(2-(q+1))/2} = u$. Hence, $\tilde{f}(x\nu) = 0$ $\Leftrightarrow \nu = \frac{g(x^{(1-q)/2})g(wx^{(1-q)/2})}{T(w)}$ for some $w \in S \setminus \{1\}$. The proof is complete. □

Introducing the set $G'_v = \left\{ \frac{g(wv)}{T(w)} \mid w \in S \setminus \{1\} \right\} \subset F$, for all $v \in S$, we have the following

Corollary 3.2. *Let the set $\mathcal{O} = \{\frac{u}{g(u)} \mid u \in S\}$ be an oval in $AG(2, q) = K$ with nucleus at 0. Let Niho bent function $f : K \to \mathbb{F}_2$ be defined by $f(\lambda u) = tr(\lambda g(u))$. Then*

$$\mathrm{Supp}(f) = \bigcup_{v\in S} vg(v)^{-1}V_1, \ \mathrm{Supp}(\tilde{f} + 1) = \bigcup_{v\in S} vG'_v.$$

4. Finding Niho bent functions and ovals in small dimensions

In this Section we present formulas to find all Niho bent functions (equivalently ovals with nucleus at 0) corresponding to a hyperoval in $AG(2, 2^m)$. In other words, we have to describe all inequivalent g-functions corresponding to a given hyperoval. Recall that, for a g-function $g(u)$, the corresponding Niho bent function is defined by $f(x) = tr(\lambda g(u))$, where $\lambda = \sqrt{x\bar{x}}$ and $u = \sqrt{x/\bar{x}}$.

Theorem 4.1. *Let $\mathcal{O}$ be an oval in K with nucleus at 0. Then*

$$\mathcal{O} = \left\{ \frac{u}{g(u)} \mid u \in S \right\},$$

where

$$g(u) = \sum_{i=0}^{q} \sum_{v\in\mathcal{O}} v^{(q-1)i/2-1} u^{i+1}.$$

Proof. Let $\mathcal{O}$ be an oval in K with nucleus at 0. Then $\mathcal{H} = \mathcal{O} \cup \{0\}$ is a hyperoval. Fix $u \in S$. Then the line uF intersects $\mathcal{H}$ in two points, 0 and νu, $\nu \in F^*$. Define

$g(u) = 1/\nu$. This allows us to determine a function $g : S \to F$ such that

$$\mathcal{O} = \left\{ \frac{u}{g(u)} \mid u \in S \right\}.$$

If $v \in \mathcal{O}$ then $v = w/g(w)$ for some $w \in S$. Hence we have

$$v^{(q-1)/2} = (w/g(w))^{(q-1)/2} = w^{(q-1)/2} = w^{(q+1-2)/2} = w^{-1} = \bar{w}.$$

Now fix $u \in S$. If $v \in \mathcal{O}$, $v = w/g(w)$ and $w = u$ then

$$\sum_{i=0}^{q} (uv^{(q-1)/2}))^i \ \frac{u}{v} = \sum_{i=0}^{q} (u\bar{w})^i \ \frac{u}{v} = \frac{u}{v} = g(u).$$

If $v \in \mathcal{O}$, $v = w/g(w)$ and $w \neq u$ then

$$\sum_{i=0}^{q} (uv^{(q-1)/2})^i \ \frac{u}{v} = \sum_{i=0}^{q} (u\bar{w})^i \ \frac{u}{v} = 0.$$

Therefore,

$$g(u) = \sum_{v \in \mathcal{O}} \sum_{i=0}^{q} (uv^{(q-1)/2}))^i \ \frac{u}{v} = \sum_{i=0}^{q} \sum_{v \in \mathcal{O}} v^{(q-1)i/2-1} u^{i+1},$$

which proves the theorem. □

Given a hyperoval in a projective plane $PG(2, q)$, an oval can be obtained by removing one of the points of the hyperoval [24]. This deleted point is the nucleus of the obtained oval. In this way, $q + 2$ ovals can be obtained, however, some of them will be equivalent under the action of the collineation group $P\Gamma L(3, q)$ of $PG(2, q)$. If P and Q are distinct points of a hyperoval $\mathcal{H}$, then ovals $\mathcal{H} \setminus \{P\}$ and $\mathcal{H} \setminus \{Q\}$ are equivalent if and only if P and Q lie in the same orbit of the collineation group $Aut(\mathcal{H})$ of $\mathcal{H}$ on $\mathcal{H}$ [3]. Hence, the number of projectively inequivalent ovals obtained from hyperoval $\mathcal{H}$ is equal to the number of orbits of $\mathcal{H}$ under the action of the collineation group $Aut(\mathcal{H})$.

Suppose that $g(u) \neq 0$ for all $u \in S$ and the set $\mathcal{H} = \left\{ \frac{u}{g(u)} \mid u \in S \right\} \cup \{0\}$ is a hyperoval in K. Define an oval $\mathcal{H} \setminus \{t/g(t)\}$, $t \in S$, whose nucleus is $t/g(t)$, and shift it by the element $t/g(t)$ in order to get an oval $\mathcal{O}_t$, whose nucleus is the origin:

$$\mathcal{O}_t = \left\{ \frac{v}{g(v)} + \frac{t}{g(t)} \mid v \in S, v \neq t \right\} \cup \left\{ \frac{t}{g(t)} \right\}.$$

Any Niho bent function associated with the hyperoval $\mathcal{H}$ is equivalent to a Niho bent function obtained from one of the ovals $\mathcal{O}_t$, and two such bent functions $f_t(x)$ and $f_s(x)$ are equivalent [3] if and only if the points $t/g(t)$ and $s/g(s)$ are in the same orbit under action of collineation group of the hyperoval $\mathcal{H}$.

Theorem 4.2. *Let $\mathcal{H}$ be a hyperoval in K defined by a function $g(u)$. Then the function $g_t(u)$ associated with the oval $\mathcal{O}_t$, whose nucleus is the origin, is equal to*

$$g_t(u) = \sum_{i=0}^{q} a_i u^{i+1},$$

where

$$a_i = g(t)t^{(q-1)i/2-1} + g(t)\sum_{v\in S, v\neq t} g(v)\big(g(t)v + tg(v)\big)^{(q-1)i/2-1}.$$

Proof. From Theorem 4.1 we have

$$\begin{aligned} a_i &= \sum_{z\in\mathcal{O}_t} z^{(q-1)i/2-1} \\ &= \left(\frac{t}{g(t)}\right)^{(q-1)i/2-1} + \sum_{v\in S, v\neq t}\left(\frac{v}{g(v)} + \frac{t}{g(t)}\right)^{(q-1)i/2-1} \\ &= \left(\frac{t}{g(t)}\right)^{(q-1)i/2-1} + \sum_{v\in S, v\neq t}\left(\frac{g(t)v + tg(v)}{g(t)g(v)}\right)^{(q-1)i/2-1} \\ &= g(t)t^{(q-1)i/2-1} + g(t)\sum_{v\in S, v\neq t} g(v)\big(g(t)v + tg(v)\big)^{(q-1)i/2-1}, \end{aligned}$$

which is our claim. □

5. Ovals and Niho bent functions in small dimensions

In this Section we will find all Niho bent functions (equivalently, ovals with nucleus at 0) corresponding to each hyperoval in $AG(2, 2^m)$, for $m \leq 6$. Essentially, we need to find all inequivalent g-functions that correspond to a given hyperoval. Recall that hyperovals are classified up to dimension $m = 6$ [10, 28, 30, 32], so it allows us to find all Niho bent functions in small dimensions $m \leq 6$.

Theorem 4.2 allows us to find all g-functions, and consequently ovals and Niho bent functions, associated with a fixed hyperoval. We use Magma [11] for calculations.

Theorem 5.1. *All g-functions, associated with inequivalent ovals in $AG(2, 2^m)$, $m \leq 6$, are listed in Tables 1 - 7 (see Appendix).*

Proof. For hyperovals up to dimension $m = 6$, the corresponding g-functions are taken from [4, 5]. Each hyperoval produces several ovals. We have to find g-functions of that ovals. Results are listed in Tables 1 - 7 (see Appendix), where the first g-function is the standard one associated with the hyperoval.

For $m = 1$, the hyperconic is only one hyperoval. Its automorphism group is transitive [24], so it gives only one orbit and only one g-function.

For $m = 2$, there is also only one hyperoval which is the hyperconic and similarly its automorphism group is transitive [24], hence we have only one g-function.

For $m = 3$, there is also only one hyperoval which is the hyperconic but here its automorphism group gives two orbits [24], hence we have two g-functions.

For $m = 4$, there are two inequivalent hyperovals which are the hyperconic and the Lunelli-Sce hyperoval. Their automorphism groups give two orbits [24] and one orbit [5, 12, 25], respectively. The automorphism group of the Lunelli-Sce hyperoval is transitive on its points.

Table 1 (see Appendix) shows the associated g-functions and Niho bent functions in dimensions $m \leq 4$.

For $m = 5$, let ε be a primitive element of K with $\varepsilon^{10}+\varepsilon^6+\varepsilon^5+\varepsilon^3+\varepsilon^2+\varepsilon+1 = 0$. Let $w = \varepsilon^{31}$, then $S = \langle w\rangle$, $|S| = 33$. Let $\omega \in S, \omega^3 = 1, \omega \neq 1$. Define $a_i = \dfrac{w^i}{g(w^i)} \in \mathcal{O}$.

In this dimension there are only 6 inequivalent hyperovals which are the hyperconic, translation, Segre, Subiaco (Payne), Cherowitzo and O'Keefe-Penttila hyperovals [10]. The hyperconic, translation and Segre hyperovals were analyzed in [6]. Their automorphism groups produce 2 orbits, 3 orbits and 2 orbits, respectively. The related g-functions are presented in Table 2 (see Appendix).

The automorphism group of the Subiaco hyperoval, defined by $g(u) = 1+u^5+\bar{u}^5+u+\bar{u}$, is equal to the Galois group $Gal(K/\mathbb{F}_2) = \langle\sigma\rangle$ [4]. It produces 6 orbits with representatives 0, 1, a_1, a_3, a_5 and a_{11}, and the associated g-functions are presented in Table 3 (see Appendix).

For the Cherowitzo hyperoval, defined by $g(u) = u^5+u^8+u^9+\omega u^{12}+\omega u^{13}+\omega u^{16}+\bar{u}^5+\bar{u}^8+\bar{u}^9+\bar{\omega}\bar{u}^{12}+\bar{\omega}\bar{u}^{13}+\bar{\omega}\bar{u}^{16}$, the automorphim group is generated by the automorphism σ^2 [4]. It gives 10 orbits with representatives 0, a_0, a_1, a_2, a_3, a_5, a_6, a_7, a_{11} and a_{22}. The corresponding g-functions are presented in Table 4 (see Appendix).

Finally, the O'Keefe-Penttila hyperoval [29], defined by $g(u) = 1+\varepsilon^{123}u^9+u^{12}+\bar{\varepsilon}^{123}\bar{u}^9+\bar{u}^{12}$ [6], has 12 orbits with representatives 0 and a_i, $0 \leq i \leq 10$. The automorphism group of the O'Keefe-Penttila hyperoval is generated by the map $\varphi_\omega(x) = \omega x$ [6]. Table 5 (see Appendix) shows the g-functions that correspond to the O'Keefe-Penttila hyperoval.

For $m = 6$, let α be a primitive element of K satisfying $\alpha^{12}+\alpha^7+\alpha^6+\alpha^5+\alpha^3+\alpha+1 = 0$. Let $w = \alpha^{63}$, then $S = \langle w\rangle$, $|S| = 65$. Define $b_i = \dfrac{w^i}{g(w^i)} \in \mathcal{O}$. Define also $\theta = \alpha^{273} \in K$ so that $\theta^4+\theta+1 = 0$, $\theta^{15} = 1$ and $\theta \in \mathbb{F}_{16}$.

For $m = 6$ there are only 4 inequivalent hyperovals which are the hyperconic, two different Subiaco hyperovals, and the Adelaide hyperoval [10, 32].

The hyperconic was analyzed in [24], and its automorphism group gives 2 orbits.

Define $\varphi_v(x) = vx$, $v \in S$, and $\sigma(x) = x^2$, $x \in K$. For the Subiaco hyperoval defined by $g(u) = 1+u^5+\bar{u}^5$, its automorphism group is equal to the semidirect product $\langle\varphi_v\rangle\langle\sigma\rangle$, where $v = w^{(q+1)/5}$ [4]. It has 3 orbits with representatives 0, b_0 and b_1.

For the Subiaco hyperoval defined by $g(u) = 1+wu^5+\bar{w}\bar{u}^5$, its automorphim group is equal to the cyclic group $\langle\varphi_{w^3}\sigma^4\rangle$ [4]. It gives 6 orbits with representatives 0, b_0, b_1, b_2, b_5 and b_7.

Finally, the automorphism group of the Adelaide hyperoval, defined by $g(u) = 1+u^{(q-1)/3}+\bar{u}^{(q-1)/3} = 1+u^{21}+\bar{u}^{21}$, is equal to the Galois group $Gal(K/\mathbb{F}_2)$ [4], and it produces 8 orbits with representatives 0, 1, b_1, b_3, b_5, b_7, b_{11} and b_{13}.

Tables 6 and 7 (see Appendix) show the g-functions corresponding to each orbit of the above hyperovals in $AG(2, 64)$. All g-functions were found by applying Theorem 4.2 and using Magma [11]. □

Dimension m	**Hyperoval**	**Functions** $g(u)$	**Niho bent function** $f(x)$
1	Hyperconic	1	$Tr(ax^3)$
2	Hyperconic	1	$Tr(ax^{10})$
3	Hyperconic	1	$Tr(ax^{36})$
		$1+u^4+\bar{u}^4$	$Tr(ax^{36}+x^{22})$
4	Hyperconic	1	$Tr(ax^{136})$
		$1+u^4+u^5+u^8+\bar{u}^4+\bar{u}^5+\bar{u}^8$	$Tr(ax^{136}+x^{106}+x^{226}+x^{76})$
	Lunelli-Sce	$1+u^5+\bar{u}^5$	$Tr(ax^{136}+x^{226})$

TABLE 1. Hyperoval and associated g-functions in small dimensions ($T(a)=1$)

Hyperoval	**Functions** $g(u)$
Hyperconic	1
	$1+u^4+u^5+u^8+u^9+u^{12}+u^{13}+u^{16}+$ $\bar{u}^4+\bar{u}^5+\bar{u}^8+\bar{u}^9+\bar{u}^{12}+\bar{u}^{13}+\bar{u}^{16}+u+\bar{u}$
Translation	$1+u^{16}+\bar{u}^{16}+u+\bar{u}$
	$1+u^8+u^9+u^{16}+$ $\bar{u}^8+\bar{u}^9+\bar{u}^{16}$
	$1+\omega u^4+\omega u^5+\omega u^8+\omega u^9+\omega u^{12}+\omega u^{13}+$ $\bar{\omega}\bar{u}^4+\bar{\omega}\bar{u}^5+\bar{\omega}\bar{u}^8+\bar{\omega}\bar{u}^9+\bar{\omega}\bar{u}^{12}+\bar{\omega}\bar{u}^{13}$
Segre	$1+\omega u^9+\bar{\omega}u^{12}+\bar{\omega}\bar{u}^9+\omega\bar{u}^{12}$
	$\omega u^4+u^5+\omega u^8+u^9+\omega u^{12}+u^{13}+$ $\bar{\omega}\bar{u}^4+\bar{u}^5+\bar{\omega}\bar{u}^8+\bar{u}^9+\bar{\omega}\bar{u}^{12}+\bar{u}^{13}+u+\bar{u}$

TABLE 2. Associated g-functions of hyperovals in $AG(2,32)$

Hyperoval	**Functions** $g(u)$
Subiaco (Payne)	$1+u^5+\bar{u}^5+u+\bar{u}$
	$1+u^5+u^8+u^{12}+u^{13}+\bar{u}^5+\bar{u}^8+\bar{u}^{12}+\bar{u}^{13}+u+\bar{u}$
	$u^4+\omega u^5+\bar{\omega}u^9+u^{12}+\bar{\omega}u^{16}+$ $\bar{u}^4+\bar{\omega}\bar{u}^5+\omega\bar{u}^9+\bar{u}^{12}+\omega\bar{u}^{16}$
	$1+\varepsilon^7u^4+\varepsilon^{50}u^5+\bar{\varepsilon}^{175}u^8+\bar{\varepsilon}^{59}u^9+\varepsilon^{729}u^{12}+\bar{\varepsilon}^{80}u^{13}+\varepsilon^{600}u^{16}+$ $\bar{\varepsilon}^7\bar{u}^4+\bar{\varepsilon}^{50}\bar{u}^5+\varepsilon^{175}\bar{u}^8+\varepsilon^{59}\bar{u}^9+\bar{\varepsilon}^{729}\bar{u}^{12}+\varepsilon^{80}\bar{u}^{13}+\bar{\varepsilon}^{600}\bar{u}^{16}+u+\bar{u}$
	$1+\varepsilon^3u^4+\bar{\varepsilon}^{11}u^5+\varepsilon^5u^8+\bar{\varepsilon}^{115}u^9+\varepsilon^{402}u^{12}+\varepsilon^{341}u^{13}+\bar{\varepsilon}^{82}u^{16}+$ $\bar{\varepsilon}^3\bar{u}^4+\varepsilon^{11}\bar{u}^5+\bar{\varepsilon}^5\bar{u}^8+\varepsilon^{115}\bar{u}^9+\bar{\varepsilon}^{402}\bar{u}^{12}+\bar{\varepsilon}^{341}\bar{u}^{13}+\varepsilon^{82}\bar{u}^{16}$
	$1+\varepsilon^{11}u^4+\varepsilon^{235}u^5+\varepsilon^{12}u^8+\varepsilon^{528}u^9+\bar{\varepsilon}^{464}u^{12}+\varepsilon^{536}u^{13}+\varepsilon^{333}u^{16}+$ $\bar{\varepsilon}^{11}\bar{u}^4+\bar{\varepsilon}^{235}\bar{u}^5+\bar{\varepsilon}^{12}\bar{u}^8+\bar{\varepsilon}^{528}\bar{u}^9+\varepsilon^{464}\bar{u}^{12}+\bar{\varepsilon}^{536}\bar{u}^{13}+\bar{\varepsilon}^{333}\bar{u}^{16}$

TABLE 3. Associated g-functions for Subiaco hyperoval in $AG(2,32)$

Hyperoval	**Functions** $g(u)$
Cherowitzo	$u^5+u^8+u^9+\omega u^{12}+\omega u^{13}+\omega u^{16}+$ $\bar{u}^5+\bar{u}^8+\bar{u}^9+\bar{\omega}\bar{u}^{12}+\bar{\omega}\bar{u}^{13}+\bar{\omega}\bar{u}^{16}$
	$1+u^4+u^5+\omega u^8+\omega u^9+\bar{\omega}u^{12}+\omega u^{13}+\omega u^{16}+$ $\bar{u}^4+\bar{u}^5+\bar{\omega}\bar{u}^8+\bar{\omega}\bar{u}^9+\omega\bar{u}^{12}+\bar{\omega}\bar{u}^{13}+\bar{\omega}\bar{u}^{16}+\bar{\omega}u+\omega\bar{u}$
	$u^4+\omega u^8+u^9+\omega u^{12}+u^{13}+u^{16}+$ $\bar{u}^4+\bar{\omega}\bar{u}^8+\bar{u}^9+\bar{\omega}\bar{u}^{12}+\bar{u}^{13}+\bar{u}^{16}+\omega u+\bar{\omega}\bar{u}$
	$1+u^4+\bar{\omega}u^5+\omega u^8+\bar{\omega}u^9+\omega u^{12}+u^{13}+\omega u^{16}+$ $\bar{u}^4+\omega\bar{u}^5+\bar{\omega}\bar{u}^8+\omega\bar{u}^9+\bar{\omega}\bar{u}^{12}+\bar{u}^{13}+\bar{\omega}\bar{u}^{16}+\bar{\omega}u+\omega\bar{u}$
	$1+\varepsilon^3u^4+\varepsilon^5u^5+\bar{\varepsilon}^{862}u^8+\varepsilon^{895}u^9+\bar{\varepsilon}^{191}u^{12}+\varepsilon^{894}u^{13}+\bar{\varepsilon}^{112}u^{16}+$ $\bar{\varepsilon}^3\bar{u}^4+\bar{\varepsilon}^5\bar{u}^5+\varepsilon^{862}\bar{u}^8+\bar{\varepsilon}^{895}\bar{u}^9+\varepsilon^{191}\bar{u}^{12}+\bar{\varepsilon}^{894}\bar{u}^{13}+\varepsilon^{112}\bar{u}^{16}+u+\bar{u}$
	$1+\varepsilon^9u^4+\bar{\varepsilon}^{81}u^5+\varepsilon^3u^8+\varepsilon^{133}u^9+\varepsilon^{411}u^{12}+\bar{\varepsilon}^{344}u^{13}+\bar{\varepsilon}^{84}u^{16}+$ $\bar{\varepsilon}^9\bar{u}^4+\varepsilon^{81}\bar{u}^5+\bar{\varepsilon}^3\bar{u}^8+\bar{\varepsilon}^{133}\bar{u}^9+\bar{\varepsilon}^{411}\bar{u}^{12}+\varepsilon^{344}\bar{u}^{13}+\varepsilon^{84}\bar{u}^{16}+\bar{\varepsilon}u+\varepsilon\bar{u}$
	$1+\varepsilon^{15}u^4+\varepsilon^{344}u^5+\varepsilon^{61}u^8+\varepsilon^{175}u^9+\varepsilon^{206}u^{12}+\varepsilon^{859}u^{13}+\bar{\varepsilon}^{26}u^{16}+$ $\bar{\varepsilon}^{15}\bar{u}^4+\bar{\varepsilon}^{344}\bar{u}^5+\bar{\varepsilon}^{61}\bar{u}^8+\bar{\varepsilon}^{175}\bar{u}^9+\bar{\varepsilon}^{206}\bar{u}^{12}+\bar{\varepsilon}^{859}\bar{u}^{13}+\varepsilon^{26}\bar{u}^{16}+\bar{\varepsilon}u+\varepsilon\bar{u}$
	$1+\varepsilon^3u^4+\bar{\varepsilon}^{140}u^5+\varepsilon^{895}u^8+\varepsilon^{333}u^9+\varepsilon^{82}u^{12}+\bar{\varepsilon}^{190}u^{13}+\varepsilon^{181}u^{16}+$ $\bar{\varepsilon}^3\bar{u}^4+\varepsilon^{140}\bar{u}^5+\bar{\varepsilon}^{895}\bar{u}^8+\bar{\varepsilon}^{333}\bar{u}^9+\bar{\varepsilon}^{82}\bar{u}^{12}+\varepsilon^{190}\bar{u}^{13}+\bar{\varepsilon}^{181}\bar{u}^{16}+u+\bar{u}$
	$1+\varepsilon^5u^4+\varepsilon^2u^5+\bar{\varepsilon}^{345}u^8+\varepsilon^{767}u^9+\varepsilon^{278}u^{12}+\bar{\varepsilon}^{510}u^{13}+\bar{\varepsilon}^{380}u^{16}+$ $\bar{\varepsilon}^5\bar{u}^4+\bar{\varepsilon}^2\bar{u}^5+\varepsilon^{345}\bar{u}^8+\bar{\varepsilon}^{767}\bar{u}^9+\bar{\varepsilon}^{278}\bar{u}^{12}+\varepsilon^{510}\bar{u}^{13}+\varepsilon^{380}\bar{u}^{16}$
	$1+\varepsilon^{11}u^4+\bar{\varepsilon}^{18}u^5+\bar{\varepsilon}^{796}u^8+\bar{\varepsilon}^{664}u^9+\varepsilon^{27}u^{12}+\bar{\varepsilon}^{78}u^{13}+\varepsilon^{433}u^{16}+$ $\bar{\varepsilon}^{11}\bar{u}^4+\varepsilon^{18}\bar{u}^5+\varepsilon^{796}\bar{u}^8+\varepsilon^{664}\bar{u}^9+\bar{\varepsilon}^{27}\bar{u}^{12}+\varepsilon^{78}\bar{u}^{13}+\bar{\varepsilon}^{433}\bar{u}^{16}$

TABLE 4. Associated g-functions for Cherowitzo hyperoval in $AG(2,32)$

Hyperoval	**Functions** $g(u)$
O'Keefe-Penttila	$1+\varepsilon^{123}u^9+u^{12}+\bar{\varepsilon}^{123}\bar{u}^9+\bar{u}^{12}$
	$1+\varepsilon^{11}u^4+\bar{\varepsilon}^{95}u^5+\varepsilon^{337}u^8+\varepsilon^{210}u^9+\bar{\varepsilon}^{111}u^{12}+\varepsilon^{599}u^{13}+\varepsilon^{213}u^{16}$ $\bar{\varepsilon}^{11}\bar{u}^4+\varepsilon^{95}\bar{u}^5+\bar{\varepsilon}^{337}\bar{u}^8+\bar{\varepsilon}^{210}\bar{u}^9+\varepsilon^{111}\bar{u}^{12}+\bar{\varepsilon}^{599}\bar{u}^{13}+\bar{\varepsilon}^{213}\bar{u}^{16}+\bar{\varepsilon}u+\varepsilon\bar{u}$
	$1+\varepsilon^{11}u^4+\bar{\varepsilon}^{49}u^5+\bar{\varepsilon}^{206}u^8+\varepsilon^{765}u^9+\bar{\varepsilon}^{438}u^{12}+\varepsilon^{135}u^{13}+\varepsilon^{506}u^{16}+$ $\bar{\varepsilon}^{11}\bar{u}^4+\varepsilon^{49}\bar{u}^5+\varepsilon^{206}\bar{u}^8+\bar{\varepsilon}^{765}\bar{u}^9+\varepsilon^{438}\bar{u}^{12}+\bar{\varepsilon}^{135}\bar{u}^{13}+\bar{\varepsilon}^{506}\bar{u}^{16}+u+\bar{u}$
	$1+\varepsilon u^4+\bar{\varepsilon}^{473}u^5+\varepsilon^{118}u^8+\varepsilon^{732}u^9+\varepsilon^{82}u^{12}+\bar{\varepsilon}^{349}u^{13}+\bar{\varepsilon}^{636}u^{16}+$ $\bar{\varepsilon}\bar{u}^4+\varepsilon^{473}\bar{u}^5+\bar{\varepsilon}^{118}\bar{u}^8+\bar{\varepsilon}^{732}\bar{u}^9+\bar{\varepsilon}^{82}\bar{u}^{12}+\varepsilon^{349}\bar{u}^{13}+\varepsilon^{636}\bar{u}^{16}$
	$1+\varepsilon^5u^4+\varepsilon^{402}u^5+\varepsilon^{136}u^8+\varepsilon^{276}u^9+\varepsilon^{372}u^{12}+\bar{\varepsilon}^{280}u^{13}+\bar{\varepsilon}^{566}u^{16}+$ $\bar{\varepsilon}^5\bar{u}^4+\bar{\varepsilon}^{402}\bar{u}^5+\bar{\varepsilon}^{136}\bar{u}^8+\bar{\varepsilon}^{276}\bar{u}^9+\bar{\varepsilon}^{372}\bar{u}^{12}+\varepsilon^{280}\bar{u}^{13}+\varepsilon^{566}\bar{u}^{16}+u+\bar{u}$
	$1+\varepsilon^{764}u^4+\varepsilon^{62}u^5+\varepsilon^{276}u^8+\bar{\varepsilon}^{88}u^9+\bar{\varepsilon}^{284}u^{12}+\varepsilon^{398}u^{13}+\bar{\varepsilon}^{862}u^{16}$ $\bar{\varepsilon}^{764}\bar{u}^4+\bar{\varepsilon}^{62}\bar{u}^5+\bar{\varepsilon}^{276}\bar{u}^8+\varepsilon^{88}\bar{u}^9+\varepsilon^{284}\bar{u}^{12}+\bar{\varepsilon}^{398}\bar{u}^{13}+\varepsilon^{862}\bar{u}^{16}$
	$1+\varepsilon^{735}u^4+\bar{\varepsilon}^{207}u^5+\bar{\varepsilon}^{567}u^8+\bar{\varepsilon}^{402}u^9+\bar{\varepsilon}^{466}u^{12}+\bar{\varepsilon}^{727}u^{13}+\bar{\varepsilon}^{56}u^{16}+$ $\bar{\varepsilon}^{735}\bar{u}^4+\varepsilon^{207}\bar{u}^5+\varepsilon^{567}\bar{u}^8+\varepsilon^{402}\bar{u}^9+\varepsilon^{466}\bar{u}^{12}+\varepsilon^{727}\bar{u}^{13}+\varepsilon^{56}\bar{u}^{16}+u+\bar{u}$
	$1+\varepsilon^{442}u^4+\bar{\varepsilon}^{305}u^5+\bar{\varepsilon}^{135}u^8+\bar{\varepsilon}^{925}u^9+\bar{\varepsilon}^{564}u^{12}+\bar{\varepsilon}^{434}u^{13}+\bar{\varepsilon}^{311}u^16+$ $\bar{\varepsilon}^{442}\bar{u}^4+\varepsilon^{305}\bar{u}^5+\varepsilon^{135}\bar{u}^8+\varepsilon^{925}\bar{u}^9+\varepsilon^{564}\bar{u}^{12}+\varepsilon^{434}\bar{u}^{13}+\varepsilon^{311}\bar{u}^{16}$
	$1+\bar{\varepsilon}^{180}u^4+\varepsilon^{182}u^5+\bar{\varepsilon}^{404}u^8+\bar{\varepsilon}^{639}u^9+\bar{\varepsilon}^{408}u^{12}+\bar{\varepsilon}^{302}u^{13}+\bar{\varepsilon}^{267}u^{16}+$ $\varepsilon^{180}\bar{u}^4+\bar{\varepsilon}^{182}\bar{u}^5+\varepsilon^{404}\bar{u}^8+\varepsilon^{639}\bar{u}^9+\varepsilon^{408}\bar{u}^{12}+\varepsilon^{302}\bar{u}^{13}+\varepsilon^{267}\bar{u}^{16}$
	$1+\varepsilon^{31}u^4+\bar{\varepsilon}^{106}u^5+\bar{\varepsilon}^{147}u^8+\bar{\varepsilon}^{597}u^9+\varepsilon^{80}u^{12}+\varepsilon^{92}u^{13}+\bar{\varepsilon}^{379}u^{16}+$ $\bar{\varepsilon}^{31}\bar{u}^4+\varepsilon^{106}\bar{u}^5+\varepsilon^{147}\bar{u}^8+\varepsilon^{597}\bar{u}^9+\bar{\varepsilon}^{80}\bar{u}^{12}+\bar{\varepsilon}^{92}\bar{u}^{13}+\varepsilon^{379}\bar{u}^{16}+u+\bar{u}$
	$1+\bar{\varepsilon}^{500}u^4+\varepsilon^{735}u^5+\bar{\varepsilon}^{602}u^8+\varepsilon^{313}u^9+\bar{\varepsilon}^{637}u^{12}+\varepsilon^{283}u^{13}+\varepsilon^{46}u^{16}+$ $\varepsilon^{500}\bar{u}^4+\bar{\varepsilon}^{735}\bar{u}^5+\varepsilon^{602}\bar{u}^8+\bar{\varepsilon}^{313}\bar{u}^9+\varepsilon^{637}\bar{u}^{12}+\bar{\varepsilon}^{283}\bar{u}^{13}+\bar{\varepsilon}^{46}\bar{u}^{16}+\bar{\varepsilon}u+\varepsilon\bar{u}$
	$1+\bar{\varepsilon}^2u^4+\varepsilon^{319}u^5+\bar{\varepsilon}^{410}u^8+\bar{\varepsilon}^{731}u^9+\varepsilon^{733}u^{12}+\bar{\varepsilon}^{278}u^{13}+\varepsilon^{368}u^{16}+$ $\varepsilon^2\bar{u}^4+\bar{\varepsilon}^{319}\bar{u}^5+\varepsilon^{410}\bar{u}^8+\varepsilon^{731}\bar{u}^9+\bar{\varepsilon}^{733}\bar{u}^{12}+\varepsilon^{278}\bar{u}^{13}+\bar{\varepsilon}^{368}\bar{u}^{16}+\varepsilon u+\bar{\varepsilon}\bar{u}$

TABLE 5. Associated g-functions for O'Keefe-Penttila hyperoval in $AG(2,32)$

Hyperoval	**Functions** $g(u)$
Hyperconic	1
	$1+u^4+u^5+u^8+u^9+u^{12}+u^{13}+u^{16}+u^{17}+u^{20}+u^{21}+u^{24}+u^{25}+u^{28}+u^{29}+u^{32}+$ $\bar{u}^4+\bar{u}^5+\bar{u}^8+\bar{u}^9+\bar{u}^{12}+\bar{u}^{13}+\bar{u}^{16}+\bar{u}^{17}+\bar{u}^{20}+\bar{u}^{21}+\bar{u}^{24}+\bar{u}^{25}+\bar{u}^{28}+\bar{u}^{29}+\bar{u}^{32}$
Subiaco	$1+u^5+\bar{u}^5$
	$1+u^4+u^5+u^9+u^{13}+u^{17}+u^{21}+u^{24}+u^{25}+u^{29}+$ $\bar{u}^4+\bar{u}^5+\bar{u}^9+\bar{u}^{13}+\bar{u}^{17}+\bar{u}^{21}+\bar{u}^{24}+\bar{u}^{25}+\bar{u}^{29}$
	$1+\bar{\alpha}^{230}u^4+\alpha^{208}u^5+\alpha^{366}u^8+\alpha^{1832}u^9+\bar{\alpha}^{922}u^{12}+\bar{\alpha}^{2171}u^{13}+\bar{\alpha}^{574}u^{16}+\bar{\alpha}^{344}u^{17}+$ $\alpha^{2299}u^{20}+\bar{\alpha}^{831}u^{21}+\alpha^{1121}u^{24}+\bar{\alpha}^{243}u^{25}+\alpha^{796}u^{28}+\alpha^{723}u^{29}+\alpha^{2028}u^{32}+$ $\alpha^{230}\bar{u}^4+\bar{\alpha}^{208}\bar{u}^5+\bar{\alpha}^{366}\bar{u}^8+\bar{\alpha}^{1832}\bar{u}^9+\alpha^{922}\bar{u}^{12}+\alpha^{2171}\bar{u}^{13}+\alpha^{574}\bar{u}^{16}+\alpha^{344}\bar{u}^{17}+$ $\bar{\alpha}^{2299}\bar{u}^{20}+\alpha^{831}\bar{u}^{21}+\bar{\alpha}^{1121}\bar{u}^{24}+\alpha^{243}\bar{u}^{25}+\bar{\alpha}^{796}\bar{u}^{28}+\bar{\alpha}^{723}\bar{u}^{29}+\bar{\alpha}^{2028}\bar{u}^{32}$
Subiaco	$1+wu^5+\bar{w}\bar{u}^5$
	$1+\bar{\alpha}^{1656}u^4+\alpha^{2472}u^5+\alpha^{398}u^8+\alpha^{2868}u^9+\alpha^{1562}u^{12}+\bar{\alpha}^{3063}u^{13}+\bar{\alpha}^{3065}u^{16}+\alpha^{533}u^{17}+$ $\alpha^{1042}u^{20}+\alpha^{1895}u^{21}+\bar{\alpha}^{2350}u^{24}+\alpha^{1235}u^{25}+\alpha^{671}u^{28}+\bar{\alpha}^{3126}u^{29}+\alpha^{2099}u^{32}+$ $\alpha^{1656}\bar{u}^4+\bar{\alpha}^{2472}\bar{u}^5+\bar{\alpha}^{398}\bar{u}^8+\bar{\alpha}^{2868}\bar{u}^9+\bar{\alpha}^{1562}\bar{u}^{12}+\alpha^{3063}\bar{u}^{13}+\alpha^{3065}\bar{u}^{16}+\bar{\alpha}^{533}\bar{u}^{17}$ $+\bar{\alpha}^{1042}\bar{u}^{20}+\bar{\alpha}^{1895}\bar{u}^{21}+\alpha^{2350}\bar{u}^{24}+\bar{\alpha}^{1235}\bar{u}^{25}+\bar{\alpha}^{671}\bar{u}^{28}+\alpha^{3126}\bar{u}^{29}+\bar{\alpha}^{2099}\bar{u}^{32}$
	$1+\bar{\alpha}^{2558}u^4+\bar{\alpha}^{2809}u^5+\alpha^{510}u^8+\bar{\alpha}^{702}u^9+\bar{\alpha}^{2039}u^{12}+\bar{\alpha}^{1719}u^{13}+\alpha^{272}u^{16}+\bar{\alpha}^{955}u^{17}+$ $\bar{\alpha}^{360}u^{20}+\bar{\alpha}^{467}u^{21}+\bar{\alpha}^{151}u^{24}+\bar{\alpha}^{639}u^{25}+\alpha^{1837}u^{28}+\bar{\alpha}^{655}u^{29}+\alpha^{274}u^{32}+$ $\alpha^{2558}\bar{u}^4+\alpha^{2809}\bar{u}^5+\bar{\alpha}^{510}\bar{u}^8+\alpha^{702}\bar{u}^9+\alpha^{2039}\bar{u}^{12}+\alpha^{1719}\bar{u}^{13}+\bar{\alpha}^{272}\bar{u}^{16}+\alpha^{955}\bar{u}^{17}+$ $\alpha^{360}\bar{u}^{20}+\alpha^{467}\bar{u}^{21}+\alpha^{151}\bar{u}^{24}+\alpha^{639}\bar{u}^{25}+\bar{\alpha}^{1837}\bar{u}^{28}+\alpha^{655}\bar{u}^{29}+\bar{\alpha}^{274}\bar{u}^{32}+\bar{\alpha}u+\alpha\bar{u}$
	$1+\bar{\alpha}^{987}u^4+\bar{\alpha}^{2940}u^5+\bar{\alpha}^{63}u^8+\bar{\alpha}^{3381}u^9+\bar{\alpha}^{1323}u^{12}+\bar{\alpha}^{3003}u^{13}+\bar{\alpha}^{945}u^{16}+\alpha^{924}u^{17}+$ $\bar{\alpha}^{2478}u^{20}+\alpha^{1575}u^{21}+\bar{\alpha}^{3192}u^{24}+\alpha^{2226}u^{25}+\bar{\alpha}^{2541}u^{28}+\bar{\alpha}^{1764}u^{29}+\bar{\alpha}^{3255}u^{32}+$ $\alpha^{987}\bar{u}^4+\alpha^{2940}\bar{u}^5+\alpha^{63}\bar{u}^8+\alpha^{3381}\bar{u}^9+\alpha^{1323}\bar{u}^{12}+\alpha^{3003}\bar{u}^{13}+\alpha^{945}\bar{u}^{16}+\bar{\alpha}^{924}\bar{u}^{17}$ $+\alpha^{2478}\bar{u}^{20}+\bar{\alpha}^{1575}\bar{u}^{21}+\alpha^{3192}\bar{u}^{24}+\bar{\alpha}^{2226}\bar{u}^{25}+\alpha^{2541}\bar{u}^{28}+\alpha^{1764}\bar{u}^{29}+\alpha^{3255}\bar{u}^{32}+u+\bar{u}$
	$1+\bar{\alpha}^{417}u^4+\bar{\alpha}^{174}u^5+\alpha^{1185}u^8+\bar{\alpha}^{3454}u^9+\bar{\alpha}^{1907}u^{12}+\alpha^{1705}u^{13}+\bar{\alpha}^{2348}u^{16}+\bar{\alpha}^{1129}u^{17}+$ $\bar{\alpha}^{1375}u^{20}+\bar{\alpha}^{2474}u^{21}+\bar{\alpha}^{721}u^{24}+\bar{\alpha}^{2239}u^{25}+\alpha^{2607}u^{28}+\bar{\alpha}^{1564}u^{29}+\bar{\alpha}^{982}u^{32}+$ $\alpha^{417}\bar{u}^4+\alpha^{174}\bar{u}^5+\bar{\alpha}^{1185}\bar{u}^8+\alpha^{3454}\bar{u}^9+\alpha^{1907}\bar{u}^{12}+\bar{\alpha}^{1705}\bar{u}^{13}+\alpha^{2348}\bar{u}^{16}+\alpha^{1129}\bar{u}^{17}+$ $\alpha^{1375}\bar{u}^{20}+\alpha^{2474}\bar{u}^{21}+\alpha^{721}\bar{u}^{24}+\alpha^{2239}\bar{u}^{25}+\bar{\alpha}^{2607}\bar{u}^{28}+\alpha^{1564}\bar{u}^{29}+\alpha^{982}\bar{u}^{32}$
	$1+\bar{\alpha}^{1077}u^4+\bar{\alpha}^{1015}u^5+\bar{\alpha}^{2228}u^8+\bar{\alpha}^{996}u^9+\bar{\alpha}^{2872}u^{12}+\bar{\alpha}^{25}u^{13}+\bar{\alpha}^{2413}u^{16}+\alpha^{94}u^{17}+$ $\bar{\alpha}^{1237}u^{20}+\alpha^{3261}u^{21}+\alpha^{1891}u^{24}+\alpha^{764}u^{25}+\alpha^{933}u^{28}+\alpha^{1197}u^{29}+\bar{\alpha}^{3578}u^{32}+$ $\alpha^{1077}\bar{u}^4+\alpha^{1015}\bar{u}^5+\alpha^{2228}\bar{u}^8+\alpha^{996}\bar{u}^9+\alpha^{2872}\bar{u}^{12}+\alpha^{25}\bar{u}^{13}+\alpha^{2413}\bar{u}^{16}+\bar{\alpha}^{94}\bar{u}^{17}$ $+\alpha^{1237}\bar{u}^{20}+\bar{\alpha}^{3261}\bar{u}^{21}+\bar{\alpha}^{1891}\bar{u}^{24}+\bar{\alpha}^{764}\bar{u}^{25}+\bar{\alpha}^{933}\bar{u}^{28}+\bar{\alpha}^{1197}\bar{u}^{29}+\alpha^{3578}\bar{u}^{32}+\bar{w}u+w\bar{u}$

TABLE 6. Associated g-functions for hyperovals in $AG(2,64)$

Hyperoval	**Functions** $g(u)$
Adelaide	$1+u^{21}+\bar{u}^{21}$
	$1+u^4+u^5+u^9+u^{12}+u^{13}+u^{16}+u^{17}+u^{20}+u^{24}+u^{25}+u^{32}+$ $\bar{u}^4+\bar{u}^5+\bar{u}^9+\bar{u}^{12}+\bar{u}^{13}+\bar{u}^{16}+\bar{u}^{17}+\bar{u}^{20}+\bar{u}^{24}+\bar{u}^{25}+\bar{u}^{32}$
	$1+\theta^2u^4+\bar{\theta}^3u^5+u^8+\theta^6u^9+\theta^{11}u^{12}+u^{13}+\bar{\theta}u^{16}+\theta^7u^{17}+$ $\bar{\theta}^{11}u^{20}+\bar{\theta}^3u^{21}+\theta^7u^{24}+\theta^3u^{25}+\bar{\theta}^{11}u^{28}+\theta^5u^{29}+\theta^7u^{32}+$ $\bar{\theta}^2\bar{u}^4+\theta^3\bar{u}^5+\bar{u}^8+\bar{\theta}^6\bar{u}^9+\bar{\theta}^{11}\bar{u}^{12}+\bar{u}^{13}+\theta\bar{u}^{16}+\bar{\theta}^7\bar{u}^{17}+$ $\theta^{11}\bar{u}^{20}+\theta^3\bar{u}^{21}+\bar{\theta}^7\bar{u}^{24}+\bar{\theta}^3\bar{u}^{25}+\theta^{11}\bar{u}^{28}+\bar{\theta}^5\bar{u}^{29}+\bar{\theta}^7\bar{u}^{32}+u+\bar{u}$
	$1+\alpha^{747}u^4+\alpha^{1887}u^5+\alpha^{767}u^8+\alpha^{436}u^9+\bar{\alpha}^{262}u^{12}+\alpha^{442}u^{13}+\alpha^{379}u^{16}+\bar{\alpha}^{169}u^{17}+$ $\alpha^{2212}u^{20}+\alpha^{767}u^{21}+\alpha^{119}u^{24}+\alpha^{162}u^{25}+\bar{\alpha}^{1266}u^{28}+\bar{\alpha}^{830}u^{29}+\alpha^{1716}u^{32}+$ $\bar{\alpha}^{747}\bar{u}^4+\bar{\alpha}^{1887}\bar{u}^5+\bar{\alpha}^{767}\bar{u}^8+\bar{\alpha}^{436}\bar{u}^9+\alpha^{262}\bar{u}^{12}+\bar{\alpha}^{442}\bar{u}^{13}+\bar{\alpha}^{379}\bar{u}^{16}+\alpha^{169}\bar{u}^{17}+$ $\bar{\alpha}^{2212}\bar{u}^{20}+\bar{\alpha}^{767}\bar{u}^{21}+\bar{\alpha}^{119}\bar{u}^{24}+\bar{\alpha}^{162}\bar{u}^{25}+\alpha^{1266}\bar{u}^{28}+\alpha^{830}\bar{u}^{29}+\bar{\alpha}^{1716}\bar{u}^{32}$
	$1+\alpha^{83}u^4+\alpha^{2346}u^5+\alpha^{3186}u^8+\bar{\alpha}^{2495}u^9+\alpha^{173}u^{12}+\bar{\alpha}^{1391}u^{13}+\bar{\alpha}^{2174}u^{16}+\bar{\alpha}^{383}u^{17}+$ $\alpha^{1969}u^{20}+\bar{\alpha}^{242}u^{21}+\bar{\alpha}^{2087}u^{24}+\bar{\alpha}^{1628}u^{25}+\alpha^{1977}u^{28}+\alpha^{862}u^{29}+\bar{\alpha}^{1388}u^{32}+$ $\bar{\alpha}^{83}\bar{u}^4+\bar{\alpha}^{2346}\bar{u}^5+\bar{\alpha}^{3186}\bar{u}^8+\alpha^{2495}\bar{u}^9+\bar{\alpha}^{173}\bar{u}^{12}+\alpha^{1391}\bar{u}^{13}+\alpha^{2174}\bar{u}^{16}+\alpha^{383}\bar{u}^{17}+$ $\bar{\alpha}^{1969}\bar{u}^{20}+\alpha^{242}\bar{u}^{21}+\alpha^{2087}\bar{u}^{24}+\alpha^{1628}\bar{u}^{25}+\bar{\alpha}^{1977}\bar{u}^{28}+\bar{\alpha}^{862}\bar{u}^{29}+\alpha^{1388}\bar{u}^{32}+u+\bar{u}$
	$1+\alpha^{1834}u^4+\bar{\alpha}^{1174}u^5+\bar{\alpha}^{2027}u^8+\alpha^{1574}u^9+\alpha^{242}u^{12}+\bar{\alpha}^{1828}u^{13}+\bar{\alpha}^{2547}u^{16}+\bar{\alpha}^{873}u^{17}+$ $\bar{\alpha}^{598}u^{20}+\alpha^{1206}u^{21}+\bar{\alpha}^{1825}u^{24}+\alpha^{1571}u^{25}+\bar{\alpha}^{920}u^{28}+\bar{\alpha}^{625}u^{29}+\bar{\alpha}^{2623}u^{32}+$ $\bar{\alpha}^{1834}\bar{u}^4+\alpha^{1174}\bar{u}^5+\alpha^{2027}\bar{u}^8+\bar{\alpha}^{1574}\bar{u}^9+\bar{\alpha}^{242}\bar{u}^{12}+\alpha^{1828}\bar{u}^{13}+\alpha^{2547}\bar{u}^{16}+\alpha^{873}\bar{u}^{17}+$ $\alpha^{598}\bar{u}^{20}+\bar{\alpha}^{1206}\bar{u}^{21}+\alpha^{1825}\bar{u}^{24}+\bar{\alpha}^{1571}\bar{u}^{25}+\alpha^{920}\bar{u}^{28}+\alpha^{625}\bar{u}^{29}+\alpha^{2623}\bar{u}^{32}$
	$1+\alpha^{431}u^4+\bar{\alpha}^{1398}u^5+\alpha^{3513}u^8+\alpha^{538}u^9+\alpha^{281}u^{12}+\alpha^{2044}u^{13}+\bar{\alpha}^{1574}u^{16}+\alpha^{2605}u^{17}+$ $\bar{\alpha}^{672}u^{20}+\alpha^{549}u^{21}+\alpha^{2601}u^{24}+\alpha^{1916}u^{25}+\bar{\alpha}^{1253}u^{28}+\bar{\alpha}^{2991}u^{29}+\bar{\alpha}^{2487}u^{32}+$ $\bar{\alpha}^{431}\bar{u}^4+\alpha^{1398}\bar{u}^5+\bar{\alpha}^{3513}\bar{u}^8+\bar{\alpha}^{538}\bar{u}^9+\bar{\alpha}^{281}\bar{u}^{12}+\bar{\alpha}^{2044}\bar{u}^{13}+\alpha^{1574}\bar{u}^{16}+\bar{\alpha}^{2605}\bar{u}^{17}+$ $\alpha^{672}\bar{u}^{20}+\bar{\alpha}^{549}\bar{u}^{21}+\bar{\alpha}^{2601}\bar{u}^{24}+\bar{\alpha}^{1916}\bar{u}^{25}+\alpha^{1253}\bar{u}^{28}+\alpha^{2991}\bar{u}^{29}+\alpha^{2487}\bar{u}^{32}$
	$1+\bar{\alpha}^{1694}u^4+\alpha^{3056}u^5+\alpha^{2414}u^8+\alpha^{353}u^9+\alpha^{1826}u^{12}+\alpha^{341}u^{13}+\alpha^{3772}u^{16}+\alpha^{1523}u^{17}+$ $\alpha^{2607}u^{20}+\bar{\alpha}^{1021}u^{21}+\bar{\alpha}^{1723}u^{24}+\bar{\alpha}^{1135}u^{25}+\alpha^{402}u^{28}+\bar{\alpha}^{2927}u^{29}+\bar{\alpha}^{562}u^{32}+$ $\alpha^{1694}\bar{u}^4+\bar{\alpha}^{3056}\bar{u}^5+\bar{\alpha}^{2414}\bar{u}^8+\bar{\alpha}^{353}\bar{u}^9+\bar{\alpha}^{1826}\bar{u}^{12}+\bar{\alpha}^{341}\bar{u}^{13}+\bar{\alpha}^{3772}\bar{u}^{16}+\bar{\alpha}^{1523}\bar{u}^{17}+$ $\bar{\alpha}^{2607}\bar{u}^{20}+\alpha^{1021}\bar{u}^{21}+\alpha^{1723}\bar{u}^{24}+\alpha^{1135}\bar{u}^{25}+\bar{\alpha}^{402}\bar{u}^{28}+\alpha^{2927}\bar{u}^{29}+\alpha^{562}\bar{u}^{32}+u+\bar{u}$

TABLE 7. Associated g-functions for Adelaide hyperoval in $AG(2,64)$

Remark 5.2. In the Tables 1 - 7 (see Appendix), $g(u)\neq 0$ for all $u\in S$. This can be achieved by adding to $g(u)$ an appropriate linear part $\langle c,u\rangle$, $c\in K$. Therefore, the obtained ovals $\left\{\dfrac{u}{g(u)} \mid u\in S\right\}$ are subsets of $AG(2,q)$.

Remark 5.3. One of the main motivations to find all g-functions for $m\leq 6$ is the following. Assume that there is a new series of g-functions (hyperovals) for many values m. For $m\leq 6$, these g-functions should be equivalent to one

of the functions in Tables 1 - 7 (see Appendix). Therefore, one could guess possible new g-functions for $m > 6$. For example, one of Adelaide ovals with $g(u) = 1 + \theta^2 u^4 + \bar{\theta}^3 u^5 + u^8 + \theta^6 u^9 + \theta^{11} u^{12} + u^{13} + \bar{\theta} u^{16} + \theta^7 u^{17} + \bar{\theta}^{11} u^{20} + \bar{\theta}^3 u^{21} + \theta^7 u^{24} + \theta^3 u^{25} + \bar{\theta}^{11} u^{28} + \theta^5 u^{29} + \theta^7 u^{32} + \bar{\theta}^2 \bar{u}^4 + \theta^3 \bar{u}^5 + \bar{u}^8 + \bar{\theta}^6 \bar{u}^9 + \bar{\theta}^{11} \bar{u}^{12} + \bar{u}^{13} + \theta \bar{u}^{16} + \bar{\theta}^7 \bar{u}^{17} + \theta^{11} \bar{u}^{20} + \theta^3 \bar{u}^{21} + \bar{\theta}^7 \bar{u}^{24} + \bar{\theta}^3 \bar{u}^{25} + \theta^{11} \bar{u}^{28} + \bar{\theta}^5 \bar{u}^{29} + \bar{\theta}^7 \bar{u}^{32} + u + \bar{u}$ could be a beginning of a new series. Recall that $\theta \in \mathbb{F}_{16}$. In addition, O'Keefe-Penttila g-function $g(u) = 1 + \varepsilon^{123} u^9 + u^{12} + \bar{\varepsilon}^{123} \bar{u}^9 + \bar{u}^{12}$ could also be a beginning of a new series.

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