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GLEASON-KAHANE-ZELAZKO THEOREM IN JORDAN BANACH ALGEBRAS

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Abstract. We review the celebrated Gleason-Kahane-Zelazko and Kowalski-Slodkowski theorems from the setting of associative Banach algebras to the wider class of nonassociative Jordan Banach algebras. We introduce the notion of almost multiplicative linear functionals in Jordan Banach algebras and prove a theorem extending a former result of B.E. Johnson for Banach algebras by employing the more recent concept of condition spectrum. We show how to rediscover the Gleason-Kahane-Zelazko theorem for Jordan Banach algebras from the corresponding version for almost multiplicative linear functionals.

1. Motivation

The original motivation to introduce the class of non-associative algebras known as Jordan algebras came from *Quantum Mechanics* (cf. [7]). Let E be a complex Hilbert space, regarded as the '*state space*' of a quantum mechanical system. Let $\mathcal{H}(E)$ be the real vector space of all bounded self-adjoint linear operators on E, interpreted as the (bounded) observables of the system. In 1932, P. Jordan observed that $\mathcal{H}(E)$ is a (non-associative) algebra via the anti-commutator product

$$(x \circ y) := \frac{1}{2}(xy + yx). \tag{1.1}$$

Unlike the (associative) operator product xy, the anti-commutator product preserves observables and has therefore a physical meaning.

The product (1.1) is clearly commutative (i.e., $x \circ y = y \circ x$) and satisfies the so-called Jordan identity

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y), \tag{1.2}$$

as is easily checked. Note that, by commutativity, (1.2) is a weak form of associativity. A commutative algebra $\mathfrak{J}$ with product $x \circ y$ (not necessarily given by an anti-commutator) is called a Jordan algebra if (1.2) holds. (This definition works over fields of characteristic $\neq 2$; we are only interested in real or complex

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algebras). Every Jordan algebra $\mathfrak{J}$ is power associative, i.e., any singly generated subalgebra of $\mathfrak{J}$ is associative. For $x \in \mathfrak{J}$, let

$$M(x)y := x \circ y \tag{1.3}$$

define the multiplication operator on $\mathfrak{J}$. Then (1.2) is equivalent to the operator identity

$$[M(x), M(x^2)] = 0.$$

Here $[a, b] : ab - ba$ is the commutator product.

On a formal level, Jordan algebras are somewhat analogous to *Lie algebras* (anticommutator and Jordan algebra identity versus commutator and Jacobi identity). On a deeper level however, there is a fundamental difference: As a consequence of the Poincarré-Witt-Birkhoff Theorem, every Lie algebra (over a field of characteristic 0) can be realized as a (Lie) subalgebra of an associative algebra endowed with the commutator product. A Jordan algebra $\mathfrak{J}$ is called *special* if it can be realized as a (Jordan) subalgebra of an associative algebra A endowed with the anticommutator product. Since the commutativity implies

$$x \circ y = \frac{(x+y)^2 - x^2 - y^2}{2},$$

it is enough to assume that $\mathfrak{J}$ is a linear subspace of A closed under squaring. It turns out that not every Jordan algebra is special. Moreover, P.M. Cohn has shown that the class of all special Jordan algebras cannot be described in terms of identities alone i.e., it is not closed under taking quotients (cf. [6], Section 1.3, Theorem 2).

This motivates the study of 'abstract' Jordan algebras not necessarily given by an anti-commutator product. Since the algebras arising in quantum mechanics are typically infinite dimensional, it is of interest to develop a structure theory for suitable real or complex Jordan Banach algebras.

This problem turned out to be difficult; it was ultimately solved by E. Alfsen, F. Shultz and E. Stormer in [1].

2. Introduction

A real or complex *Jordan Banach algebra* is a Banach space $\mathfrak{J}$ equipped with a bilinear mapping $(a, b) \longrightarrow (a \circ b)$ called the 'Jordan product' satisfying the following axioms:

(J1) $\|a \circ b\| \leq \|a\|\|b\|$, for all $a, b \in \mathfrak{J}$;
(J2) $a \circ b = b \circ a$, for all $a, b \in \mathfrak{J}$ (commutativity);
(J3) $(a^2 \circ b) \circ a = (a \circ b) \circ a^2$, for all $a, b \in \mathfrak{J}$ (Jordan identity).

Note that (J3) is a form of weak associativity. In this note we shall consider unital complex Jordan Banach algebras. We say that $\mathfrak{J}$ is unital if there exists an element $\mathbf{1} \in \mathfrak{J}$ (called the unit of $\mathfrak{J}$) such that $\mathbf{1} \circ a = a$ for all $a \in \mathfrak{J}$. Jordan algebras are well understood when equipped with the U map defined as follows: Given an element $a \in \mathfrak{J}$, the map U_a is the linear mapping on $\mathfrak{J}$ defined by

$$U_a(b) := 2(a \circ b) \circ a - a^2 \circ b$$

for all $b \in \mathfrak{J}$.

A natural example can be derived from any associative Banach algebra A equipped with the natural Jordan product given by

$$a \circ b := \frac{1}{2}(ab + ba)$$

where ab stands for the associative product in A. Jordan Banach subalgebras of Banach algebras are called *special*. For instance, the hermitian part of each C^*-algebra is a real Jordan Banach subalgebra but not an associative subalgebra. We recall that the subalgebra, $\mathfrak{J}_a$, generated by a single element a and the unit element of a complex Jordan Banach algebra $\mathfrak{J}$, is an associative and commutative Banach algebra. Consequently, the usual *holomorphic functional calculus* applies naturally in the setting of complex Jordan Banach algebras (cf. [4], Theorem 4.1.88 and Theorem 4.1.93).

An element a in a unital Jordan Banach algebra $\mathfrak{J}$ is called *invertible* if we can find an element $b \in \mathfrak{J}$ satisfying $a \circ b = \mathbf{1}$ and $a^2 \circ b = a$. The element b is unique and will be denoted by a^{-1} (cf. [4], Definition 4.1.2). We also denote by $\mathrm{Inv}(\mathfrak{J})$ the set of invertible elements in $\mathfrak{J}$. With these basic notions in mind, we clearly can define the *exponential* of each element $a \in \mathfrak{J}$ by the holomorphic functional calculus or by the power series

$$e^a = \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

which yields an invertible element in $\mathfrak{J}$, lying in the Jordan Banach subalgebra generated by a and $\mathbf{1}$. It is worth mentioning that a basic spectral theory for Jordan Banach algebras similar to the one available for associative Banach algebras is readily available (as can be found in [2] and Chapter 4 in [4]).

The Jordan *spectrum* for an element $a \in \mathfrak{J}$ is the nonempty compact set

$$\mathrm{Sp}(a) := \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1} \notin \mathrm{Inv}(\mathfrak{J})\}.$$

We recall that contrarily to properties of associative Banach algebras affirming that the set of invertible elements is a group, for Jordan Banach algebras even the Jordan product of two invertible elements might be no longer invertible. But, expressions of the form

$$U_a(b) := 2(a \circ b) \circ a - a^2 \circ b$$

yield an invertible element whenever a and b are invertible in $\mathfrak{J}$.

Another important result in Banach algebras characterizes the *principal component* of invertible elements as

$$\mathcal{U}_1(A) = \{e^{x_1} \cdot e^{x_2} \cdots e^{x_n} : x_i \in A\}.$$

Of course such a formulation cannot hold for Banach Jordan algebras because the product of invertible elements is not necessarily invertible in $\mathfrak{J}$. But instead we have the following algebraic characterization of the principal component $\mathfrak{J}_{\mathbf{1}}^{-1}$ (i.e the *connected component* containing the identity $\mathbf{1}$) of the set of invertible elements $\mathrm{Inv}(\mathfrak{J})$, in a unital complex Jordan Banach algebras (see the main theorem in [13]):

$$\mathfrak{J}_1^{-1} := \{U_{\exp(a_1)} \cdots U_{\exp(a_n)}(1) : a_i \in \mathfrak{J}, n \in \mathbb{N}\}$$

3. Background on Jordan Banach algebras

3.1. **Power associativity.** Power associativity is one of the basic properties of Jordan Banach algebras, more concretely, for each Jordan Banach algebra $\mathfrak{J}$, the Jordan Banach subalgebra $\mathfrak{J}_a$, generated by $a \in \mathfrak{J}$, is associative. In other words, by setting $a^0 = 1, a^{n+1} = a \circ a^n$, we get $a^m \circ a^n = a^{m+n}$ for all $n, m \in \mathbb{N} \setminus \{0\}$ (cf. [4], Proposition 2.4.13). Therefore $\mathfrak{J}_a$ is a commutative and associative Banach algebra.

3.2. **Invertibility.** Invertibility is deeply related to U maps since an element $a \in \mathfrak{J}$ is invertible if and only if U_a is a bijective mapping; and in such a case $U_a^{-1} = U_{a^{-1}}$ (cf. [4], Definition 4.1.2 and Theorem 4.1.3). We denote by $\mathrm{Inv}(\mathfrak{J})$ for the set of all invertible elements in $\mathfrak{J}$. It is well-known that $\mathrm{Inv}(\mathfrak{J})$ is open (cf. [4], Theorem 4.1.7) and hence $\mathrm{Inv}(\mathfrak{J})$ is locally connected and all its connected components are open. The Jordan product of two invertible elements is not in general an invertible element. However, an element of the form $U_x(y)$ is invertible if and only if both x and y are invertible (cf. [4], Theorem 4.1.3(iv)), and in that case

$$(U_x(y))^{-1} = U_x^{-1}(y^{-1})$$

In particular, we obtain that

$$(x^{-1})^2 = (x^2)^{-1} = x^{-2}.$$

Consequently, the square of of the n-th power ($n \in \mathbb{N}$) of each invertible element is an invertible element.

3.3. **Spectrum.** The spectrum, also called the Jordan-spectrum of an element a, in a unital Complex Jordan Banach algebra $\mathfrak{J}$ (denoted simply by Sp(a)) is the set of all complex numbers λ for which $a - \lambda \mathbf{1}$ is not invertible. As for Banach algebras, $\mathrm{Sp}(a)$ is a non-empty compact subset of the complex plane (cf. [4], Theorem 4.1.17). Working in the power associative Banach algebra $\mathfrak{J}_a$, it is clear that an element a in $\mathfrak{J}$ is invertible in the Jordan sense if and only if it is invertible in the usual sense. In particular, we use the notation $\mathrm{Sp}(a)$ instead of $\mathrm{Sp}_{\mathfrak{J}}(a)$.

3.4. **Full subalgebras.** A subalgebra $\mathfrak{L}$ of $\mathfrak{J}$ is called *strongly associative* if $[M(a), M(b)] = 0$ for any $a, b \in \mathfrak{L}$, *spectral* if $\mathrm{Sp}_{\mathfrak{L}}(a) \cup \{0\} = \mathrm{Sp}_{\mathfrak{J}}(a) \cup \{0\}$ for each $a \in \mathfrak{L}$, and *full* (or *inverse closed*) if $\mathfrak{L}$ contains the inverse of each element of $\mathfrak{L}$ that is invertible in $\mathfrak{J}$. Note that each strongly associative subalgebra is associative. For any $a \in \mathfrak{J}$ there is a maximal strongly associative subalgebra containing a which is inverse closed whenever $\mathfrak{J}$ is unital see [4]; this subalgebra is spectral.

4. Gleason-Kahane-Zelazko theorems in Banach algebras

Let A denote a complex Banach algebra with unit, and let M be a *maximal ideal* of A. Then M is a closed subspace, and the quotient Banach algebra A/M is a field which, by the Gelfan-Mazur Theorem, is isometrically isomorphic to $\mathbb{C}$. Thus the quotient map $\varphi : A \longrightarrow A/M$ is a complex *homomorphism* (multiplicative linear functional) on A. It follows that M is a (closed) linear subspace of codimension 1 in A which contains no invertible elements (since no proper ideal can contain invertible elements). The remarkable fact that this property characterizes maximal ideals was discovered by Gleason (cf. [5]) and, independently, Kahane and Zelazko (cf. [8]).

Theorem 4.1. [5, 8] *Let A be a commutative Banach algebra with unit. A linear subspace M of codimension 1 in A is a maximal ideal if and only if it contains no invertible elements.*

Proof. We have already noted that a maximal ideal has the properties stated in the theorem. Suppose that the linear subspace M has codimension 1 in A and contains no invertible elements. Then it contains no element near the identity $\mathbf{1}$, so its closure $\overline{M}$ is a proper subspace. Since M has codimension 1, $M = \overline{M}$ so M is closed. Let φ the continuous linear functional in A such that $M = \ker \varphi$ and $\varphi(\mathbf{1}) = 1$. We prove that

$$\varphi(xy) = \varphi(x)\varphi(y), \quad x, y \in A, \tag{4.1}$$

from which it follows immediately that M is an ideal. To this end, fix $x \in A$ and consider the analytic function f defined by

$$f(\lambda) = \varphi(e^{\lambda x}) = \sum_{n=0}^{\infty} \frac{\varphi(x^n)}{n!} \lambda^n. \tag{4.2}$$

Since $|\varphi(x^n)| \leq \|\varphi\| \|x\|^n$, then f is entire and satisfies

$$|f(\lambda)| \leq \|\varphi\| e^{\|x\| |\lambda|}.$$

Moreover, since $\exp(\lambda x)$ is invertible in A, $\quad f(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$, and $f(0) = \varphi(\mathbf{1}) = 1$. Therefore, by a classical version of Hadamard's theorem,

$$f(\lambda) = e^{\alpha \lambda} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \lambda^n \tag{4.3}$$

for some $\alpha \in \mathbb{C}$. Comparing (4.2) and (4.3), we have $\varphi(x^n) = \alpha^n$ for all n. Hence, in particular,

$$\varphi(x^2) = \varphi(x)^2. \tag{4.4}$$

Since this holds for each $x \in A$, it follows that for all $x, y \in A$, we get

$$\varphi((x+y)^2) = (\varphi(x) + \varphi(y))^2$$

which, after simplification, reduces to

$$\varphi(xy) = \varphi(x)\varphi(y), \tag{4.5}$$

as required. $\square$

The characterization of multiplicative linear functionals in Banach algebras, given independently by Gleason and Kahane-Zelazko, is formulated in the following theorem.

Theorem 4.2. [5, 8, 15] *Let A be a (non-necessarily unital nor commutative) complex Banach algebra. Let $f : A \longrightarrow \mathbb{C}$ be a linear selection from the spectrum, i.e., $f(x) \in \mathrm{Sp}(x)$ for every $x \in A$. Then f is multiplicative.*

In [9], the authors weaken the assumptions of this theorem in the following way:

Theorem 4.3. [9] *Let A be a complex Banach algebra (not necessarily unital nor commutative). Let $f : A \longrightarrow \mathbb{C}$ satisfy*

(i) $f(0) = 0$,
(ii) $f(x) - f(y) \in \mathrm{Sp}(x - y)$ *for* $x, y \in A$.

Then f is multiplicative and linear.

Our aim in this note is to review the recent extensions of the Gleason-Kahane-Zelazko and Kowalski-Slodkowski theorems to the more general setting of complex Jordan Banach algebras obtained by Escolano, Peralta and Villena in [3]. In the just quoted reference the proof of the Gleason-Kahane-Zelazko theorem for complex Jordan Banach algebras is obtained by restricting to the localized subalgebra generated by one element. In this note, we present another proof, closer to the original ideas, based mainly on Hadamard's theorem on entire functions of exponential type. As for the Kowalski-Slodkowski theorem, we also follow the lines indicated in [3], where essentially the only tool that needs to be adapted to the nonassociative setting is Lemma 2.1 in [9].

5. Multiplicative linear functionals

Definition 5.1. A *multiplicative linear functional* on a complex Jordan Banach algebra $\mathfrak{J}$ is a non-zero linear functional ϕ on $\mathfrak{J}$ such that $\phi(x \circ y) = \phi(x)\phi(y)$ for all $x, y \in \mathfrak{J}$, i.e. a non-zero *homomorphism* of $\mathfrak{J}$ into $\mathbb{C}$.

Proposition 5.2. *Let ϕ be a linear multiplicative linear functional on $\mathfrak{J}$. Then ϕ is continuous and $\|\phi\| \leq 1$. If $\mathfrak{J}$ is unital with unit $\boldsymbol{1}$, then $\phi(\boldsymbol{1}) = 1$.*

Remark 5.3. If $\mathfrak{J}$ is non-unital, we can always extend ϕ to a linear multiplicative functional on the unitization.

Let $\mathfrak{J}$ have a unit element $\mathbf{1}$ and let ϕ be a multiplicative linear functional on $\mathfrak{J}$. Then $a - \phi(\mathbf{1})a \in \ker \phi \subset \mathrm{Sing}(\mathfrak{J})$ and so

$$\phi(a) \in \mathrm{Sp}(a) \quad (a \in \mathfrak{J}). \tag{5.1}$$

We show below that (5.1) distinguishes the multiplicative linear functionals amongst the linear functionals on $\mathfrak{J}$. A linear mapping between two Jordan algebras is multiplicative if and only if it preserves squares (i.e. it is a Jordan functional). We rediscover next the Gleason-Kahane-Zelazko theorem in the setting of Jordan Banach algebras established in [3]. Here we present an alternative proof closer to the original arguments in the associative setting.

Theorem 5.4 (Gleason-Kahane-Zelazko [3], Theorem 3.2)**.** *Let $\mathfrak{J}$ be a Jordan Banach algebra and let ϕ be a linear functional on $\mathfrak{J}$. The following conditions are equivalent:*

(i) $\phi(1) = 1$ *and* $\ker \phi \subset \mathrm{Sing}(\mathfrak{J})$;
(ii) $\phi(a) \in \mathrm{Sp}(a) \quad (a \in \mathfrak{J})$;
(iii) ϕ *is multiplicative.*

Proof. It is clear that $(i) \Leftrightarrow (ii)$ and $(iii) \Rightarrow (i)$. Let condition (i) hold. Given $x \in \mathfrak{J}, \lambda \in \mathbb{C}, \|x\| \leq 1, |\lambda| > 1$ we have $1 - \frac{x}{\lambda} \in \mathrm{Inv}(\mathfrak{J})$ and so $1 - \frac{\phi(x)}{\lambda} \neq 0$, hence $\phi(x) \neq \lambda$ and $|\phi(x)| \leq 1$. Therefore, ϕ is continuous and $\|\phi\| \leq 1$. Given $a \in \mathfrak{J}$ and let F be defined on $\mathbb{C}$ by

$$F(z) = \phi(\exp(za)) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \phi(a^n) z^n. \tag{5.2}$$

Then F is an entire function and

$$|F(z)| \leq 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \|a\|^n |z|^n = \exp(\|a\||z|), \quad (z \in \mathbb{C} \tag{5.3}$$

so that F has order at most 1. Since $\exp(za) \in \mathrm{Inv}(\mathfrak{J})$ we have $F(z) \neq 0 \quad (z \in \mathbb{C})$. By Hadamard's theorem, there exists $\alpha \in \mathbb{C}$ such that

$$F(z) = \exp(\alpha z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \alpha^n z^n. \tag{5.4}$$

Comparing coeffiecients in (5.3) and (5.4) we obtain $\phi(a) = \alpha$ and $\phi(a^2) = \alpha^2$. Therefore ϕ is a Jordan homomorphism, so it is multiplicative since any linear map between two Jordan algebras preserving the squares is multiplicative. □

Remark 5.5. On a complex unital Jordan Banach algebra $\mathfrak{J}$, every multiplicative linear functional ϕ satisfies the condition $\phi(1) = 1 = \|\phi\|$. A multiplicative linear functional on a complex Jordan Banach algebra $\mathfrak{J}$ can have arbitrarily small norm. To see this, let ϕ be a multiplicative linear functional on $\mathfrak{J}$, let $t \geq 1$ and let $p(a) = \|a\| + t|\phi(a)| \quad (a \in \mathfrak{J})$. Then p is a Banach algebra norm on $\mathfrak{J}$ equivalent to $\|\cdot\|$ and $|\phi(a)| \leq t^{-1} p(a) \quad (a \in \mathfrak{J})$.

6. Kowalski-Slodkowski Theorem

Definition 6.1. Let X be a complex linear space. A map $\Phi : X \to \mathbb{C}$ is complex (real) linear (shortly $\mathbb{C}$-linear or $\mathbb{R}$-linear) if it is additive and homogeneous with respect to complex (real) scalars.

We reproduce here the proof of Lemma 2.1 of [9] for Banach algebras: If A is a complex Banach algebra, every $\mathbb{R}$-linear spectral-valued functional $\Phi : A \to \mathbb{C}$ is $\mathbb{C}$-linear.

Lemma 6.2 ([9] Lemma 2.1)**.** *Let A be a complex Banach algebra and let Φ be an $\mathbb{R}$-linear selection of the spectrum. Then Φ is $\mathbb{C}$-linear.*

Proof. We have $e^{ir}\Phi(e^{-ix}) \in e^{ir}\mathrm{Sp}(e^{-ir}x) = \mathrm{Sp}(x)$ for every $r \in \mathbb{R}$, and $x \in A$. Since Φ is $\mathbb{R}$-linear,

$$e^{ir}\Phi(e^{-ir}x) = \frac{\Phi(x) - i\Phi(ix}{2} + e^{2ir}\frac{\Phi(x) + i\Phi(ix)}{2}; \tag{6.1}$$

so for a fixed $x \in A$ the set

$$\Gamma_x = \{e^{ir}\Phi(e^{-ir}x) : r \in \mathbb{R}\} \tag{6.2}$$

is a circle contained in $\mathrm{Sp}(x)$ with center $\frac{1}{2}(\Phi(x) - i\Phi(ix))$. It is easy to verify that the numbers $\Re\Phi(x) + i\Im(-i\Phi(ix))$ and $\Re(-i\Phi(ix)) + i\Im\Phi(x)$ are in Γ_x and so in $\mathrm{Sp}(x)$. Thus the functionals

$$\Phi_1(x) = \Re\Phi(x) + i\Im(-i\Phi(ix)) = \Re\Phi(x) - i\Re\Phi(ix) \tag{6.3}$$

and

$$\Phi_2(x) = \Re(-i\Phi(ix)) + i\Im\Phi(x) = \Im\Phi(ix) - i\Im\Phi(x) \tag{6.4}$$

are selections from the spectrum.

Moreover, from $\Phi_k(ix) = i\Phi_k(x), k = 1, 2$, they are $\mathbb{C}$-linear. By the Gleason-Kahane-Zelazko theorem they are multiplicative on A. We shall now show that $\Phi_1 = \Phi_2$. Otherwise we would have, for a certain element $a \in A$,

$$\Phi_1(a) = 1, \quad \Phi_2(a) = 0. \tag{6.5}$$

Consider $h(z) = e^{\frac{1}{2}\pi iz} - 1$. The function h operates in every Banach algebra, (not necessarily unital). By formulas (6.3) and (6.4) it follows that

$$\begin{aligned}\Phi(h(a)) &= \Re\Phi(h(a)) + i\Im\Phi(h(a)) \\ &= \Re\Phi_1(h(a)) + i\Im\Phi_2(h(a)) \\ &= \Re h(\Phi_1(a)) + i\Im h((\Phi_2(a)) \\ &= \Re h(1) + i\Im h(0) \\ &= -1 \in \mathrm{Sp}(h(a)) = h(\mathrm{Sp}(a)) \subset h(\mathbb{C}) = \mathbb{C} \setminus \{-1\}\end{aligned}$$

which is impossible. Having $\Phi_1 = \Phi_2$, we see by formulas (6.3) and (6.4) that

$$\Re\Phi(x) = \Re(-i\Phi(ix))$$

and

$$\Im\Phi(x) = \Im(-i\Phi(ix)),$$

which implies that $\Phi(x) = -i\Phi(ix)$ and Φ is $\mathbb{C}$-linear. □

If $\mathfrak{J}$ is a complex Jordan Banach algebra and $\Phi : \mathfrak{J} \to \mathbb{C}$ is an $\mathbb{R}$-linear spectral-valued functional, by restricting Φ to the Jordan Banach subalgebra $\mathfrak{J}_a$, generated by a single element $a \in \mathfrak{J}$, we conclude by Lemma 6.2 that $\Phi_{|\mathfrak{J}_a}$ (and thus Φ) is $\mathbb{C}$-linear. Thus we obtain the following Jordan Banach algebra version of the previous lemma as established in [3], Lemma 3.4.

Lemma 6.3. *([3], Lemma 3.4) Let $\mathfrak{J}$ be a complex Jordan Banach algebra, and let $\Phi : \mathfrak{J} \to \mathbb{C}$ be an $\mathbb{R}$-linear spectral-valued functional (i.e. a selection from the spectrum). Then Φ is $\mathbb{C}$-linear.*

Next, we extend Kowalski-Slodkowski theorem to Jordan Banach algebras.

Theorem 6.4. *([3], Theorem 3.3) Let $\mathfrak{J}$ be a unital complex Jordan Banach algebra. Let $f : \mathfrak{J} \longrightarrow \mathbb{C}$ satisfy*

(i) $f(0) = 0$,
(ii) $f(x) - f(y) \in \mathrm{Sp}(x - y)$ *for* $x, y \in \mathfrak{J}$.

Then f is multiplicative and linear.

Proof. The proof is similar to the proof of Theorem 1.2 in [9], with Lemma 2.1 replaced by Lemma 6.3. □

7. ϵ-Invertibility in complete normed non-associative algebras

Motivated by the concept of ϵ-condition spectrum studied in [10], in the setting of associative Banach algebras, we define and prove some basic properties of an analogous ϵ-condition spectrum for Jordan Banach algebras. In particular, we show that the ϵ-condition spectrum $\mathrm{CSp}_\epsilon(a)$ of a is a non-empty compact subset of $\mathbb{C}$ containing $\mathrm{Sp}(a)$. Since $\mathrm{Sp}(a)$ is a purely algebraic concept comparatively to $\mathrm{CSp}_\epsilon(a)$ which is a geometric concept, one should expect that these two sets possess some different properties. We mention for instance that the ϵ-condition spectrum has no isloated points and has always only a finite number of components; in addition each component contains an element of the spectrum. The following are two extensions of the spectrum to the context of Jordan algebras.

Definition 7.1 (ϵ-pseudospectrum)**.** Let $\mathfrak{J}$ be a complex unital Jordan Banach algebra and let $\epsilon > 0$. The ϵ-pseudospectrum of an element $a \in \mathfrak{J}$ is denoted as

$$\mathrm{Sp}_\epsilon(a) = \{\lambda \in \mathbb{C} : \lambda - a \, \textit{is not invertible or} \parallel (\lambda - a)^{-1} \parallel \geq \frac{1}{\epsilon}\}.$$

Remark 7.2. The spectrum is independent of the norm, whereas pseudospectra depend on the norm.

Definition 7.3 (ϵ-condition spectrum)**.** Let $\mathfrak{J}$ be a complex unital Jordan Banach algebra with unit $\mathbf{1}$ and let $\epsilon > 0$. The ϵ-condition spectrum of an element $a \in \mathfrak{J}$ is denoted as

$$\mathrm{CSp}_\epsilon(a) = \{\lambda \in \mathbb{C} : \lambda - a \, \textit{is not invertible or} \parallel (\lambda - a) \parallel \parallel (\lambda - a)^{-1} \parallel \geq \frac{1}{\epsilon}\}.$$

Because of the above convention, $\mathrm{CSp}_\epsilon(a) \supseteq \mathrm{Sp}(a)$ for $0 < \epsilon < 1$.

Proposition 7.4. *Let $\mathfrak{J}$ be a normed unital Jordan algebra and let a be in $\mathfrak{J}$. Then the closed $\mathfrak{J}$-full subalgebra of $\mathfrak{J}$ generated by a is associative and commutative.*

Remark 7.5. In classical associative Banach algebras, *inverse closed subalgebras* parallel the notion of *full subalgebras* defined for Jordan algebras in section 3.4. Indeed, if $\mathfrak{A}$ is a complex unital Banach algebra and $\mathfrak{B}$ is a unital subalgebra of $\mathfrak{A}$, we say that $\mathfrak{B}$ is inverse closed in $\mathfrak{A}$ if every element of $\mathfrak{B}$, which is invertible in $\mathfrak{A}$, is also invertible in $\mathfrak{B}$, that is $\mathfrak{B} \cap \mathrm{Inv}(\mathfrak{A}) \subseteq \mathrm{Inv}(\mathfrak{B})$.

Theorem 7.6. *Let $\mathfrak{J}$ be a complex unital Jordan Banach algebra with unit 1, and $\mathfrak{L}$ a full subalgebra of $\mathfrak{J}$. Then for all $l \in \mathfrak{L}$, $\mathrm{Sp}(l, \mathfrak{L}) = \mathrm{Sp}(l, \mathfrak{J})$. Also, for all $\epsilon > 0$, $\mathrm{Sp}_\epsilon(l, \mathfrak{L}) = \mathrm{Sp}_\epsilon(l, \mathfrak{J})$. Moreover, for all $0 < \epsilon < 1$, $\mathrm{CSp}_\epsilon(l, \mathfrak{L}) = \mathrm{CSp}_\epsilon(l, \mathfrak{J})$.*

Proof. Obviously, for each $l \in \mathfrak{L}$, $\mathrm{Sp}(l, \mathfrak{J}) \subseteq \mathrm{Sp}(l, \mathfrak{L})$. Next, let $l \in \mathfrak{L}$, $\lambda \in \mathbb{C}$ and $\lambda \notin \mathrm{Sp}(l, J)$. Then $\lambda - l$ is invertible in $\mathfrak{J}$ and hence also in $\mathfrak{L}$. Thus $\lambda \notin \mathrm{Sp}(l, \mathfrak{L})$. Hence $\mathrm{Sp}(l, \mathfrak{L}) \subseteq \mathrm{Sp}(l, \mathfrak{J})$. This shows that $\mathrm{Sp}(l, \mathfrak{J}) \subseteq \mathrm{Sp}(l, \mathfrak{L})$.

Since the norms in $\mathfrak{L}$ and $\mathfrak{J}$ are the same, it follows from the definition of pseudo spectrum and condition spectrum that $\mathrm{Sp}_\epsilon(l, \mathfrak{J}) = \mathrm{Sp}_\epsilon(l, \mathfrak{L})$ for all $\epsilon > 0$ and $\mathrm{CSp}_\epsilon(l, \mathfrak{J}) = \mathrm{CSp}_\epsilon(l, \mathfrak{L})$ for all $0 < \epsilon < 1$. □

Theorem 7.7. *We have the following properties:*

(1) $\mathrm{CSp}_\epsilon(0) = \{0\}$ *and* $\mathrm{CSp}_\epsilon(1) = \{1\}$.
(2) *If* $0 < \epsilon_1 < \epsilon_2 < 1$ *then* $\mathrm{CSp}_{\epsilon_1}(a) \subseteq \mathrm{CSp}_{\epsilon_2}(a)$ *for every* $a \in J$.
(3) $\mathrm{Sp}(a) \subseteq \mathrm{CSp}_\epsilon(a)$ *for every* $a \in \mathfrak{J}$. *In fact*

$$\mathrm{Sp}(a) = \bigcap_{0<\epsilon<1} \mathrm{CSp}_\epsilon(a)$$

(4) $\mathrm{CSp}_\epsilon(a)$ *is a nonempty compact subset of* $\mathbb{C}$ *for every* $a \in J$.
(5) *The map* $a \longmapsto \mathrm{CSp}_\epsilon(a)$ *is an upper semicontinuous multivalued function from* J *to the collection of all compact subsets of* $\mathbb{C}$.
(6) $\mathrm{CSp}_\epsilon(\alpha + \beta a) = \alpha + \beta \mathrm{CSp}_\epsilon(a)$ *for all* $\alpha, \beta \in \mathbb{C}$.

Proof. Properties (1), (2) and (3) follow from the definition of the ϵ-condition spectrum. The rest of the proof, we refer to [10]. □

It is known that the spectral radius of an element $a \in \mathfrak{J}$ is defined as

$$\rho(a) := \sup\left\{|\lambda| : \lambda \in \mathrm{Sp}(a)\right\}$$

and similarly we define the ϵ-condition spectral radius of an element $a \in \mathfrak{J}$ as

$$r(a) := \sup\left\{|\lambda| : \lambda \in \mathrm{CSp}_\epsilon(a)\right\}.$$

Theorem 7.8. *For* $a \in \mathfrak{J}$, *we have the following inequality between the spectral radius and the* ϵ*-condition spectral radius:*

$$\rho(a) \leq \mathrm{r}_\epsilon(a) \leq \frac{1+\epsilon}{1-\epsilon}\|a\|.$$

Proof. Since $\mathrm{Sp}(a) \subseteq \mathrm{CSp}_\epsilon(a)$, we have $\rho(a) \leq \mathrm{r}_\epsilon(a)$. To prove the remaining part of the inequality, consider $\lambda \in \mathrm{CSp}_\epsilon(a)$. If $|\lambda| \leq \|a\|$, then clearly $|\lambda| \leq \frac{1+\epsilon}{1-\epsilon}\|a\|$. Suppose $|\lambda| > \|a\|$, then $\lambda - a$ is invertible and $\|(\lambda - a)^{-1}\| \leq \frac{1}{|\lambda| - \|a\|}$. Hence

$$1 \leq \epsilon\|(\lambda - a)^{-1}\|\|\lambda - a\| \leq \epsilon\frac{|\lambda| + \|a\|}{|\lambda| - \|a\|}.$$

On simplification, we get

$$|\lambda| \leq \frac{1+\epsilon}{1-\epsilon}\|a\|.$$

□

8. Almost multiplicative functions on Jordan Banach algebras

Definition 8.1. Let $\delta > 0$. A linear map $\phi : \mathfrak{J} \longrightarrow \mathbb{C}$ is said to be δ-almost multiplicative if

$$|\phi(a \circ b) - \phi(a)\phi(b)| \leq \delta\|a\|\|b\|$$

for all $a, b \in \mathfrak{J}$.

8.1. **Main results.** We saw previously that, for every multiplicative functional ϕ, the value of ϕ, at any element of $\mathfrak{J}$, belongs to the spectrum of that element. Similarly, the value of an almost multiplicative functional at an element belongs to the condition spectrum of that element.

Theorem 8.2. *Let $\mathfrak{J}$ be a complex Jordan Banach algebra with unit $\mathbf{1}$ and let ϕ be a δ-almost multiplicative linear functional on $\mathfrak{J}$ with $\phi(\mathbf{1}) = 1$. Then $\phi(a) \in \mathrm{CSp}_\delta(a)$.*

Proof. Let $a \in \mathfrak{J}$ and $\phi(a) = \lambda$. If $\lambda - a$ is not invertible, then $\lambda \in \mathrm{Sp}(a) \subseteq \mathrm{CSp}_\delta(a)$. Then the conclusion follows from (i). Next, assume $\lambda - a$ is invertible. Then

$$1 = |\phi(\mathbf{1})| = |\phi(\lambda - a)\phi((\lambda - a)^{-1})| \leq \delta\|\lambda - a\|\|(\lambda - a)^{-1}\|.$$

That is, $\|\lambda - a\|\|(\lambda - a)^{-1}\| \geq \frac{1}{\delta}$, which implies $\lambda(= \phi(a)) \in \mathrm{CSp}_\delta(a)$. □

The following lemma gives a sufficient condition for a linear function to be almost multiplicative from its behavior on the unit sphere.

Lemma 8.3. *Let $\mathfrak{J}$ be a Jordan Banach algebra and $\phi : \mathfrak{J} \longrightarrow \mathbb{C}$ be a linear map. If $|\phi(a^2) - (\phi(a))^2| \leq \delta_1$, for all $a \in \mathfrak{J}$ with $\|a\| = 1$ then ϕ is $2\delta_1$-almost multiplicative.*

Proof. Noting that the inequality holds trivially with $a = 0$ and replacing a nonzero a by $\frac{a}{\|a\|}$, we obtain

$$|\phi(a^2) - (\phi(a))^2| \leq \delta_1\|a\|^2$$

for all $a \in \mathfrak{J}$. Next note that for all $a, b \in \mathfrak{J}$, we have

$$4\left(\phi(a \circ b) - \phi(a)\phi(b)\right) = \phi((a + b)^2) - (\phi(a + b))^2 - \phi((a - b)^2) + (\phi(a - b))^2.$$

Hence for all $a, b \in \mathfrak{J}$ with $\|a\| = \|b\| = 1$, we get

$$|\phi(a \circ b) - \phi(a)\phi(b)| \leq \frac{1}{4}\delta_1\left(\|a + b\|^2 + \|a - b\|^2\right) \leq \frac{1}{4}\delta_1(4 + 4) = 2\delta_1.$$

Hence for arbitrary $a, b \in \mathfrak{J}$, we get

$$|\phi(a \circ b) - \phi(a)\phi(b)| \leq 2\delta_1\|a\|\|b\|.$$

□

The next theorem which connects the condition spectrum with almost multiplicative linear functionals can be seen as an approximate version of the Gleason-Kahane-Zelazko theorem.

Theorem 8.4. *([10], Theorem 5) Let $\mathfrak{J}$ be unital complex Jordan Banach algebra. Let $0 < \epsilon < \frac{1}{3}$ and $\phi : \mathfrak{J} \to \mathbb{C}$ be a linear function. If $\phi(a) \in \mathrm{CSp}_\epsilon(a)$ for every $a \in \mathfrak{J}$, then ϕ is δ-almost multiplicative, where*

$$\delta = \frac{4}{\ln(\frac{1}{\epsilon})}\left(1 + \frac{2}{((\ln 2)/3)^2}\right).$$

Proof. The proof is the same as the proof of Theorem 5 in [10] because everything takes place in the power associative algebra generated by one element $a \in \mathfrak{J}$. This is of course a Banach algebra when endowed with the norm of J and then Theorem 5 of [10] holds verbatim. □

Using Theorem 8.4, we can deduce the classical Gleason-Kahane-Zelazko theorem for Jordan Banach algebras.

Corollary 8.5 (Gleason-Kahane-Zelazko)**.** *Let $\mathfrak{J}$ be a unital complex Jordan Banach algebra and $\phi : \mathfrak{J} \to \mathbb{C}$ a linear functional. If $\phi(a) \in \mathrm{Sp}(a)$ for every $a \in \mathfrak{J}$, then ϕ is multiplicative.*

Proof. Since $\mathrm{Sp}(a) \subseteq \mathrm{CSp}_\epsilon(a)$ for every $0 < \epsilon < 1$ by (i), then $\phi(a) \in \mathrm{CSp}_\epsilon(a)$, for all $a \in \mathfrak{J}$, and $0 < \epsilon < \frac{1}{3}$. Applying Theorem 10, we see that ϕ is δ-almost multiplicative. Note that $\delta \to 0^+$ as $\epsilon \to 0^+$. Hence ϕ is multiplicative. □

9. Generalization of The Gleason-Kahane-Zelazko and Kowalski-Slodkowski theorems

A spherical version of the Gleason-Kahane-Zelazko theorem was established as Proposition 2.2 in [12]. In what follows, $\mathbb{T}$ denotes the unit circle in the complex plane, i.e $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Proposition 9.1. *([12], Proposition 2.2) Let $F : A \to \mathbb{C}$ be a continuous linear functional, where A is a unital complex Banach algebra. Suppose that $F(a) \in \mathbb{T}\mathrm{Sp}(a)$, for every $a \in A$. Then the mapping $\overline{F(1)}F$ is multiplicative.*

Moreover, a spherical variant of the Kowalski-Slodkowski theorem was also established in the same paper [12] as Proposition 3.2.

Proposition 9.2. *([12], Proposition 3.2) Let A be a complex Banach algebra, and let $\Delta : A \to \mathbb{C}$ be a mapping satisfying the following properties:*

(a) *Δ is 1-homogeneous, i.e $\Delta(\lambda x) = \lambda\Delta(x)$, for every $x \in A$, $\lambda \in \mathbb{C}$.*
(b) *$\Delta(x) - \Delta(y) \in \mathbb{T}\mathrm{Sp}(x - y)$, for every $x, y \in A$.*

Then Δ is linear, and there exists $\lambda_0 \in \mathbb{T}$ such that $\lambda_0\Delta$ is multiplicative.

In [14], the hypothesis (a) in Proposition 9.2 is relaxed to $\Delta(0) = 0$, so that it becomes closer to the one in the original Kowalski-Slodkowski theorem.

Theorem 9.3. *([14], Theorem 2.2) Let A be a unital complex Banach algebra. Suppose that a map $\Delta : A \to \mathbb{C}$ satisfies the conditions:*

(a) *$\Delta(0) = 0$;*
(b) *$\Delta(x) - \Delta(y) \in \mathbb{T}\mathrm{Sp}(x - y)$, for every $x, y \in A$.*

Then Δ is complex-linear or conjugate-linear and $\overline{\Delta(1)}\Delta$ is multiplicative.

Remark 9.4. These spherical variants of the Gleason-Kahane-Zelazko and Kowalski-Slodkowski theorems obtained in [12, 14] are also valid for Jordan Banach algebras as indicated, for instance, on page 17 in [3].

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