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ACYCLIC AND TREE UNIQUE COLOURINGS IN RANDOM GRAPHS

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Abstract. In this paper we study acyclic colouring in the random subgraph G of the complete graph K_n on n vertices where each edge is present with probability p, independent of the other edges. We show that unlike the ordinary chromatic number, the acyclic chromatic number exhibits a phase transition from sub-linear to linear growth as the edge probability increases, even in the sparse regime and obtain estimates for the critical exponent. We show that allowing for a small relaxation in the acyclic colouring condition allows for sub-linear growth for all values of the edge probability exponent. We also study a generalization of unique colourings where we impose the condition that at least one vertex in any copy of a given tree has a unique colour and obtain bounds for the minimum number of colours needed for such a colouring.

1. Introduction

Chromatic number of a graph has important applications in various fields and the acyclic chromatic number χ_{acyc} essentially studies the minimum number of colours needed to ensure that the graph has no bichromatic cycle. The systematic study of acyclic chromatic number was initiated in [2] where the Local Lemma was used to establish the growth order in terms of the maximum vertex degree and also provide examples of graphs that nearly achieve the bound. Recently, entropy compression was used in [13] [5] to further improve the results of [2].

In this paper, we study acyclic colouring in random graphs by obtaining estimates on the critical exponent at which the acyclic chromatic number transitions from sublinear to linear growth (with respect to the number of vertices n). We then show that by allowing for an arbitrary small fraction of cycles to violate the acyclic condition, the resulting "fractional" acyclic chromatic number always remains sublinear in the number of vertices n.

Next we study tree unique colourings in random graphs. Conflict-free and injective colourings of random graphs are important problems from both theoretical and application perspectives. Conflict-free colouring [12] of a graph requires that the closed neighbourhood of each vertex contains a unique colour and at the other

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end, injective colouring [9] implies that all the vertices in the open neighbourhood receive unique colours. In this paper, we study a generalization of the above definitions by imposing the condition that at least one vertex in a given tree has a unique colour. We obtain bounds for the minimum number of colours needed for such a colouring in terms of the size t of the tree.

We remark that the material in this paper is also present in our arxiv versions [7] [8].

The paper is organized as follows: In Section 2, we study bounds for the acyclic chromatic number of random graphs and in Section 3, we establish bounds for tree unique colourings of random graphs.

2. Acyclic Colouring

Let K_n be the complete graph on n vertices and let $Y_f, f \in K_n$ indexed by the edge set of K_n, be independent and identically distributed (i.i.d.) Bernoulli random variables satisfying

$$\mathbb{P}(X_f = 1) = p = 1 - \mathbb{P}(X_f = 0) \tag{2.1}$$

for some $0 \leq p \leq 1$. The graph G formed by all edges satisfying$X_f = 1$ is the homogenous random graph with edge probability p.

We have some standard definitions. A path $P = (u_1, \ldots, u_t)$ in G is a sequence such that vertex u_i is adjacent to u_{i+1} for each $1 \leq i \leq t-1$ and the number of edges in P is the length of P. If u_1 and the last vertex u_t are also adjacent, then P is said to be a *cycle* of length t. Let $k \geq 1$ be an integer.

We define a proper $k-$colouring of G to be map $z : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, k\}$ satisfying $z(a) \neq z(b)$ for any edge (a, b) in G. The chromatic number $\chi(G)$ is the smallest integer k such that G admits a proper $k-$colouring.

Definition 2.1. If z is a proper colouring of G, then for $0 < \eta < 1$ and integer $c \geq 3$, we say that z is $(\eta, c)-$*acyclic* if a fraction of at least $(1 - \eta)$ of the all the cycles of length c or more in G, contain at least c distinct colours.

The minimum number of colours needed for a $(\eta, c)-$acyclic colouring of G is defined to be the $(\eta, c)-$*acyclic chromatic number* and denoted as $\chi_{\eta,c}(G)$.

For $\eta = 0$ and $c = 3$, the above definition is simply the usual acyclic chromatic number (see for e.g. [2]). The parameter η could be interpreted as a “relaxation” parameter whose effect is described in Theorem 2.2 below.

Increasing the number of edges only increases the chromatic number and so both $\chi(G)$ and $\chi_{\eta,c}(G)$ increase with p. It is also well-known that $\chi(G)$ remains sub-linear (in the number of vertices n) even when p is a constant (see for e.g. [6]), with high probability, i.e. with probability converging to one as $n \rightarrow \infty$. In our main result below, we see how the acyclic chromatic number undergoes a phase transition from sublinear to linear even in the sparse regime by estimating the following critical exponent:

$$\beta_{crit}(\eta, c) := \inf \left\{ \beta > 0 : \frac{\chi_{\eta,c}(G)}{n} \longrightarrow 0 \text{ in probability} \right\} \tag{2.2}$$

Throughout constants do not depend on n.

Theorem 2.2. *The following properties hold:*
(a) *For every integer constant $c \geq 3$, we have that*

$$\frac{1}{4h(c)-10} \leq \beta_{crit}(0,c) \leq 1 - \frac{1}{c-1}$$

where $h(c) = c$ if c is even and $h(c) = c+1$ if c is odd.
(b) *For every $0 < \eta < 1$ and $c \geq 3$, we have that $\beta_{crit}(\eta, c) = 0$.*

Part (a) considers the acyclic colouring with $\eta = 0$ (i.e. no relaxations) and the positivity of β_{crit} demonstrates the phase transition. In part (b), we consider $\eta > 0$ and show that $\beta_{crit} = 0$ no matter how small η is: thus the relaxed acyclic chromatic number is *always* sublinear in n with high probability.

In our proof of Theorem 2.2, we use the following results regarding deviation estimates of independent Bernoulli random variables and the Lovász Local Lemma, collected together for convenience.

Lemma 2.3. (a) *Let $Y_i, 1 \leq i \leq t$ be independent Bernoulli random variables satisfying $\mathbb{P}(Y_i = 1) = 1 - \mathbb{P}(Y_i = 0)$. If $Z_t = \sum_{i=1}^{t} Y_i$ and $\mu_t = \mathbb{E}Z_t$, then for any $0 < \epsilon < \frac{1}{2}$ we have that*

$$\mathbb{P}\left(|Z_t - \mu_t| \geq \epsilon\mu_t\right) \leq 2\exp\left(-\frac{\epsilon^2}{4}\mu_t\right). \tag{2.3}$$

(b) *Let $A_1, \ldots, A_t$ be events in an arbitrary probability space. Let Γ be the dependency graph for the events $\{A_i\}$, with vertex set $\{1, 2, \ldots, t\}$ and edge set E; i.e. assume that each A_i is independent of the family of events $A_j, (i,j) \notin E$. If there are reals $0 \leq y(i) < 1$ such that $\mathbb{P}(A_i) \leq y(i)\prod_{(i,j)\in E}(1 - y(j))$, for each i, then*

$$\mathbb{P}\left(\bigcap_i A_i^c\right) \geq \prod_{1\leq i\leq n}(1 - y(i)) > 0.$$

For proofs of Lemma 2.3(a) and (b), we refer respectively to Corollary A.1.14, pp. 312 and Lemma 5.1.1, pp. 64 of [1].

The final auxiliary result used in our proof of Theorem 2.2 involves estimating the probability of occurrence of small cliques in G. We recall that a clique of size k is a complete subgraph of G containing k vertices. Defining $F_k = F_k(n)$ to be the event that G contains a clique of size k, we have:

Lemma 2.4. *For every integer $k \geq 1$ there is a constant $C = C(k) > 0$ such that*

$$\mathbb{P}(F_k) \geq 1 - kn^{k-2}e^{-Cn^2p^{2k-3}} \tag{2.4}$$

for all n large.

Proof of Lemma 2.4: The main idea is as in Lemma $6-7$ of [6] and for completeness, we give a short proof here. Defining $t_k(n) := \mathbb{P}(F_k^c(n))$ to be the probability that G *does not* contain a clique of size k, we obtain a recursion involving $t_k(n)$. Let $m(G)$ be the number of edges in G so that $\mathbb{E}m(G) = \binom{n}{2}p$. Using the deviation estimate (2.3), we then see

$$\mathbb{P}\left(m(G) \geq \frac{n^2p}{4}\right) \geq 1 - e^{-Dn^2p} \tag{2.5}$$

for some constant $D > 0$.

Suppose $m(G) \geq \frac{n^2p}{4}$ occurs henceforth. By standard double counting, the sum of vertex degrees is twice the number of edges and so there is a vertex v satisfying $d(v) \geq \frac{np}{2}$, where $d(v)$ is the degree of v. Thus

$$\begin{aligned} t_k(n) &\leq \mathbb{P}\left(m(G) \leq \frac{n^2p}{4}\right) + \mathbb{P}\left(F_k^c \bigcap \bigcup_{1\leq v\leq n} \left\{d(v) \geq \frac{np}{2}\right\}\right) \\ &\leq e^{-Dn^2p} + \sum_{1\leq v\leq n} \mathbb{P}\left(F_k^c \bigcap \left\{d(v) \geq \frac{np}{2}\right\}\right) \end{aligned} \tag{2.6}$$

by (2.5).

If $N(v)$ is the set of neighbours of v in G then

$$\mathbb{P}\left(F_k^c \bigcap \left\{d(v) \geq \frac{np}{2}\right\}\right) = \sum_T \mathbb{P}\left(F_k^c \cap \{N(v) = T\}\right), \tag{2.7}$$

where the summation is over all sets T of size $\#T \geq \frac{np}{2}$ and not containing the vertex v. If $F_k^c \cap \{N(v) = T\}$ occurs then so does the event $J_{k-1}(T)$ that there is no clique of size $k-1$ in the induced subgraph of G with vertex set T. By definition, $J_{k-1}(T)$ is independent of the set of neighbours of v and so

$$\begin{aligned} \mathbb{P}\left(F_k^c \cap \{N(v) = T\}\right) &\leq \mathbb{P}\left(J_{k-1}(T) \cap \{N(v) = T\}\right) \\ &= \mathbb{P}(J_{k-1}(T))\mathbb{P}(N(v) = T) \\ &\leq t_{k-1}(n_1)\mathbb{P}(N(v) = T), \end{aligned} \tag{2.8}$$

where $n_1 := \frac{np}{2}$ and the final estimate in (2.8) is true since T has size at least n_1. Substituting into (2.7) we obtain

$$\mathbb{P}\left(F_k^c \bigcap \left\{d(v) \geq \frac{np}{2}\right\}\right) \leq t_{k-1}(n_1) \sum_S \mathbb{P}(N(v) = T) \leq t_{k-1}(n_1)$$

and so from (2.6), we get that

$$t_k(n) \leq n t_{k-1}(n_1) + e^{-Dn^2p}. \tag{2.9}$$

Setting $n_i := n\left(\frac{p}{2}\right)^i$ for $i \geq 0$ and applying (2.9) recursively, we then see

$$\begin{aligned} t_k(n) &\leq n(n_1 t_{k-2}(n_2) + e^{-Dn_1^2p}) + e^{-Dn^2p} \\ &\leq n^2 t_{k-2}(n_2) + ne^{-Dn_1^2p} + e^{-Dn^2p} \\ &\leq \ldots \\ &\leq n^{k-2} t_2(n_{k-2}) + \sum_{i=0}^{k-3} n^i e^{-Dn_i^2p}. \end{aligned}$$

The term $t_2(n_{k-2})$ is simply the probability that there is no edge in the random graph with vertex set $\{1, 2, \ldots, n_{k-2}\}$ and so using (2.5), we get

$$t_2(n_{k-1}) \leq e^{-Dn_{k-1}^2p}.$$

Thus

$$t_k(n) \leq \sum_{i=0}^{k-2} n^i e^{-Dn_i^2 p} \leq kn^{k-2} e^{-Dn_{k-2}^2 p} \leq kn^{k-2} e^{-Cn^2 p^{2k-3}}$$

for some constant $C > 0$. This completes the proof of Lemma 2.4. ∎

Proof of Theorem 2.2(a): We upper bound $\chi_{0,c}(G)$ by using powers of G and we begin with some preliminary computations. For any vertex v, the expected number of neighbours is $\mathbb{E}d(v) = (n-1)p$ and so using the deviation estimate (2.3), we get that

$$\mathbb{P}\left(d(v) \geq \frac{np}{2}\right) \geq 1 - e^{-Dnp}$$

for some constant $D > 0$, not depending on the choice of v. Defining

$$E_{deg} := \bigcap_{v=1}^{n} \left\{\frac{np}{2} \leq d(v) \leq 2np\right\}$$

we then get from the union bound that

$$\mathbb{P}(E_{deg}) \geq 1 - ne^{-Dnp} \longrightarrow 1, \tag{2.10}$$

again since $p = \frac{1}{n^\beta}$ for some $0 < \beta < 1$, strictly.

We assume henceforth that E_{deg} occurs. For integer $k \geq 1$, the k^{th} power graph G_k of the graph G is obtained by joining any two vertices a and b by an edge if the length of the shortest path between a and b in G is at most k. Any proper colouring of G_{c-1} is a proper $(0,c)-$acyclic colouring of G, since any path of length c will contain all distinct colours. Thus

$$\chi_{0,c}(G) \leq \Delta(G_{c-1}) + 1$$

where $\Delta(G_{c-1})$ is the maximum vertex degree in G_{c-1}. Since E_{deg} occurs, the maximum degree of a vertex in G is at most $2np$ and so

$$\Delta(G_{c-1}) \leq (2np)^{c-1}.$$

This implies that

$$\chi_c(G) \leq \Delta_G^{c-1} \leq (2np)^{c-1} = (2n^{1-\beta})^{c-1}.$$

Choosing $\beta > 1 - \frac{1}{c-1}$ strictly, we then get from (2.10) that $\frac{\chi_{0,c}(G)}{n} \longrightarrow 0$ in probability and so $\beta_{crit}(0,c) \leq 1 - \frac{1}{c-1}$.

For the lower bound on $\beta_{crit}(0,c)$, we assume that c is even, set $p = \frac{1}{n^\beta}$ with $\beta < \frac{1}{4c-10}$ strictly and show that with high probability, every $t-$proper colouring of G with $t \leq \frac{n}{4}$, contains a cycle of length c with less than $c-1$ distinct colours. For c odd, an analogous analysis as below holds with $\frac{c}{2}$ replaced by $\frac{c+1}{2}$.

As before, we begin with some preliminary computations. We say that a partition $V = \{V_1, \ldots, V_t\}$ of $\{1, 2, \ldots, n\}$ into t disjoint sets is *good* there are at least $z \geq \frac{np}{100 \log n}$ sets in $\{V_j\}$, each containing at least two vertices. Let V be a good partition and suppose that $W_1, \ldots, W_z$ are the sets in V each containing at least two vertices. We use a deterministic rule to choose two vertices $w_{i,1}$ and $w_{i,2}$ from each W_i and construct the random graph $H(V)$ with vertex set $\{r_1, r_2, \ldots, r_z\}$ as follows.

An edge (r_a, r_b) with endvertices r_a and r_b is said in $H(V)$ if and only if all the four edges

$$E_{a,b} := \{(w_{a,1}, w_{a,2}), (w_{a,1}, w_{b,1}), (w_{a,2}, w_{b,1}), (w_{a,2}, w_{b,2})\}$$

is present in G. Letting E_{path} be the event that $H(V)$ contains a path of length $\frac{c}{2}$ for *every* good partition V, we compute the probability of the occurrence of E_{path} as follows. For a fixed good partition V, any edge (r_a, r_b) is present in $H(V)$ with probability p^4 and the total number of edges in the set $\bigcup_{1\leq a<b\leq z} E_{a,b}$ is $4\binom{z}{2}$.

We therefore use Lemma 2.4 with n and p replaced by z and p^4 respectively, to get that $H(V)$ contains an open path of length $k = \frac{c}{2}$ with probability at least

$$1 - z^{c/2} e^{-D_1 z^2 p^{4(c-3)}} \geq 1 - n^{c/2} \exp\left(-D_2 \frac{n^2 p^{4c-10}}{(\log n)^2}\right). \tag{2.11}$$

for some constants $D_1, D_2 > 0$. Recalling that $p = \frac{1}{n^\beta}$ and choosing $\beta < \frac{1}{4c-10}$ strictly, the final term in (2.11) is at least $1 - e^{-2D_3 n (\log n)^2}$ for some constants $D_3 > 0$. Fixing such a β and using the fact that there are at most $n^n = e^{n\log n}$ choices for the partition V, we get from the union bound that

$$\mathbb{P}(E_{path}) \geq 1 - e^{n\log n} e^{-2D_3 n(\log n)^2} \geq 1 - e^{-D_3 n (\log n)^2} \tag{2.12}$$

for all n large.

The estimate (2.12) is useful in extracting proper colourings of G that are also good, thereby allowing for a contradiction argument, as described below. Let $V = \{V_1, \ldots, V_t\}$ be a partition of the vertex set $\{1, 2, \ldots, n\}$ representing a $t-$proper colouring of G so that each V_i is stable (i.e., no edge of G has both endvertices in V_i) and all the vertices in V_i have the same colour i. We denote V_i to be a colour *class* of V and estimate below the probability of the event E_{size} that each colour class in a proper colouring of G contains at most $\frac{8\log n}{p}$ vertices.

From the discussion in the above paragraph we see that each V_i is a stable set and so the probability that some $V_i, 1 \leq i \leq t$ has size $t \geq \frac{8\log n}{p}$ is

$$\begin{aligned} \mathbb{P}(E^c_{size}) &\leq \binom{n}{t} \cdot (1-p)^{\binom{t}{2}} \\ &\leq e^{t\log n} \exp\left(-\frac{pt^2}{4}\right) \\ &\leq \exp\left(-\frac{pt^2}{8}\right) \end{aligned} \tag{2.13}$$

where the first inequality in (2.13) is true since $\binom{n}{t} \leq n^t = e^{t\log n}$ and $(1-p)^{\binom{t}{2}} \leq e^{-p\binom{t}{2}} \leq \exp\left(-\frac{pt^2}{4}\right)$ and the second inequality in (2.13) follows from the fact that $t \geq \frac{8\log n}{p}$.

Suppose E_{size} occurs so that each $V_i, 1 \leq i \leq t$ has size at most $\frac{8\log n}{p}$. This necessarily implies that $t \geq \frac{np}{9\log n}$ and we now count the number of "bad" sets in V that have exactly one vertex in them. If $V_1, \ldots, V_x$ are the sets in V containing exactly one vertex each, then the remaining $n-x$ vertices of G must be "squeezed"

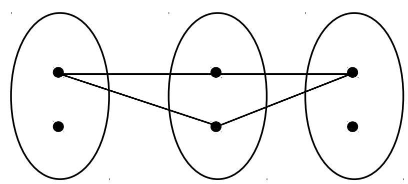

FIGURE 1. Obtaining a cycle of length c containing $\frac{c}{2}+1$ distinct colours for $c = 4$. The solid lines in the figure correspond to the cycle.

into the remaining $t - x$ sets $V_{x+1}, \ldots, V_t$. By the pigeonhole principle, at least one set in V therefore would have size at least $\frac{n-x}{t-x} \geq \frac{10 \log n}{p}$ if

$$x \geq \frac{t - \frac{np}{10 \log n}}{1 - \frac{p}{10 \log n}} =: x_0,$$

which cannot happen since E_{size} occurs. Thus $x \leq x_0$ and there are at least

$$y_0 := t - x_0 \geq (n-t)\frac{p}{10 \log n}\left(1 - \frac{p}{10 \log n}\right)^{-1} \tag{2.14}$$

sets in V each containing at least two vertices.

Summarizing the discussion in the previous two paragraphs we have the following: If E_{bad} is the event that there exists a proper $(0,c)-$acyclic colouring of G containing $t \leq \frac{n}{8}$ colours and E_{path} is the event defined prior to (2.12), then the occurrence of $E_{bad} \cap E_{size} \cap E_{path}$ necessarily implies that there exists a proper $(0,c)-$acyclic colouring $V = \{V_1, \ldots, V_t\}$ containing $\frac{8 \log n}{p} \leq t \leq \frac{n}{8}$ colour classes. Moreover, from (2.14), we also get that there at least

$$\frac{7np}{80 \log n}\left(1 - \frac{p}{10 \log n}\right)^{-1} \geq \frac{7np}{100 \log n} =: z \tag{2.15}$$

colour classes in V, each containing at least two vertices.

From the description prior to (2.12), we then get that the graph $H(V)$ corresponding to the partition V contains a path of length $\frac{c}{2}$. But this necessarily implies that G contains a cycle of length c with $\frac{c}{2}+1$ distinct colours, as illustrated in Figure 1 for the case $c = 4$. This leads to a contradiction and so

$$\mathbb{P}(E_{bad}) \leq \mathbb{P}(E^c_{path} \cup E^c_{size}) \leq \mathbb{P}(E^c_{path}) + \mathbb{P}(E^c_{size}) \longrightarrow 0$$

as $n \to \infty$ by (2.12) and (2.13). This completes the proof of Theorem 2.2(a). ■

Using Lemma 2.3, we now prove Theorem 2.2(b).

Proof of Theorem 2.2(b): To estimate the fractional acyclic chromatic number $\chi_{ayc}(\eta, c)$, we first show that for any constant integer $s \geq c+1$, the cycles of length at most $s-1$ in G constitute less than a fraction η of the total number of cycles in G. Next, we use the local lemma described in Lemma 2.3 to estimate the number of colours needed to obtain a proper colouring of G in which each path of length g or more, contains at least c distinct colours. Finally, we choose g large enough so that $\chi_{\eta,c} = o(n)$ with high probability. We provide the details below.

Any cycle Q of length k in K_n is present in the graph G with probability p^k and there are $\binom{n}{k}$ choices for the vertex set of Q. Thus if N_k is the number of cycles of length k in the random graph G, then $\mathbb{E}N_k = \binom{n}{k}p^k$. Using the bounds $\binom{n}{k} \leq n^k$ and

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} \geq \frac{(n-k+1)^k}{k!} \geq \frac{n^k}{2^k k!}$$

for all n large, we obtain the bounds

$$\frac{(np)^k}{2^k k!} \leq \mathbb{E}N_k \leq (np)^k. \tag{2.16}$$

To obtain a deviation bound for N_k, we let $Q_0 = (1, 2, \dots, k, 1)$ be the cycle of length k and use Lemma 3.5 of [10] to get that

$$var(N_k) \leq C_1 \frac{(\mathbb{E}N_k)^2}{\Phi_0}$$

for some constant $C_1 > 0$ where

$$\Phi_0 = \min_{H \subseteq Q_0} n^{v_H} p^{e_H}$$

and the minimum is over all non-empty subgraphs H of Q_0 containing $v_H \geq 2$ vertices, $e_H \geq 1$ edges and no isolated vertices. Any such H is a collection of vertex disjoint paths $P_1, \dots, P_y$ where each P_j contains $v_j \geq 2$ vertices and $v_j - 1$ edges and so

$$n^{v_H} p^{e_H} = \prod_{j=1}^{y} n^{v_j} p^{v_j - 1} \geq n^2 p.$$

This in turn implies that $\Phi_0 = n^2 p$ and so

$$var(N_k) \leq C_1 \frac{(\mathbb{E}N_k)^2}{n^2 p}. \tag{2.17}$$

Recalling the bounds for $\mathbb{E}N_k$ in (2.16), we now define

$$E_k := \left\{ \frac{(np)^k}{2^{k+1} k!} \leq N_k \leq 2(np)^k \right\}$$

and get from the (2.17) and the Chebychev's inequality that

$$\mathbb{P}\left(E_k\right) \geq 1 - \frac{C_2}{n^2 p}$$

for some constant $C_2 > 0$. For constant $g \geq c+1$ to be determined later, we then define $E_{cyc} := \bigcap_{c \leq k \leq g} E_k$ and get from the union bound that

$$\mathbb{P}(E_{cyc}) \geq 1 - \frac{C_3}{n^2 p} \longrightarrow 1 \tag{2.18}$$

as $n \to \infty$, for some constant $C_3 > 0$. If E_{cyc} occurs, then the total number of cycles of length at most $g-1$ is at most $\sum_{k=3}^{g-1} 2(np)^k \leq C_4(np)^{g-1}$ for some constant $C_4 > 0$ and the number of cycles of length g is at least $\frac{(np)^g}{2^{g+1}g!}$. Thus the number of cycles of length at most $g-1$, is at most a fraction $\frac{C_5}{np} \leq \eta$ of the total number of cycles in G, for some constant $C_5 > 0$. This completes the first step in our proof.

In the second step, we use the probabilistic method to obtain a proper colouring in which each *path* of length g or more has at least c colours. This in turn ensures that each cycle of length g or more has at least c colours. Recalling the event E_{deg} defined before, we see that if E_{deg} occurs, then the degree of each vertex in G lies between $\frac{np}{2}$ and $2np$. We henceforth assume that $E_{deg} \cap E_{cyc}$ occurs and let $1 \leq \frac{np}{2} \leq \Delta \leq 2np$ be the maximum vertex degree in G. For integer $\theta = \theta(n) \geq 2\Delta c \geq 2c$ to be determined later, let $X_1, \ldots, X_n$ be independent and uniformly distributed in $\{1, 2, \ldots, \theta\}$, also independent of G and let $\mathbb{P}_X$ denote the distribution of $(X_1, \ldots, X_n)$.

For $1 \leq u < v \leq n$ let $A_{u,v}$ be the event that $X_u = X_v$. For distinct integers $U := \{u_1, \ldots, u_g\}$, let B_U be the event that the set $\{X_{u_1}, \ldots, X_{u_g}\}$ has cardinality at most $c-1$. There are constants $D_l, 1 \leq l \leq c-1$ such that

$$\mathbb{P}_X(A_{u,v}) = \frac{1}{\theta} =: \frac{y(u,v)}{2} \text{ and } \mathbb{P}_X(B_U) \leq \sum_{l=1}^{c-1} \binom{\theta}{l} \frac{D_l}{\theta^g} \leq \frac{D\theta^{c-1}}{\theta^g} =: \frac{y(U)}{2}. \quad (2.19)$$

We now use Lemma 2.3 with weights $y(.)$ as defined above. We construct the dependency graph Γ whose vertices are either edges of G which we call as type A vertex or paths of length g in the graph G which we call as type B vertex. We recall that the maximum degree of any vertex is at most $2np$. Therefore for any vertex $u \in K_n$, there are at most $g(2np)^g$ paths in G containing u as an endvertex. This implies any type A vertex in Γ is adjacent to at most $2np$ type A vertices and at most $2g(2np)^g$ type B vertices. Similarly, any type B vertex is adjacent to at most $2gnp$ type A vertices and at most $g^2(2np)^g$ type B vertices. Consequently, if we ensure that

$$\frac{1}{\theta} \leq \frac{2}{\theta}\left(1-\frac{2}{\theta}\right)^{2np}\left(1-\frac{D\theta^{c-1}}{\theta^g}\right)^{2g(2np)^g} \quad (2.20)$$

and

$$\frac{D\theta^{c-1}}{\theta^g} \leq \frac{2D\theta^{c-1}}{\theta^g}\left(1-\frac{2}{\theta}\right)^{2gnp}\left(1-\frac{D\theta^{c-1}}{\theta^g}\right)^{g^2(2np)^g}, \quad (2.21)$$

then Lemma 2.3 with weight assignments as in (2.19) guarantees the existence of a proper colouring of G using θ colours, in which every path of length g contains at least c distinct colours.

Setting $\theta = 8D\left(g^2(2np)^g\right)^{\frac{1}{g-c+1}}$ and using $(1-x)^a \geq 1-ax$, we see that both (2.20) and (2.21) are satisfied for all n large and so summarizing, we get that if $E_{deg} \cap E_{cyc}$ occurs, then $\chi_{\eta,c}(G) \leq C(np)^{\frac{g}{g-c+1}}$. Finally, choosing $g = g(\beta, c)$ large enough so that $(np)^{\frac{g}{g-c+1}} = n^{\frac{(1-\beta)g}{g-c+1}} \leq n^{1-\frac{\beta}{2}}$ and using (2.18) and (2.10), we get that $\frac{\chi_{\eta,c}(G)}{n} \longrightarrow 0$ in probability. This completes the proof of Theorem 2.2(b). ■

3. Tree Unique Colourings

In this section, we study tree unique colourings of the random graph $G = G(n,p)$ with edge probability p as defined in (2.1). We begin with some definitions. For integer $k \geq 1$ let f be a $k-$colouring of G, not necessarily proper. For a constant integer $t \geq 2$ let T be any deterministic labelled tree on t vertices and with vertex set $V(T)$. We say that a subgraph H of G with vertex set $V(H)$ is *isomorphic* to T if there exists a bijection $g : V(H) \to V(T)$ such that any two vertices u and v in H are adjacent if and only if $g(u)$ and $g(v)$ are adjacent in T.

Definition 3.1. We say that f is a $T-$unique colouring of G if every subgraph H of G isomorphic to T contains at least one unique colour.

We let $\chi_T(G)$ denote the minimum number of colours needed for a $T-$unique colouring of G.

If T is simply the single edge on $t = 2$ vertices, then Definition 3.1 reduces to the usual notion of proper colouring described before.

For general T, we have the following result regarding bounds for $\chi_T(G)$.

Theorem 3.2. *If $p = o(1)$ and $\frac{np^{2t-1}}{\log n} \longrightarrow \infty$ as $n \to \infty$, then*

$$\mathbb{P}\left(\frac{D_1 np^{2t-1}}{\log n} \leq \chi_T(G) \leq D_2(np)^{2\left(1-\frac{1}{t}\right)}\right) \geq 1 - e^{-D_3(\log n)^2} - ne^{-D_3 np} \tag{3.1}$$

for some positive constants $D_i, 1 \leq i \leq 3$ and all n large.

In particular, allowing T to be the star graph on t vertices, we see from Theorem 3.2 that even $1-$unique colouring with *bounded* sub-neighbourhood constraints requires at least of the order of $\frac{np^{2t-1}}{\log n}$ colours.

Proof of Theorem 3.2: We begin with the proof of the upper bound. The expected degree of any vertex in G is $(n-1)p$ and so if E_{deg} denotes the event that each vertex in G has degree at most $2np$, then the deviation estimate (2.3) together with the union bound implies that

$$\mathbb{P}(E_{deg}) \geq 1 - ne^{-C_1 np} \tag{3.2}$$

for some constant $C_1 > 0$.

Suppose E_{deg} occurs and let $X_1, \ldots, X_n$ be independent random variables (also independent of G) that are uniformly chosen from the set $\{1, 2, \ldots, q\}$ for some integer $q \geq 1$ to be determined later. Assign colour X_i to vertex i and let $\mathbb{P}_X$ denote the distribution of $(X_1, \ldots, X_n)$. Let T be the set of all subgraphs of G isomorphic to T and for a tree $\tau \in T$, let A_τ be the event that τ has a unique colour.

Recalling that t denotes the number of vertices in T and assuming that $t = 2z$ is even for simplicity, we now estimate the $\mathbb{P}_X-$probability that τ does not have a unique colour. Indeed, if the tree τ containing $2z$ vertices does not have a unique colour, then at most z colours have been used to colour the vertices of τ. The number of ways of choosing $l \leq z$ colours is $\binom{q}{l}$ and the number of ways of assigning these l colours to the vertices of τ is at most l^{2z}. Therefore if $q \geq 2z$ then $\mathbb{P}_X(A_\tau^c) \leq \sum_{1 \leq l \leq z} \frac{\binom{q}{l} l^{2z}}{q^{2z}} \leq \frac{z\binom{q}{z} z^{2z}}{q^{2z}}$ due to the monotonicity of the binomial coefficient and the fact that $q \geq 2z$. Further using the estimate $\binom{q}{z} \leq \left(\frac{qe}{z}\right)^z$, we then get that

$$\mathbb{P}_X(A^c_\tau) \leq z\left(\frac{qe}{z}\right)^z \cdot \frac{z^{2z}}{q^{2z}} = z\left(\frac{ze}{q}\right)^z := \frac{y(\tau)}{2}. \quad (3.3)$$

By definition, the events A_τ and A_λ are dependent only if the trees τ and λ share a common vertex, which we denote as $\tau \cap \lambda \neq \emptyset$. Since E_{deg} occurs, the degree of each vertex is at most $2np$ and so the number of subgraphs of G isomorphic to T and containing the vertex 1 is at most $C_2(np)^{t-1} = C_2(np)^{2z-1}$ for some large constant $C_2 > 0$. This in turn implies that $\tau \cap \lambda \neq \emptyset$ for at most $C_2(np)^{2z-1}$ trees $\lambda \in T$ and so from (3.3), we see that there is a constant $C_3 > 0$ such that

$$\begin{aligned} y(\tau)\prod_{\lambda:\lambda\cap\tau\neq\emptyset}(1-y(\lambda)) &\geq 2z\left(\frac{ze}{q}\right)^z\left(1-2z\left(\frac{ze}{q}\right)^z\right)^{C_2(np)^{2z-1}} \\ &\geq 2z\left(\frac{ze}{q}\right)^z\left(1-C_3\frac{(np)^{2z-1}}{q^z}\right) \\ &\geq z\left(\frac{ze}{q}\right)^z \\ &\geq \mathbb{P}_X(A_\tau), \end{aligned}$$

provided $q^z \geq 2C_3(np)^{2z-1}$ or equivalently $q \geq C_4(np)^{2-\frac{1}{z}}$ for some constant $C_4 > 0$. Fixing such a q, the local lemma (Lemma 2.3) then implies that there is a $T-$unique colouring of G with positive $\mathbb{P}_X-$probability. From (3.2), we then obtain the upper bound in (3.1).

For the lower bound, we use Lemma 2.4. Let C be the constant in Lemma 2.4 and for a deterministic set S containing $s := \frac{2\log n}{Cp^{2t-1}}$ vertices, let $E_{tree}(S)$ be the event that the subgraph G_S of G induced by the vertices of S, itself contains a subgraph isomorphic to T. From Lemma 2.4, we see that $\mathbb{P}(E^c_{tree}(S)) \leq e^{-Cs^2p^{2t-1}}$ and so if $E_{tree} := \bigcap_S E_{tree}(S)$ where the intersection is over all sets S containing s vertices, the union bound implies that $\mathbb{P}(E^c_{tree}) \leq \binom{n}{s}e^{-Cs^2p^{2t-1}}$. Using $\binom{n}{s} \leq n^s = e^{s\log n}$ we then get that

$$\mathbb{P}(E^c_{tree}) \leq e^{s\log n - Cs^2p^{2t-1}} = e^{-s\log n} \leq e^{-C_5(\log n)^2} \quad (3.4)$$

for some constant $C_5 > 0$, since $s = \frac{2\log n}{Cp^{2t-1}} \geq \frac{2\log n}{C}$. If E_{tree} occurs and f is a $T-$unique colouring of G, then each colour is used on at most s vertices and therefore $\chi_T(G) \geq \frac{n}{s} \geq \frac{Cnp^{2t-1}}{2\log n}$. The estimate (3.4) then obtains the lower bound in (3.1). ■

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