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INDUCED UPPER-CONVECTED MAXWELL FLUID FLOW IN A CYLINDER OSCILLATING LONGITUDINALLY AND TORSIONALLY WITH SLIP AT THE BOUNDARY

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Abstract. This study examines the time dependent incompressible fluid flow of a non-Newtonian upper-convected Maxwell, (UCM), fluid within a very long cylinder, which is oscillating in both the longitudinal and torsional directions, with slip occurring at the boundary. Based on the proposed governing equations, analytical expressions are obtained for the velocity of the fluid, stresses, work done, the cylinder's drag and the drag coefficient, which are examined graphically.

1. Introduction

A rod, infinite in length, rotating in a Newtonian fluid was examined in the study of Stokes in 1886 [15]. In this pioneering work, an exact solution was presented for the velocity field. This work was followed by the study of Casarella and Laura [4] in which a cylindrical cable is undergoing longitudinal and torsional oscillations in a Newtonian fluid. Based on the proposed governing equations, the velocity components and viscous drag forces acting on the cylinder's wall were obtained by analytical solutions. However, with much interest in the study of non-Newtonian fluids and its applications, Rajagopal [12] utilized the non-Newtonian fluid, second-grade fluid, and derived exact solutions to the proposed equations.

It should be noted that relatively few authors have considered the internal problem. The corresponding internal problem to that proposed by Casarella and Laura [4] was explored by the researchers, Ramkissoon and Majumdar [13]. The work of Ramkissoon and Majumdar [13] was extended by Calmelet-Eluhu and Majumdar [3] who used a Micropolar fluid instead, and examined the fluid's velocity, micro-rotation, drag force and shear stresses acting on the cylinder's wall. An upper-convected Maxwell fluid was used by Rahaman [11] to examine this flow and to make comparisons with its Newtonian counterpart.

Owen and Rahaman [8] also considered the internal flow problem using an Oldroyd-B fluid to derive analytical expressions for the shear-stresses, velocity components, drag on the cylinder and work done. The study of Murthy and

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Bahali [7] used a micropolar fluid, in which the cylinder oscillated with different frequencies to obtain analytical expressions for the velocity components and microrotation components in terms of modified Bessel's functions, along with the drag force acting on the wall of the cylinder. Phillips and Rahaman [10] examined a Newtonian fluid which is subjected to different amplitudes of oscillations of a cylinder to obtain analytical expressions for shear stresses, the velocity of the fluid, drag coefficient, drag forces and work done by these drag forces. Ramlal and Rahaman [14] also considered different frequencies using a Newtonian fluid. Kumar, Sil and Prajapati [6] examined the rotating flow of a non-Newtonian MHD fluid through a porous media with Hall effect in a magnetic field and obtained exact solutions for straight parallel flows.

It should be noted that these studies did not include the effect of slip occurring on the wall of a cylinder which should be taken into consideration since even before the linear stress-strain relationship becomes invalid, the no-slip condition becomes invalid [5]. Thus, the main objective of this research is to investigate the unsteady flow of an incompressible non-Newtonian upper-convected Maxwell fluid, subject to longitudinal and torsional oscillations, by including slip occurring on the wall of the cylinder. Analytical expressions for fluid's velocity, cylinder's drag, shear stresses, drag coefficient and work done are obtained and some illustrated graphically to draw conclusions on the effects of slip. Although this work includes a periodic oscillation of a boundary, no issues were obtained as addressed by Takhirov [16] on periodic boundaries.

2. Statement of the Problem

The rheological equation of state for the upper-convected Maxwell fluid is given by [9]

$$\underline{T} = -p\underline{I} + \underline{S} \tag{2.1}$$

where

$$\underline{S} + \lambda \check{\underline{S}} = 2\mu \underline{D} \tag{2.2}$$

and $\underline{T}$ is the total stress, p is the isotropic pressure, $\underline{S}$ is the extra stress tensor, λ is the relaxation time, μ is the viscosity coefficient, $\underline{D}$ is the deformation rate tensor, and $\check{\underline{S}}$ is the upper-convected derivative of the extra stress tensor defined by,

$$\check{\underline{S}} = \check{S}_{ij} = \frac{\partial S_{ij}}{\partial t} + q_m S_{ij,m} - S_{im} q_{j,m} - S_{mj} q_{i,m} \tag{2.3}$$

The unsteady flow of an incompressible upper-convected Maxwell fluid undergoing longitudinal and torsional oscillations contained in an infinitely long cylinder, of radius 'a' with slip occurring at the cylinder's wall, is examined. The cylindrical polar coordinates, (R, θ, z), are used due to the nature of the flow, with the axis of the cylinder coinciding with the z-axis. At $R = a$, the velocity of the cylinder, $\underline{q}_a$, is given by,

$$\underline{q}_a = q_0 \, cos\alpha \, cos(\Omega t)\hat{\theta} + q_0 \, sin\alpha \, cos(\Omega t)\hat{z} \tag{2.4}$$

where q_0, α and Ω are real constants [4, 13].

The oscillations are purely torsional for $\alpha = 0$ or π and purely longitudinal for $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. It is fair to assume that there is no radial component of the fluid's velocity due to the motion of the cylinder. Additionally, an axisymmetric flow about the z-axis is assumed. Therefore, the form of the fluid's velocity is,

$$\underline{q} = v(R,t)\hat{\theta} + w(R,t)\hat{z} \tag{2.5}$$

The following continuity equation,

$$\nabla \cdot \underline{q} = 0 \tag{2.6}$$

is satisfied by Equation (2.5) since the fluid is incompressible [2].

For the slip at the boundary of the cylinder, Basset's slip condition is used [1]. This condition states that the tangential stress, T_t, exerted by a solid upon the fluid is in the same direction as that of the relative velocity and proportional to it, thus, for the oscillating cylinder,

$$T_t = \beta(u - u_a) \tag{2.7}$$

where β is the slip parameter, known as the 'Coefficient of Sliding Friction', u and u_a are velocity components of the fluid and boundary respectively. Further, when $\beta = 0$ there is perfect slip, and when $\beta \to \infty$ there is no-slip.

Using Equation (2.2) - Equation (2.5) in Equation (2.7) gives at $R = a$, on the cylinder's wall,

$$\beta[v(R,t) - q_0\, cos\alpha\, cos(\Omega t)] = \mu\left(\frac{\partial v}{\partial R} - \frac{v}{R}\right) \tag{2.8}$$

and

$$\beta[w(R,t) - q_0\, sin\alpha\, cos(\Omega t)] = \mu\,\frac{\partial w}{\partial R} \tag{2.9}$$

On eliminating the stresses from Equation (2.2) and the dynamic equation,

$$\nabla \cdot \underline{S} - \nabla p = \rho\frac{d\underline{q}}{dt} \tag{2.10}$$

where ρ is the density, gives the equation of motion for the $\hat{\theta}$-component of the velocity as,

$$\upsilon\left(\frac{1}{R}\frac{\partial v}{\partial R} + \frac{\partial^2 v}{\partial R^2} - \frac{v}{R^2}\right) = \frac{\partial v}{\partial t} + \lambda\frac{\partial^2 v}{\partial t^2} \tag{2.11}$$

and that of the $\hat{z}$-component as,

$$\upsilon\left(\frac{1}{R}\frac{\partial w}{\partial R} + \frac{\partial^2 w}{\partial R^2}\right) = \frac{\partial w}{\partial t} + \lambda\frac{\partial^2 w}{\partial t^2} \tag{2.12}$$

where $\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity.

It is observed in Equation (2.11) and Equation (2.12) that on taking the elastic parameter, $\lambda = 0$, one gets the governing equations for classical fluids [13].

3. Main Results

It is assumed that the $\hat{\theta}$-component of the velocity of the fluid takes the form,

$$v(R,t) = Re[f(R)e^{i\Omega t}] \tag{3.1}$$

where Re pertains to the part which is real, and substituting into Equation (2.11), using Equation (2.8) and that the velocity remains finite as $R \to 0$, it follows that,

$$v(R,t) = Re\left[\frac{I_1(\gamma R)\, q_0\, cos\alpha}{\left(1+\frac{2\mu}{a\beta}\right) I_1(\gamma a) - \frac{\mu\gamma}{\beta} I_0(\gamma a)} e^{i\Omega t}\right] \tag{3.2}$$

where

$$\gamma = \sqrt{\frac{i\Omega - \lambda\Omega^2}{\nu}} \tag{3.3}$$

In a similar manner, using Equation (2.9) and Equation (2.12), one gets the $\hat{z}$-component of the fluid's velocity as,

$$w(R,t) = Re\left[\frac{I_0(\gamma R)\, q_0\, sin\alpha}{I_0(\gamma a) - \frac{\mu\gamma}{\beta} I_1(\gamma a)} e^{i\Omega t}\right] \tag{3.4}$$

It is noted that as $\beta \to \infty$ in Equation (3.2) and Equation (3.4), one gets the corresponding results obtained in the case of no-slip at the boundary by [11]. Further, on also taking the elastic parameter, $\lambda = 0$, the classical result for the Newtonian case is obtained [13].

Using Equation (2.2), the tangential stresses gives at $R = a$, on the cylinder's wall,

$$S_{R\theta|_{R=a}} = Re\left[\frac{\gamma I_0(\gamma a) - \frac{2}{a} I_1(\gamma a)}{\left(1+\frac{2\mu}{a\beta}\right) I_1(\gamma a) - \frac{\mu\gamma}{\beta} I_0(\gamma a)} \frac{\mu q_0\, cos\alpha}{1+i\Omega\lambda} e^{i\Omega t}\right] \tag{3.5}$$

and

$$S_{Rz|_{R=a}} = Re\left[\frac{\gamma I_1(\gamma a)}{I_0(\gamma a) - \frac{\mu\gamma}{\beta} I_1(\gamma a)} \frac{\mu q_0\, sin\alpha}{1+i\Omega\lambda} e^{i\Omega t}\right] \tag{3.6}$$

The tangential drag, $\underline{D}$, acting on the wall of the cylinder per unit length is given by [13],

$$\underline{D} = -2\pi a(S_{R\theta}\hat{\theta} + S_{Rz}\hat{z})|_{R=a} \tag{3.7}$$

Using Equation (3.5) and Equation (3.6) here gives,

$$\begin{aligned}\underline{D} \;=\; -2\pi a\mu q_0 Re\Bigg[\Bigg(\Bigg(&\frac{\gamma I_0(\gamma a) - \frac{2}{a} I_1(\gamma a)}{\left(1+\frac{2\mu}{a\beta}\right) I_1(\gamma a) - \frac{\mu\gamma}{\beta} I_0(\gamma a)} cos\alpha\Bigg)\hat{\theta} \\ &+ \left(\frac{\gamma I_1(\gamma a)}{I_0(\gamma a) - \frac{\mu\gamma}{\beta} I_1(\gamma a)} sin\alpha\right)\hat{z}\Bigg)\frac{e^{i\Omega t}}{1+i\Omega\lambda}\Bigg]\end{aligned} \tag{3.8}$$

On taking the slip parameter, $\beta \to \infty$, for the no-slip case in Equation (3.5), Equation (3.6) and Equation (3.8) one gets the results given by [11]. Further,

taking the elastic parameter, $\lambda = 0$, the classical result for the Newtonian case is obtained [13].

An alternative form for the tangential drag on the cylinder given by Equation (3.7) is,

$$\underline{D} = -2\pi a S(cos\delta\,\hat{\theta} + sin\delta\,\hat{z}) \tag{3.9}$$

where

$$S = \sqrt{S_{R\theta}^2 + S_{Rz}^2},\ S_{R\theta} = S\,cos\delta,\ S_{Rz} = S\,sin\delta \tag{3.10}$$

with

$$tan\,\delta = \frac{S_{Rz}}{S_{R\theta}} \tag{3.11}$$

The work done, W, by the drag force $\underline{D}$ on the fluid per half-cycle of motion is given by [13],

$$W = -\int_0^{\frac{\pi}{\Omega}} \underline{D} \cdot \underline{q}_a\, dt \tag{3.12}$$

On using Equation (2.4) and Equation (3.8) gives,

$$\begin{aligned} W \;=\; & \frac{\pi^2 a\mu q_0^2}{\Omega} Re\left[\left(\frac{\gamma I_0(\gamma a) - \frac{2}{a} I_1(\gamma a)}{\left(1 + \frac{2\mu}{a\beta}\right) I_1(\gamma a) - \frac{\mu\gamma}{\beta} I_0(\gamma a)} cos^2\alpha \right.\right. \\ & \left.\left. + \frac{\gamma I_1(\gamma a)}{I_0(\gamma a) - \frac{\mu\gamma}{\beta} I_1(\gamma a)} sin^2\alpha\right) \frac{1}{1 + i\Omega\lambda}\right] \end{aligned} \tag{3.13}$$

Taking the slip parameter, $\beta \to \infty$, for the no-slip case gives the result by [11]. Using Equation (3.9) in Equation (3.12) gives the alternative expression for the work done as,

$$W = 2\pi a q_0 \int_0^{\frac{\pi}{\Omega}} S\,cos(\delta - \alpha)\,cos(\Omega t)\,dt \tag{3.14}$$

The drag coefficient, C, is deduced by comparing the work done on the fluid with a hypothesized drag force of the form [13],

$$\underline{D}_H = -C q_a^n (cos\alpha\,\hat{\theta} + sin\alpha\,\hat{z}) \tag{3.15}$$

On using Equation (2.4) in Equation (3.15) gives,

$$\underline{D}_H = -C q_0^n cos^n(\Omega t)(cos\alpha\,\hat{\theta} + sin\alpha\,\hat{z}) \tag{3.16}$$

On using Equation (3.16) and Equation (2.4) in Equation (3.12) gives,

$$W = C q_0^{n+1} \int_0^{\frac{\pi}{\Omega}} cos^{n+1}(\Omega t)\,dt \tag{3.17}$$

On equating Equation (3.14) to Equation (3.17) gives the drag coefficient as,

$$C = \frac{2\pi a}{q_0^n} \frac{\int_0^{\frac{\pi}{\Omega}} S\,cos(\delta - \alpha)\,cos(\Omega t)\,dt}{\int_0^{\frac{\pi}{\Omega}} cos^{n+1}(\Omega t)\,dt} \tag{3.18}$$

Since C is a dimensionless quantity, and it can be shown that S is proportional to q_0, then $n = 1$ will provide the only meaningful case. As a result, Equation (3.18) becomes,

$$C = \frac{2\pi a}{q_0} \frac{\int_0^{\frac{\pi}{\Omega}} S\, cos(\delta - \alpha)\, cos(\Omega t)\, dt}{\int_0^{\frac{\pi}{\Omega}} cos^2(\Omega t)\, dt} \tag{3.19}$$

4. Graphs and Discussion

To have a better understanding of the effects of slip and elasticity of the UCM fluid, some graphs have been generated using some of the values by [11] and [13], in particular, $a = 1$, $\upsilon = 0.1$, $\Omega = 3.6$, $\lambda = 0.3$ and various values for β.

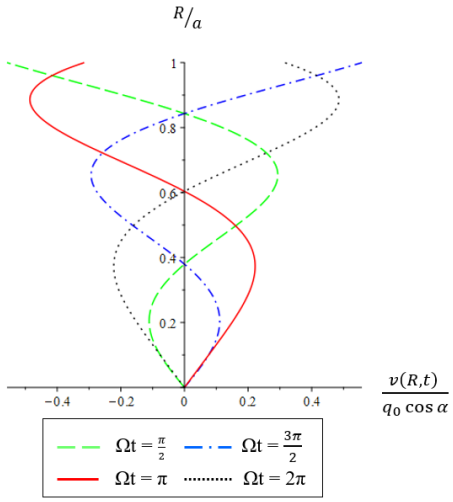


FIGURE 1. UCM $\hat{\theta}$-component of velocity for $\frac{a\beta}{\mu} = 5$

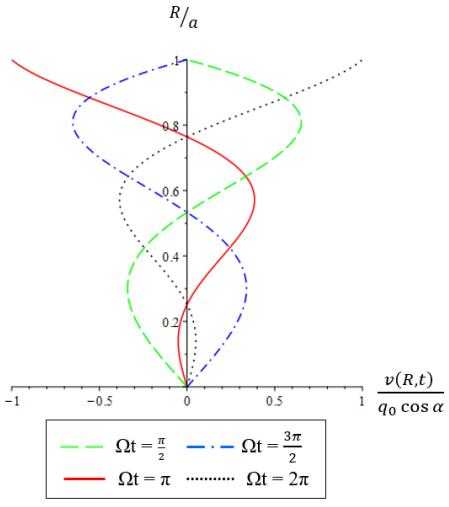


FIGURE 2. UCM $\hat{\theta}$-component of velocity for $\frac{a\beta}{\mu} \to \infty$

Figure 1 gives the $\hat{\theta}$-component of velocity with slip, whereas Figure 2 gives the same velocity component at the same times without slip. As is observed, the amplitude of the oscillations in the no-slip case is greater than that observed for slip. It is noted that for the no-slip case, Figure 2 gives the same graph obtained by [11].

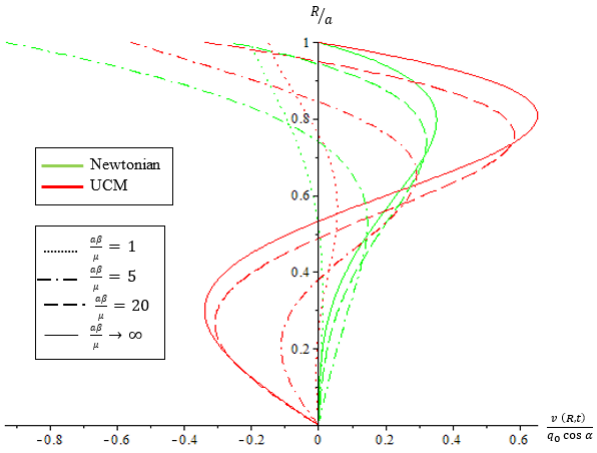


FIGURE 3. $\hat{\theta}$-component of velocity for $\Omega t = \frac{\pi}{2}$

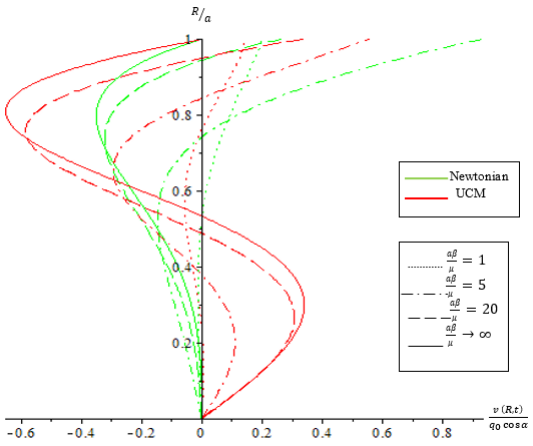


FIGURE 4. $\hat{\theta}$-component of velocity for $\Omega t = \frac{3\pi}{2}$

Figures 3 and 4 give comparisons of the $\hat{\theta}$-component of velocity for UCM and a Newtonian fluid, at different times for various values of the slip parameter. As observed, the amplitude of oscillations is greater for UCM fluids when compared to its Newtonian counterpart.

Figure 5 gives the $\hat{z}$-component of velocity with slip, whereas Figure 6 gives the same velocity component at the same times without slip. As is observed, and similar to the $\hat{\theta}$-component, the amplitude of the oscillations in the no-slip case is greater than that observed for slip. It is noted that for the no-slip case, figure 6 gives the same graph obtained by [11].

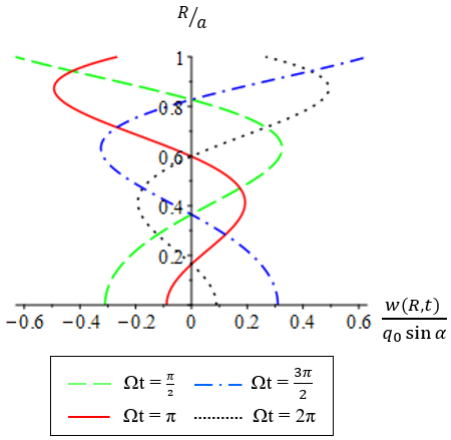


FIGURE 5. UCM $\hat{z}$-component of velocity for $\frac{\alpha\beta}{\mu} = 5$

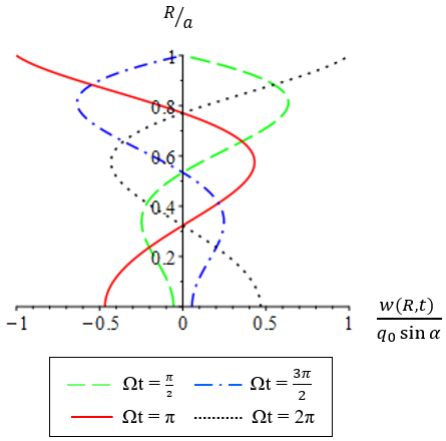


FIGURE 6. UCM $\hat{z}$-component of velocity for $\frac{\alpha\beta}{\mu} \to \infty$

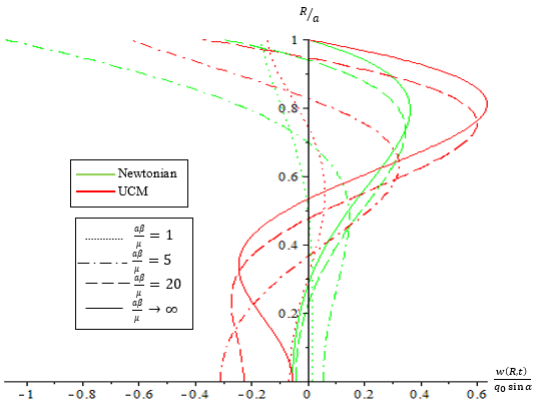


FIGURE 7. $\hat{z}$-component of velocity for $\Omega t = \frac{\pi}{2}$

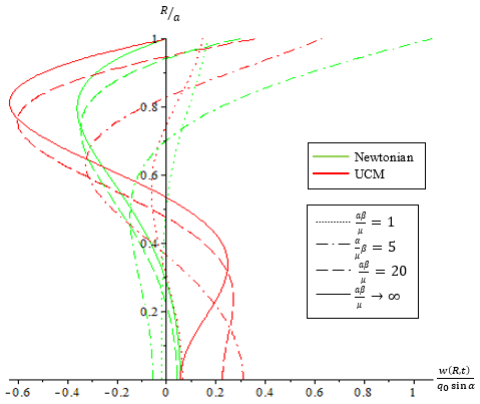


FIGURE 8. $\hat{z}$-component of velocity for $\Omega t = \frac{3\pi}{2}$

Figures 7 and 8 give comparisons of the $\hat{z}$-component of velocity for UCM and Newtonian fluids, at different times for various values of the slip parameter. As observed, and like the $\hat{\theta}$-component, the amplitude of oscillations is greater for UCM fluids when compared to that of a Newtonian fluid.

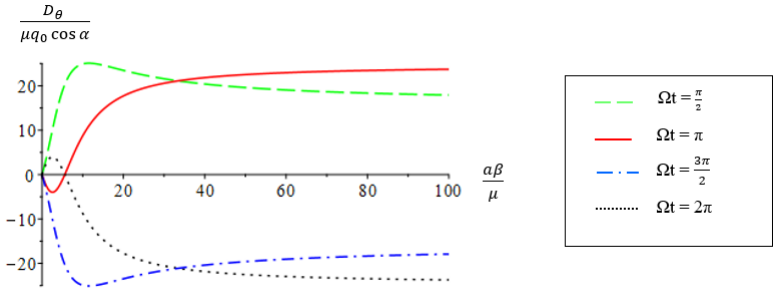


FIGURE 9. $\hat{\theta}$-component of Drag for UCM fluids

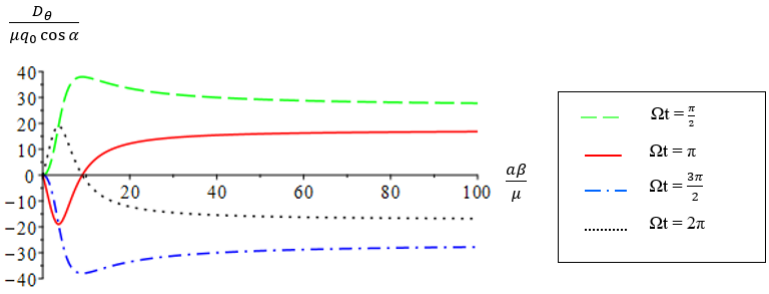


FIGURE 10. $\hat{\theta}$-component of Drag for Newtonian fluids

Figures 9 to 12 depict how slip impacts each component of Drag for both UCM and Newtonian fluids. It is observed for the times taken, that as slip decreases, the Drag appears to stabilize, and that for UCM fluids is less than that for a Newtonian fluid.

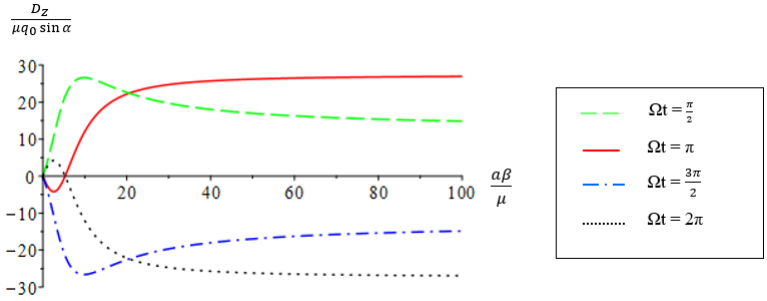


FIGURE 11. $\hat{z}$-component of Drag for UCM fluids

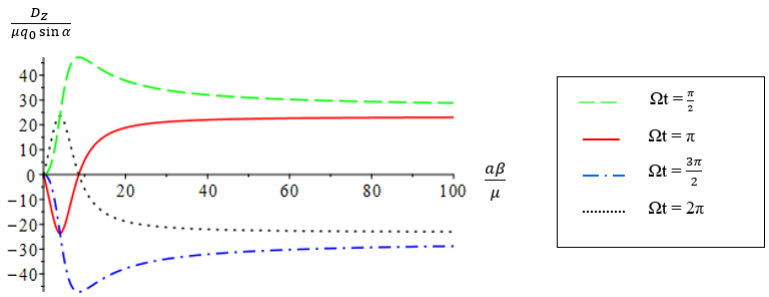


FIGURE 12. $\hat{z}$-component of Drag for Newtonian fluids

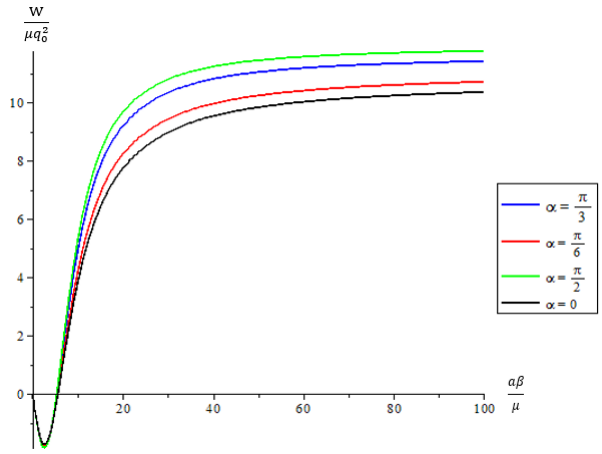


FIGURE 13. Work done for UCM fluids

Figure 13 shows the work done for UCM fluids as $\frac{\alpha\beta}{\mu}$ increases, which corresponds to decreasing slip, for various values of the oscillatory parameter, including purely torsional to purely longitudinal oscillations, while Figure 14 shows the same for a Newtonian fluid. It is observed in each case, the work done increases as the oscillations change from purely torsional to purely longitudinal, but that for a UCM fluid flow is greater than that of a Newtonian fluid.

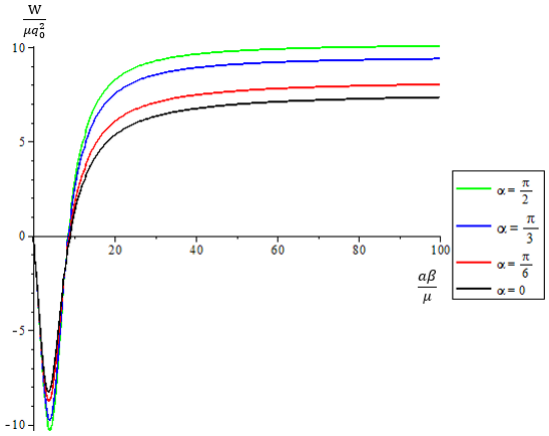


FIGURE 14. Work done for Newtonian fluids

5. Conclusion

The greater the slip, the less influence the moving cylinder has on the velocity components of the UCM fluid. Further, the velocity of the UCM fluid is greater than that of its Newtonian counterpart for the times examined. The Drag for UCM fluids is less than that of a Newtonian fluid, and as the slip decreases, the drag in each case appears to stabilize, with that of the UCM fluid continuing to be less than that for a Newtonian fluid. The work done varies as slip decreases and as the oscillations change from purely torsional to purely longitudinal.

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