

NEW SHARP JORDAN TYPE INEQUALITIES AND THEIR APPLICATIONS

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ABSTRACT. In this paper, we prove that for $x \in (0, \pi/2)$

$$(\cos p_0 x)^{1/p_0} < \frac{\sin x}{x} < \left(\cos \frac{x}{3}\right)^3$$

with the best constants $p_0 \approx 0.34731$ and $1/3$. Moreover, for $x \in (0, c) \subseteq (0, \pi/2)$ the double inequality

$$\beta_p(c)(\cos px)^{1/p} < \frac{\sin x}{x} < (\cos px)^{1/p}$$

holds if $p \in (0, 1/3]$, where $\beta_p(c) = c^{-1}(\cos pc)^{-1/p}$ and 1 are the best possible. Its reverse one holds if $p \in [1/2, 1)$. As applications, some precise estimates for sine integral and Catalan constant are given.

1. INTRODUCTION

The classical Jordan's inequality [10] states that for $x \in (0, \pi/2)$

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1. \quad (1.1)$$

Some new developments on refinements, generalizations and applications of Jordan's inequality can be found in [14] and related references therein.

In [15], Qi et al. proved that

$$\cos^2 \frac{x}{2} < \frac{\sin x}{x} \quad (1.2)$$

holds for $x \in (0, \pi/2)$. Recently, Klén et al. [8, Theorem 2.4] showed that the function $p \mapsto (\cos px)^{1/p}$ is decreasing on $(0, 1)$ and proved that for $x \in (-\sqrt{27/5}, \sqrt{27/5})$

$$\cos^2 \frac{x}{2} \leq \frac{\sin x}{x} \leq \cos^3 \frac{x}{3} \leq \frac{2 + \cos x}{3} \quad (1.3)$$

hold.

The aim of this paper is to give the sharp bounds $(\cos px)^{1/p}$ for $(\sin x)/x$, that is, for $x \in (0, \pi/2)$, to determine the best $p, q \in (0, 1)$ such that

$$(\cos px)^{1/p} < \frac{\sin x}{x} < (\cos qx)^{1/q} \quad (1.4)$$

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hold.

Our main results are contained the following two theorems.

Theorem 1.1. *Let $p, q \in (0, 1)$. Then the inequalities*

$$(\cos px)^{1/p} < \frac{\sin x}{x} < (\cos qx)^{1/q} \quad (1.5)$$

hold for $x \in (0, \pi/2)$ if and only if $p \in [p_0, 1)$ and $q \in (0, 1/3]$, where $p_0 \approx 0.3473$ is the unique root of equation

$$f_p\left(\frac{\pi}{2}\right) = \ln \frac{2}{\pi} - \frac{1}{p} \ln \left(\cos \frac{p\pi}{2}\right) = 0 \quad (1.6)$$

on $(0, 1)$, here f_p is defined by (1.9).

Moreover, we have

$$\left(\cos \frac{x}{3}\right)^\alpha < \frac{\sin x}{x} < \left(\cos \frac{x}{3}\right)^3, \quad (1.7)$$

$$(\cos p_0 x)^{1/p_0} < \frac{\sin x}{x} < (\cos p_0 x)^{1/(3p_0^2)}, \quad (1.8)$$

where the exponents

$$\alpha = \frac{2(\ln \pi - \ln 2)}{\ln 4 - \ln 3} \approx 3.1395, \quad 3 \quad \text{and} \quad 1/p_0 \approx 2.8793, \quad 1/(3p_0^2) \approx 2.7634$$

are the best constants.

Theorem 1.2. *For $p \in (0, 1)$, let f_p be the function defined on $(0, \pi/2]$ by*

$$f_p(x) = \ln \frac{\sin x}{x} - \frac{1}{p} \ln(\cos px). \quad (1.9)$$

Then f_p is decreasing if $p \in (0, 1/3]$ and increasing if $p \in [1/2, 1)$.

Moreover, if $p \in (0, 1/3]$, $x \in (0, c) \subseteq (0, \pi/2)$, then

$$\beta_p(c)(\cos px)^{1/p} < \frac{\sin x}{x} < (\cos px)^{1/p} \quad (1.10)$$

with the best possible constants $\beta_p(c) = c^{-1}(\cos pc)^{-1/p} \sin c$ and 1. (1.10) is reversed if $p \in [1/2, 1)$.

Putting $p = 1/3$, $c = \pi/2$, $\pi/4$ in Theorem 1.2, we have

Corollary 1.3. (i) *For $x \in (0, \pi/2)$*

$$\beta_{1/3}\left(\frac{\pi}{2}\right) \cos^3 \frac{x}{3} < \frac{\sin x}{x} < \cos^3 \frac{x}{3} \quad (1.11)$$

with the best constants $\beta_{1/3}(\pi/2) = 16\sqrt{3}/(9\pi) \approx 0.9801$ and 1.

(ii) *For $x \in (0, \pi/4)$ the inequalities*

$$\beta_{1/3}\left(\frac{\pi}{4}\right) \cos^3 \frac{x}{3} < \frac{\sin x}{x} < \cos^3 \frac{x}{3} \quad (1.12)$$

hold, where $\beta_{1/3}(\pi/4) = 16(3\sqrt{3} - 5)/\pi \approx 0.9990$ and 1 are the best possible.

Putting $p = 1/2$, $c = \pi/2$, $\pi/4$ in Theorem 1.2, we get

Corollary 1.4. *For $x \in (0, \pi/2)$, the inequalities*

$$\left(\cos \frac{x}{2}\right)^2 < \frac{\sin x}{x} < \frac{4}{\pi} \left(\cos \frac{x}{2}\right)^2 \quad (1.13)$$

hold, where $4/\pi$ and 1 are the best possible constants.

2. LEMMAS

Lemma 2.1 ([17], [1]). *Let $f, g : [a, b] \mapsto \mathbb{R}$ be two continuous functions which are differentiable on (a, b) . Further, let $g' \neq 0$ on (a, b) . If f'/g' is increasing (or decreasing) on (a, b) , then so are the functions*

$$x \mapsto \frac{f(x) - f(b)}{g(x) - g(b)} \quad \text{and} \quad x \mapsto \frac{f(x) - f(a)}{g(x) - g(a)}.$$

Lemma 2.2 ([2]). *Let a_n and b_n ($n = 0, 1, 2, \dots$) be real numbers and let the power series $A(t) = \sum_{n=1}^{\infty} a_n t^n$ and $B(t) = \sum_{n=1}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, \dots$, and a_n/b_n is strictly increasing (or decreasing) for $n = 0, 1, 2, \dots$, then the function $A(t)/B(t)$ is strictly increasing (or decreasing) on $(0, R)$.*

Lemma 2.3 ([7, pp.227-229]). *We have*

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \quad |x| < \pi, \quad (2.1)$$

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n} - 1}{(2n)!} 2^{2n} |B_{2n}| x^{2n-1}, \quad |x| < \pi/2, \quad (2.2)$$

where B_n is the Bernoulli number.

Lemma 2.4. *For $p \in (0, 1)$, let F_p be the function defined $(0, \pi/2]$ by*

$$F_p(x) = \frac{\ln \frac{\sin x}{x}}{\ln(\cos px)}. \quad (2.3)$$

Then F_p is strictly increasing for $p \in (0, 1/\sqrt{5}]$ and decreasing for $p \in [1/2, 1)$. Moreover, for $x \in (0, \pi/2)$, we have

$$\frac{\ln 2 - \ln \pi}{\ln(\cos(p\pi/2))} \ln(\cos px) < \ln \frac{\sin x}{x} < \frac{1}{3p^2} \ln(\cos px) \quad (2.4)$$

if $p \in (0, 1/\sqrt{5}]$. The inequalities (2.4) are reversed if $p \in [1/2, 1)$.

Proof. For $x \in (0, \pi/2]$, we define $f(x) = \ln((\sin x)/x)$ and $g(x) = \ln(\cos px)$, where $p \in (0, 1)$. Note that $f(0^+) = g(0^+) = 0$, then $F_p(x)$ can be written as

$$F_p(x) = \frac{f(x) - f(0^+)}{g(x) - g(0^+)}.$$

Differentiation and using (2.1) and (2.2) yield

$$\frac{f'(x)}{g'(x)} = \frac{p \left(\frac{1}{x} - \cot x\right)}{\tan px} = \frac{\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}}{\sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n)!} p^{2n-2} 2^{2n} |B_{2n}| x^{2n-1}} := \frac{\sum_{n=1}^{\infty} a_n x^{2n-1}}{\sum_{n=1}^{\infty} b_n x^{2n-1}},$$

where

$$a_n = \frac{2^{2n}}{(2n)!} |B_{2n}|, \quad b_n = \frac{2^{2n} - 1}{(2n)!} p^{2n-2} 2^{2n} |B_{2n}|.$$

Clearly, if the monotonicity of a_n/b_n is proved, then by Lemma 2.2 it is deduced the monotonicity of f'/g' , and then the monotonicity of the function F_p easily follows from Lemma 2.1. For this purpose, since $a_n, b_n > 0$ for $n \in \mathbb{N}$, we only need to show that b_n/a_n is decreasing if $0 < p \leq 1/\sqrt{5}$ and increasing if $1/2 \leq p < 1$. Indeed, elementary computation yields

$$\begin{aligned} \frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} &= (2^{2n+2} - 1) p^{2n} - (2^{2n} - 1) p^{2n-2} \\ &= (4^{n+1} - 1) p^{2n-2} \left(p^2 - \frac{1}{4} + \frac{3}{4(4^{n+1} - 1)} \right), \end{aligned}$$

from which it is easy to obtain that for $n \in \mathbb{N}$

$$\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} \begin{cases} \leq 0 & \text{if } p^2 < \frac{1}{5}, \\ > & \text{if } p^2 \geq \frac{1}{4}. \end{cases}$$

It is seen that b_n/a_n is decreasing if $0 < p \leq 1/\sqrt{5}$ and increasing if $1/2 \leq p < 1$, which together with $a_n, b_n > 0$ for $n \in \mathbb{N}$ leads to a_n/b_n is strictly increasing if $0 < p \leq 1/\sqrt{5}$ and decreasing if $1/2 \leq p < 1$.

By the monotonicity of the function F_p and the facts that

$$F_p(0^+) = \frac{1}{3p^2} \quad \text{and} \quad F_p\left(\frac{\pi}{2}\right) = \frac{\ln 2 - \ln \pi}{\ln(\cos(p\pi/2))},$$

the inequalities (2.4) follow immediately. $\square$

Lemma 2.5. For $p \in (0, 1)$, let f_p be the function defined on $(0, \pi/2]$ by (1.9).

(i) If $f_p(x) < 0$ holds for all $x \in (0, \pi/2)$ then $p \in (0, 1/3]$.

(ii) If $f_p(x) > 0$ for all $x \in (0, \pi/2)$, then $p \in [p_0, 1)$, where $p_0 \approx 0.3473$ is the unique root of equation (1.6) on $(0, 1)$.

Proof. Firstly, we show that there is a unique $p_0 \in (0, 1)$ to satisfy (1.6) such that $f_p(\pi/2) < 0$ for $p \in (0, p_0)$ and $f_p(\pi/2) > 0$ for $p \in (p_0, 1)$. Indeed, as mentioned previous (see [8, Theorem 2.3]), the function $p \mapsto p^{-1} \ln(\cos(p\pi/2))$ is decreasing on $(0, 1)$, and therefore $p \mapsto f_p(\pi/2)$ is increasing on $(0, 1)$. Since

$$\begin{aligned} f_{1/3}\left(\frac{\pi}{2}\right) &= \ln \frac{2}{\pi} - 3 \ln \frac{\sqrt{3}}{2} < 0, \\ f_{1/2}\left(\frac{\pi}{2}\right) &= \ln \frac{2}{\pi} - 2 \ln \frac{\sqrt{2}}{2} > 0, \end{aligned}$$

so the equation ((1.6) has a unique root p_0 on $(0, 1)$ and $p_0 \in (1/3, 1/2)$ such that $f_p(\pi/2) < 0$ for $p \in (0, p_0)$ and $f_p(\pi/2) > 0$ for $p \in (p_0, 1)$. Numerical calculation reveals that $p_0 \approx 0.3473$.

Secondly, if inequality $f_p(x) < 0$ holds for $x \in (0, \pi/2)$, then we have

$$\begin{cases} \lim_{x \rightarrow 0^+} x^{-2} f_p(x) = \lim_{x \rightarrow 0^+} x^{-2} (\ln \frac{\sin x}{x} - \frac{1}{p} \ln(\cos px)) = \frac{1}{2}p - \frac{1}{6} \leq 0, \\ f_p\left(\frac{\pi}{2}\right) = \ln \frac{2}{\pi} - \frac{1}{p} \ln\left(\cos \frac{p\pi}{2}\right) \leq 0. \end{cases}$$

Solving the inequalities for p yields

$$p \in (0, 1/3] \cap (0, p_0] = (0, 1/3].$$

In the same way, if inequality $f_p(x) > 0$ for all $x \in (0, \pi/2)$, then

$$p \in (1/3, 1) \cap [p_0, 1) = [p_0, 1),$$

which completes the proof. $\square$

3. PROOFS OF MAIN RESULTS

Proof of Theorem 1.1. (i) We first prove the second inequality of (1.5) holds if and only if $q \in (0, 1/3]$.

The necessity follows from Lemma 2.5. It remains to show that the condition $q \in (0, 1/3]$ is sufficient. Since $q \in (0, 1/3] \subset (0, 1/\sqrt{5}]$, by the second inequality of (2.4) it is obtained that

$$\ln \frac{\sin x}{x} < \frac{1}{3q^2} \ln(\cos qx) = \frac{1}{3q} \ln(\cos qx)^{1/q} \leq \ln(\cos qx)^{1/q},$$

which proves the sufficiency.

(ii) Now we show that the first inequality of (1.5) holds if and only if $p \in [p_0, 1)$.

The necessity is due to Lemma 2.5. We prove the first inequality of (1.5) holds if $p \in [p_0, 1)$. In fact, in view of $p_0 \in (0, 1/\sqrt{5}]$ it follows from the first inequality of (2.4) that

$$\ln \frac{\sin x}{x} > \frac{\ln 2 - \ln \pi}{\ln(\cos(p_0\pi/2))} \ln(\cos p_0x) = \frac{p_0(\ln 2 - \ln \pi)}{\ln(\cos(p_0\pi/2))} \ln(\cos p_0x)^{1/p_0} = \ln(\cos p_0x)^{1/p_0},$$

where the last equality holds is due to p_0 is the unique root of equation (1.6). And, by the monotonicity of the function $p \mapsto (\cos px)^{1/p}$, we conclude that for $p \in [p_0, 1)$

$$\frac{\sin x}{x} > (\cos p_0x)^{1/p_0} \geq (\cos px)^{1/p},$$

which implies the sufficiency.

(iii) Lastly, put $p = 1/3$ and p_0 in (2.4) lead to (1.7) and (1.8), respectively. $\square$

Proof of Theorem 1.2. Differentiation and using (2.1) and (2.2) yield

$$\begin{aligned} f'_p(x) &= \left(\cot x - \frac{1}{x} \right) + \tan px \\ &= - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1} + \sum_{n=1}^{\infty} \frac{2^{2n} - 1}{(2n)!} p^{2n-1} 2^{2n} |B_{2n}| x^{2n-1} \\ &= \sum_{n=1}^{\infty} \frac{(2^{2n} - 1) 2^{2n}}{(2n)!} |B_{2n}| \left(p^{2n-1} - \frac{1}{2^{2n} - 1} \right) x^{2n-1} \\ &: = \sum_{n=2}^{\infty} c_n d_n x^{2n-1}, \end{aligned}$$

where

$$\begin{aligned} c_n &= \frac{(2^{2n} - 1) 2^{2n}}{(2n)!} |B_{2n}| \frac{p^{2n-1} - \frac{1}{2^{2n-1}}}{p - \left(\frac{1}{2^{2n-1}}\right)^{1/(2n-1)}} > 0, \\ d_n &= p - h(n), \quad h(n) = \left(\frac{1}{2^{2n} - 1}\right)^{1/(2n-1)} \end{aligned}$$

for $n \geq 1$ and $p \in (0, 1)$.

Let us consider the function $g : (1/2, \infty) \mapsto (0, \infty)$ defined by

$$g(x) = \left(\frac{1}{2^{2x} - 1}\right)^{1/(2x-1)}. \quad (3.1)$$

Differentiation leads to

$$\begin{aligned} \frac{2(2x-1)^2}{g(x)} g'(x) &= \ln(2^{2x} - 1) - \frac{(2x-1)2^{2x} \ln 2}{(2^{2x} - 1)} := g_1(x), \\ g_1'(x) &= \frac{2^{2x+1} \ln^2 2}{(2^{2x} - 1)^2} (2x-1) > 0, \end{aligned}$$

which reveals that g_1 is increasing on $(1/2, \infty)$, and therefore $g_1(x) > g_1(1/2^+) = 0$. It follows that $g'(x) > 0$, then, g is increasing on $(1/2, \infty)$, hence for $n \geq 1$

$$\frac{1}{3} = g(1) \leq g(n) \leq g(\infty) = \lim_{n \rightarrow \infty} \left(\frac{1}{2^{2n} - 1}\right)^{1/(2n-1)} = \frac{1}{2},$$

and then, $d_n = p - g(n) \leq 0$ if $p \in (0, 1/3]$ and $d_n = p - g(n) \geq 0$ if $p \in [1/2, 1)$. Thus, $f_p'(x) < 0$ if $p \in (0, 1/3]$ and $f_p'(x) > 0$ if $p \in [1/2, 1)$, which proves the monotonicity of f_p on $(0, \pi/2]$.

Hence, for $p \in (0, 1/3]$, $x \in (0, c) \subseteq (0, \pi/2)$ we have

$$\ln \beta_p(c) = f_p(c) < f_p(x) < \lim_{x \rightarrow 0^+} f_p(x) = 0,$$

which yields (1.10).

Likewise, for $p \in [1/2, 1)$ then $f_p'(x) > 0$, (1.10) is reversed, which completes the proof. $\square$

4. APPLICATIONS

As simple applications of main results, we present some precise estimations for certain special functions and constants in this section.

4.1. The estimate for the sine integral. The sine integral is defined by

$$Si(t) = \int_0^t \frac{\sin x}{x} dx.$$

There has some results, for example, Qi [13] showed that

$$1.3333 \approx \frac{4}{3} < Si\left(\frac{\pi}{2}\right) < \frac{\pi + 1}{3} \approx 1.3805;$$

the following two estimations are due to Wu [18], [19]:

$$\begin{aligned} 1.3569 &\approx \frac{\pi + 5}{6} < Si\left(\frac{\pi}{2}\right) < \frac{\pi + 1}{3} \approx 1.3805; \\ 1.3688 &\approx \frac{92 - \pi^2}{60} < Si\left(\frac{\pi}{2}\right) < \frac{8 + 4\pi}{15} \approx 1.3711. \end{aligned}$$

Now we give a general result.

Proposition 4.1. *For $t \in (0, \pi/2)$, we have*

$$\begin{aligned} &\frac{2(3\sqrt{3} - 5)}{\pi} \left(t + \sin t + 9 \sin \frac{t}{3} + \frac{9}{2} \sin \frac{2t}{3} \right) \\ &< Si(t) < \frac{1}{8} \left(t + \sin t + 9 \sin \frac{t}{3} + \frac{9}{2} \sin \frac{2t}{3} \right). \end{aligned} \quad (4.1)$$

Particularly, we have

$$1.3696 \approx \frac{(3\sqrt{3}-5)(2\pi+9\sqrt{3}+22)}{2\pi} < Si\left(\frac{\pi}{2}\right) < \frac{2\pi+9\sqrt{3}+22}{32} \approx 1.3710. \quad (4.2)$$

Proof. By (1.12), for $x \in (0, \pi/2)$ we have

$$\beta_{1/3} \left(\frac{\pi}{4} \right) \cos^3 \frac{x}{6} < \frac{\sin \frac{x}{2}}{\frac{x}{2}} < \cos^3 \frac{x}{6},$$

where $\beta_{1/3}(\pi/4) = 16(3\sqrt{3} - 5)/\pi$. Multiplying both sides by $\cos(x/2)$ leads to

$$\beta_{1/3} \left(\frac{\pi}{4} \right) \cos^3 \frac{x}{6} \cos \frac{x}{2} < \frac{\sin x}{x} = \frac{\sin \frac{x}{2}}{\frac{x}{2}} \cos \frac{x}{2} < \cos^3 \frac{x}{6} \cos \frac{x}{2}. \quad (4.3)$$

Integrating both sides over $[0, t]$ yields

$$\beta_{1/3} \left(\frac{\pi}{4} \right) \int_0^t \cos^3 \frac{x}{6} \cos \frac{x}{2} dx < \int_0^t \frac{\sin x}{x} dx < \int_0^t \cos^3 \frac{x}{6} \cos \frac{x}{2} dx.$$

From the following

$$\int_0^t \cos \frac{x}{2} \cos^3 \frac{x}{6} dx = \frac{1}{8}t + \frac{1}{8} \sin t + \frac{9}{8} \sin \frac{t}{3} + \frac{9}{16} \sin \frac{2t}{3}$$

(4.1) follows.

Simple computation yields (4.2). □

4.2. The estimate for the Catalan constant. The Catalan constant [5]

$$K = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941772190...$$

is a famous mysterious constant appearing in many places in mathematics and physics. Its integral representations contain the following [3]

$$K = \int_0^1 \frac{\arctan x}{x} dx = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} dx = \frac{\pi^2}{16} - \frac{\pi}{4} \ln 2 + \int_0^{\pi/4} \frac{x^2}{\sin^2 x} dx. \quad (4.4)$$

We present two estimations for K below.

Proposition 4.2. *We have*

$$0.9120 \approx \frac{1}{8}(4 + 3 \ln 3) < K < \frac{3\sqrt{3}\pi}{128}(4 + 3 \ln 3) \approx 0.9305, \quad (4.5)$$

$$0.9156 \approx \frac{\pi^2}{16} - \frac{\pi}{4} \ln 2 + \frac{1376-792\sqrt{3}}{5} < K < \frac{37+6\sqrt{3}}{320}\pi^2 - \frac{\pi}{4} \ln 2 \approx 0.9173. \quad (4.6)$$

Proof. By inequalities (1.11) we get for $(0, \pi/2)$

$$\frac{1}{\cos^3 \frac{x}{3}} < \frac{x}{\sin x} < \frac{9\pi}{16\sqrt{3}} \frac{1}{\cos^3 \frac{x}{3}},$$

Integrating both sides over $[0, \pi/2]$ yields

$$\int_0^{\pi/2} \frac{dx}{\cos^3 \frac{x}{3}} < \int_0^{\pi/2} \frac{x}{\sin x} dx < \int_0^{\pi/2} \frac{dx}{\cos^3 \frac{x}{3}}.$$

From the

$$\int_0^{\pi/2} \frac{dx}{\cos^3 \frac{x}{3}} = 1 + \frac{3 \ln 3}{4}$$

and the second formula in (4.4) (4.5) follows.

Now we prove (4.6). From (1.12) it is deduced that

$$\frac{1}{\cos^6 \frac{x}{3}} < \frac{x^2}{\sin^2 x} < \frac{1}{\beta_{1/3}^2(\pi/4)} \frac{1}{\cos^6 \frac{x}{3}},$$

where $\beta_{1/3}(\pi/4) = 16(3\sqrt{3} - 5)/\pi$. Integrating over $[0, \pi/4]$ yields

$$\int_0^{\pi/4} \frac{dx}{\cos^6 \frac{x}{3}} < \int_0^{\pi/4} \frac{x^2}{\sin^2 x} dx < \frac{1}{\beta_{1/3}^2(\pi/4)} \int_0^{\pi/4} \frac{dx}{\cos^6 \frac{x}{3}}.$$

Direct computation gives

$$\int_0^{\pi/4} \frac{dx}{\cos^6 \frac{x}{3}} = \frac{1376 - 792\sqrt{3}}{5},$$

it follows from the third formula of (4.4) that (4.6) holds. $\square$

4.3. An inequality related to Schwab-Borchardt mean. The Schwab-Borchardt mean of two numbers $a \geq 0$ and $b > 0$, denoted by $SB(a, b)$, is defined as [2, Theorem 8.4], [6, 3, (2.3)]

$$SB(a, b) = \begin{cases} \frac{\sqrt{b^2 - a^2}}{\arccos(a/b)} & \text{if } a < b, \\ a & \text{if } a = b, \\ \frac{\sqrt{a^2 - b^2}}{\operatorname{arccosh}(a/b)} & \text{if } a > b. \end{cases}$$

The properties and certain inequalities involving Schwab-Borchardt mean can be found in [11], [12]. We now establish a new inequality for this mean.

For this purpose, we need to show the following lemma.

Lemma 4.3. *The second inequality of (1.7), that is,*

$$\frac{\sin x}{x} < \cos^3 \frac{x}{3} \quad (4.7)$$

also holds for $x \in (0, 3\pi/2)$.

Proof. Let us define

$$h(x) = x - \left(\cos \frac{x}{3}\right)^{-3} \sin x.$$

Differentiation yields

$$h'(x) = 1 - \frac{\sin x}{\cos^4 \frac{x}{3}} \sin \frac{x}{3} - \frac{\cos x}{\cos^3 \frac{x}{3}} = \tan^4 \frac{x}{3} > 0.$$

Hence $h(x) > h(0) = 0$ for $x \in (0, 3\pi/2)$, which implies the desired result. $\square$

A hyperbolic function version of the second inequality of (1.7) is also true. Indeed, by setting $x = \ln \sqrt{a/b}$ in the Lin's inequality [9] that for positive numbers $a, b > 0$ with $a \neq b$

$$\frac{a - b}{\ln a - \ln b} < \left(\frac{a^{1/3} + b^{1/3}}{2} \right)^3,$$

the inequality

$$\frac{\sinh x}{x} < \cosh^3 \frac{x}{3} \quad (4.8)$$

easily follows.

Now we can prove the following result.

Proposition 4.4. *For $t > 0$, we have*

$$(4t^2 - 1) SB(t, 1) < 3t^3. \quad (4.9)$$

Proof. Replacing x for $3x$ in (4.7) and next using duplication formula for sine function $\sin 3x = (4 \cos^2 x - 1) \sin x$ we have

$$\frac{\sin 3x}{x} = \frac{\sin x}{x} (4 \cos^2 x - 1) < 3 \cos^3 x \quad (4.10)$$

holds for $x \in (0, \pi/2)$. Letting $\cos x = t$ in (4.10) we get

$$\frac{\sqrt{1-t^2}}{\arccos t} (4t^2 - 1) < 3t^3,$$

that is, (4.9) holds for $t \in (0, 1)$.

Similarly, letting $\cosh(x/3) = t$ and next using duplication formula for sinh function lead us to

$$\frac{\sqrt{t^2-1}}{\operatorname{arccosh} t} (4t^2 - 1) < 3t^3,$$

that is, (4.9) holds for $t \in (1, \infty)$, this proves the desired result. $\square$

We close this paper by giving a remark on the inequalities (1.3).

Remark 4.5. The value range of variable x such that (1.3) holds can be extended to $(0, \pi)$. In fact, the first inequality of (1.3) or (1.2) is equivalent to $\tan(x/2) > x/2$, which holds for $x/2 \in (0, \pi/2)$, that is, $x \in (0, \pi)$. The second one of (1.3) holds for $x \in (0, 3\pi/2)$ due to Lemma in this section. While the third one of (1.3) holds for $x \in (-\infty, \infty)$, since

$$\cos^3 \frac{x}{3} - \frac{2 + \cos x}{3} = -\frac{1}{3} \left(\cos \frac{x}{3} + 2 \right) \left(\cos \frac{x}{3} - 1 \right)^2 < 0.$$

Consequently, the value range of variable x such that (1.3) holds can be extended to

$$(0, \pi) \cap (0, 3\pi/2) \cap (-\infty, \infty) = (0, \pi),$$

which slightly improves (1.3).

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