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REPRESENTATIONS OF ALGEBRAS SATISFYING A TRAIN IDENTITY OF DEGREE 2 AND EXPONENT 3

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Abstract. In this paper, we are interested in the study of representations of the algebras A satisfying the equation $(a^3)^2 = \omega(a)^3 a^3, \quad \forall a \in A$. We give a characterization of these representations. We also give the condition under which the A-module is associative. When the Peirce component $A_{\frac{1}{2}} = 0$, we give examples of sub A-modules. We show that the irreducible A-submodules are not necessarily of dimension 1.

1. Introduction and preliminaries

Let A be an algebra belonging to a class $\mathcal{C}$ of commutative algebra on a commutative field K and M a space vector on K. According to Eilenberg [4] a linear map $\mu : A \to End_K(M)$ is a representation of A in the class $\mathcal{C}$ if the extension $S = A \oplus M$ of M, with the multiplication given by

$$(a+m)(b+n) = ab + \mu(a)(n) + \mu(b)(m), \forall a, b \in A, m, n \in M \tag{1.1}$$

belongs to class $\mathcal{C}$. We also say that M is an A-module and we denote the action $\mu(a)(m)$ by $a \cdot m$. A representation is said to be irreducible if $M \neq 0$ and there exists no proper subspace of M which is stable by $\mu(a)$ for all $a \in A$. In other words, the A-module M does not admit its own submodules. Furthermore, a representation is said to be of finite dimension r if $dim(M) = r$.

Let K be a commutative field and A be a non-necessarily associative commutative K-algebra. A *baric* algebra is any non-necessarily associative commutative K-algebra A such that there exists a non-zero morphism of algebras ω from A to K. For such an algebra, ω is called the *weight function*.

In [2, 3, 6, 7] the authors studied the representations of certain classes of baric algebras, namely Bernstein algebras, train algebras of rank 3, power associative train algebras of rank 4. In [8, 9], authors studied evolution algebras satisfying a train identity of degree 2 and exponent 3 and the structure of a train identity of degree 2 and exponent 4. The objective of this paper is to study the representations of baric K-algebras satisfying a train identity of degree 2 and exponent 3. Let (A, ω) be a baric K-algebra satisfying a train identity of degree 2 and

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exponent 3. Let M be a vector space over K and $\mu : A \to End_K(M)$ be a linear map. We say that μ is a representation of A if the extension $S = A \oplus M$ of M with the multiplication (1.1), provided with the homomorphism $\overline{\omega}$ defined by $\overline{\omega}(a+m) = \omega(a)$ for all $a \in A$ and $m \in M$, is a baric K-algebra satisfying an identity train of degree 2 and exponent 3. In the first section, we give the characterization of the representation of a baric K-algebra satisfying a train identity of degree 2 and exponent 3. Then we show under certain conditions that the irreducible associative modules are of dimension 1. Finally in the third section, we give examples of sub-modules.

2. Representations of algebras satisfying a train identity of degree 2 and exponent 3

In this paper, K is an infinite commutative field with characteristics different from 2 and 3. We say that a baric K-algebra (A, ω) satisfies a train identity of degree 2 and exponent 3 if (A, ω) satisfies

$$(a^3)^2 = \omega(a)^3 a^3, \quad \forall a \in A. \tag{2.1}$$

A baric K-algebra (A, ω) satisfying a train identity of degree 2 and exponent 3 admits a unique weight function and has nonzero idempotents. If e is an idempotent of A then the Peirce decomposition of A is $A = Ke \oplus \ker \omega$ where $\ker \omega = A_{\frac{1}{2}} \oplus A_0 \oplus A_{-\frac{1}{2}}$ and $A_i = \{a_i \in A \,|\, ea_i = ia_i\}$ with $i \in \{\frac{1}{2}, 0, -\frac{1}{2}\}$. Moreover, in [1, theorem 3.4], the authors show that these subspaces A_i satisfy the following relations

$$\begin{gathered} A_{\frac{1}{2}}^2 \subseteq A_0 \oplus A_{-\frac{1}{2}},\ A_0^2 \subseteq A_{\frac{1}{2}} \oplus A_0,\ A_{-\frac{1}{2}}^2 \subseteq A_{\frac{1}{2}}, \\ A_{\frac{1}{2}} A_0 \subseteq A_{\frac{1}{2}} \oplus A_{-\frac{1}{2}},\ A_{\frac{1}{2}} A_{-\frac{1}{2}} \subseteq A_{\frac{1}{2}} \oplus A_0,\ A_0 A_{-\frac{1}{2}} \subseteq A_{\frac{1}{2}}. \end{gathered} \tag{2.2}$$

Lemma 2.1. *Let (A, ω) be a K-algebra satisfying a train identity of degree 2 and exponent 3, and $\mu : A \to End_K(M)$ a linear map. Then μ is a representation of A if and only if for any $a \in A$,*

$$2\mu_{a^3}(\mu_{a^2} + 2\mu_a^2) = \omega(a)^3(\mu_{a^2} + 2\mu_a^2), \tag{2.3}$$

where $\mu_a := \mu(a) \in End(M)$.

Proof. By definition, μ is a representation of A if and only if all $a + m \in A \oplus M$ satisfies identity (2.1), i.e. $((a+m)^3)^2 = \overline{\omega}(a+m)^3(a+m)^3$. By a direct calculation it follows that $(a+m)^3 = a^3 + \mu_{a^2}(m) + 2\mu_a^2(m)$, $((a+m)^3)^2 = a^3a^3 + 2\mu_{a^3}(\mu_{a^2}(m) + 2\mu_a^2(m))$ and $\overline{\omega}(a+m)^3 = \omega(a)^3$. So $((a+m)^3)^2 = \overline{\omega}(a+m)^3(a+m)^3$ is equivalent to $2\mu_{a^3}(\mu_{a^2}(m) + 2\mu_a^2(m)) = \omega(a)^3(\mu_{a^2}(m) + 2\mu_a^2(m))$. □

Example 2.2. Consider a real commutative algebra (A, ω) with basis $\{e, e_1, e_2\}$, whose multiplication table is given by $e^2 = e$, $ee_1 = \frac{1}{2}e_1$, $ee_2 = -\frac{1}{2}e_2$, $e_1e_2 = e_1$, $e_1^2 = 0$, $e_2^2 = 0$, and $\omega : A \to \mathbb{R}$ given by $\omega(e) = 1$ and $\omega(e_1) = \omega(e_2) = 0$. The algebra (A, ω) is a baric $\mathbb{R}$-algebra satisfying a train identity of degree 2 and exponent 3 [1, theorem 6.1]. Let M be a $\mathbb{R}$-vector space of dimension 3. For all

$a = \alpha_0 e + \alpha_1 e_1 + \alpha_2 e_2 \in A$, the linear map $\mu : A \to End_{\mathbb{R}}(M)$ defined by

$$\mu_a = \begin{pmatrix} \frac{1}{2}\alpha_0 & 0 & \alpha_1 \\ 0 & \frac{1}{2}\alpha_0 & \alpha_1 \\ \alpha_2 & -\alpha_2 & 0 \end{pmatrix}, \tag{2.4}$$

is an irreducible 3-dimensional representation of A.

Proof. Let $a = \alpha_0 e + \alpha_1 e_1 + \alpha_2 e_2 \in A$. We have $\omega(a) = \alpha_0$,

$$2\mu_{a^3} = \begin{pmatrix} \alpha_0^3 & 0 & 2(\alpha_0^2\alpha_1 + \alpha_0\alpha_1\alpha_2 + 2\alpha_1\alpha_2^2) \\ 0 & \alpha_0^3 & 2(\alpha_0^2\alpha_1 + \alpha_0\alpha_1\alpha_2 + 2\alpha_1\alpha_2^2) \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

$$2\mu_{a^2} = \begin{pmatrix} \frac{1}{2}\alpha_0^2 & 0 & \alpha_0\alpha_1 + 2\alpha_1\alpha_2 \\ 0 & \frac{1}{2}\alpha_0^2 & \alpha_0\alpha_1 + 2\alpha_1\alpha_2 \\ -\alpha_0\alpha_2 & \alpha_0\alpha_2 & 0 \end{pmatrix}, \tag{2.6}$$

$$2\mu_a^2 = \begin{pmatrix} \frac{1}{2}\alpha_0^2 + 2\alpha_1\alpha_2 & -2\alpha_1\alpha_2 & \alpha_0\alpha_1 \\ 2\alpha_1\alpha_2 & \frac{1}{2}\alpha_0^2 - 2\alpha_1\alpha_2 & \alpha_0\alpha_1 \\ \alpha_0\alpha_2 & -\alpha_0\alpha_2 & 0 \end{pmatrix}. \tag{2.7}$$

It follows that

$$(2\mu_{a^3} - \omega(a)^3 I_M)(\mu_{a^2} + 2\mu_a^2) = 0 \tag{2.8}$$

and therefore μ is a 3-dimensional representation of A. We have

$$\mu_e = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mu_{e_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ et } \mu_{e_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}. \tag{2.9}$$

By setting $M = \langle m_1, m_2, m_3 \rangle$ it follows that

$$\begin{cases} \mu_e(m_1) = \frac{1}{2}m_1 \\ \mu_e(m_2) = \frac{1}{2}m_2 \\ \mu_e(m_3) = 0 \end{cases}, \begin{cases} \mu_{e_1}(m_1) = 0 \\ \mu_{e_1}(m_2) = 0 \\ \mu_{e_1}(m_3) = m_1 + m_2 \end{cases} \text{ et } \begin{cases} \mu_{e_2}(m_1) = m_3 \\ \mu_{e_2}(m_2) = -m_3 \\ \mu_{e_2}(m_3) = 0 \end{cases}. \tag{2.10}$$

Subspaces stable by μ_a are in particular stable by μ_e. As μ_e is diagonalizable, the stable subspaces by μ_e are exactly the subspaces generated by the eigenvectors of μ_e. So $\langle m_1 \rangle$, $\langle m_2 \rangle$, $\langle m_3 \rangle$, $\langle m_1, m_2 \rangle$, $\langle m_1, m_3 \rangle$ and $\langle m_2, m_3 \rangle$ are the only non-trivial subspaces, stable by μ_e. None of these subspaces is stable under both μ_{e_1} and μ_{e_2}. And therefore μ is an irreducible representation of dimension 3 of A. □

Lemma 2.3. *Let (A, ω) be a baric K-algebra satisfying a train identity of degree 2 and exponent 3, and $\mu : A \to End_K(M)$ be a representation of A. Then for all $a, b, c \in A$ we have:*

$$6\mu_{J(a,a,b)} J_\mu(a,a) + 4\mu_{J(a,a,a)} J_\mu(a,b) = 3\omega(J(a,a,b)) J_\mu(a,a) + 2\omega(J(a,a,a)) J_\mu(a,b), \tag{2.11}$$

$$\begin{aligned} 6\mu_{J(a,b,c)}J_\mu(a,a) + 6\mu_{J(a,a,b)}J_\mu(a,c) + 6\mu_{J(a,a,c)}J_\mu(a,b) + 2\mu_{J(a,a,a)}J_\mu(b,c) = \\ 3\omega(J(a,b,c))J_\mu(a,a) + 3\omega(J(a,a,b))J_\mu(a,c) + \\ 3\omega(J(a,a,c))J_\mu(a,b) + \omega(J(a,a,a))J_\mu(b,c), \end{aligned} \tag{2.12}$$

where $J(a,b,c) = a(bc) + b(ca) + c(ab)$ is the Jacobian of a, b, c and $J_\mu(a,b) = \mu_{ab} + \mu_a\mu_b + \mu_b\mu_a$.

Proof. The identity (2.11) is obtained by a partial linearization of the identity (2.3) and the identity (2.12) is obtained by a partial linearization of the identity (2.11). $\square$

Proposition 2.4. *Let (A,ω) be a baric K-algebra satisfying a train identity of degree 2 and exponent 3, $e \neq 0$ an idempotent of A and $\mu : A \to End_K(M)$ a representation of A. Then $M = M_{\frac{1}{2}} \oplus M_0 \oplus M_{-\frac{1}{2}}$ where $M_i = \{m_i \in M \mid \mu_e(m_i) = im_i\}$ with $i \in \{\frac{1}{2}, 0, -\frac{1}{2}\}$.*

Proof. Let e be an idempotent of A. By replacing a by e in the identity (2.3) we obtain $4\mu_e^3 - \mu_e = 0$. Therefore, there is a single-rooted split polynomial $P(X) = 4X^3 - X = 4X(X + \frac{1}{2})(X - \frac{1}{2})$ such that $P(\mu_e) = 0$. It follows that μ_e is diagonalizable and we have $M = M_{\frac{1}{2}} \oplus M_0 \oplus M_{-\frac{1}{2}}$ where $M_i = \{m_i \in M \mid \mu_e(m_i) = im_i\}$ with $i \in \{\frac{1}{2}, 0, -\frac{1}{2}\}$. $\square$

Theorem 2.5. *Let (A,ω) be a baric K-algebra satisfying a train identity of degree 2 and exponent 3, $e \neq 0$ an idempotent of A and $\mu : A \to End_K(M)$ a representation of A. Then the action of A on M satisfies the following relations:*

$$\begin{aligned} e\cdot M_{\frac{1}{2}} = M_{\frac{1}{2}},\ e\cdot M_0 = \{0\},\ e\cdot M_{-\frac{1}{2}} = M_{-\frac{1}{2}}, \\ A_0\cdot M_{\frac{1}{2}} \subseteq M_{\frac{1}{2}} \oplus M_{-\frac{1}{2}},\ A_0\cdot M_0 \subseteq M_{\frac{1}{2}} \oplus M_0,\ A_0\cdot M_{-\frac{1}{2}} \subseteq M_{\frac{1}{2}}, \\ A_{-\frac{1}{2}}\cdot M_{\frac{1}{2}} \subseteq M_{\frac{1}{2}} \oplus M_0,\ A_{-\frac{1}{2}}\cdot M_0 \subseteq M_{\frac{1}{2}},\ A_{-\frac{1}{2}}\cdot M_{-\frac{1}{2}}\oplus \subseteq M_{\frac{1}{2}}, \\ A_{\frac{1}{2}}\cdot M_{\frac{1}{2}} \subseteq M_0 \oplus M_{-\frac{1}{2}},\ A_{\frac{1}{2}}\cdot M_0 \subseteq M_{\frac{1}{2}} \oplus M_{-\frac{1}{2}},\ A_{\frac{1}{2}}\cdot M_{-\frac{1}{2}} \subseteq M_{\frac{1}{2}} \oplus M_0. \end{aligned} \tag{2.13}$$

Proof. As μ is a representation of A then $S = A \oplus M$ equipped with $\overline{\omega}$ is a baric K-algebra satisfying a train identity of degree 2 and exponent 3. If e is an idempotent of A then the Peirce decomposition of S relatively to e is $S = Ke \oplus S_{\frac{1}{2}} \oplus S_0 \oplus S_{-\frac{1}{2}}$ where $S_i = \{a + m \in S \mid e(a+m) = i(a+m)\} = A_i \oplus M_i$ with $i \in \{0, \frac{1}{2}, -\frac{1}{2}\}$. The multiplication table of the Peirce components of S gives us the action of A_i on M_j for $i, j \in \{0, \frac{1}{2}, -\frac{1}{2}\}$. $\square$

3. Associative modules

Definition 3.1. Let A be a commutative K-algebra and μ be a representation of A. We say that the A-module M is *associative* if for all $a, b \in A$ and $m \in M$ we have $ab\cdot m = a\cdot(b\cdot m) = b\cdot(a\cdot m)$.

Remark 3.2. Let (A,ω) be a baric K-algebra satisfying a train identity of degree 2 and exponent 3, and $\mu : A \to End_K(M)$ be a representation of A. The A-module

M is not in general associative. Since $\frac{1}{4}m_{\frac{1}{2}} = e\cdot(e\cdot m_{\frac{1}{2}}) \neq e\cdot m_{\frac{1}{2}} = \frac{1}{2}m_{\frac{1}{2}}$ and $\frac{1}{4}m_{-\frac{1}{2}} = e\cdot(e\cdot m_{-\frac{1}{2}}) \neq e\cdot m_{-\frac{1}{2}} = -\frac{1}{2}m_{-\frac{1}{2}}$.

In the proposition 3.4, we give a necessary and sufficient condition for M to be an associative A-module.

Lemma 3.3. *Let (A,ω) be a baric K-algebra satisfying a train identity of degree 2 and exponent 3, and $\mu : A \to End_K(M)$ be a representation of A. If $M = M_0$ then $a_0b_0\cdot m_0 + a_0\cdot(b_0\cdot m_0) + b_0\cdot(a_0\cdot m_0) = 0$, $a_{\frac{1}{2}}b_{\frac{1}{2}}\cdot m_0 = 0$ and $a_{\frac{1}{2}}b_{-\frac{1}{2}}\cdot m_0 = 0$, $\forall a_0, b_0 \in A_0$, $a_{\frac{1}{2}}, b_{\frac{1}{2}} \in A_{\frac{1}{2}}$, $b_{-\frac{1}{2}} \in A_{-\frac{1}{2}}$, and $m_0 \in M$.*

Proof. We assume that $M = M_0$. Then the action of A on M_0 tells us

$$A_0\cdot M_0 \subseteq M_0,\ A_{\frac{1}{2}}\cdot M_0 = 0 \text{ and } A_{-\frac{1}{2}}\cdot M_0 = 0. \tag{3.1}$$

Let $a_0, b_0 \in A_0$, $a_{\frac{1}{2}}, b_{\frac{1}{2}} \in A_{\frac{1}{2}}$, $b_{-\frac{1}{2}} \in A_{-\frac{1}{2}}$, and $m_0 \in M_0$. By setting $a = e$, $b = a_0$ and $c = b_0$ in the identity (2.12) we obtain $J_\mu(a_0, b_0)(m_0) \in M_{\frac{1}{2}}$. Since $M_{\frac{1}{2}} = \{0\}$, it follows that $a_0b_0\cdot m_0 + a_0\cdot(b_0\cdot m_0) + b_0\cdot(a_0\cdot m_0) = 0$. Moreover by setting $a = e$, $b = a_{\frac{1}{2}}$ and $c = b_{\frac{1}{2}}$ on the one hand and $a = e$, $b = a_{\frac{1}{2}}$ and $c = b_{-\frac{1}{2}}$ on the other hand in the identity (2.12) and taking into account the action of A on M_0, we obtain $a_{\frac{1}{2}}b_{\frac{1}{2}}\cdot m_0 \in M_{\frac{1}{2}}$ and $a_{\frac{1}{2}}b_{-\frac{1}{2}}\cdot m_0 \in M_{\frac{1}{2}}$. Since $M_{\frac{1}{2}} = \{0\}$ it follows that $a_{\frac{1}{2}}b_{\frac{1}{2}}\cdot m_0 = 0$ and $a_{\frac{1}{2}}b_{-\frac{1}{2}}\cdot m_0 = 0$. □

Proposition 3.4. *Let (A,ω) be a baric K-algebra satisfying an identity train of degree 2 and exponent 3 and $\mu : A \to End_K(M)$ a representation of A. Then M is an associative A-module if and only if $M = M_0$ and $a_0\cdot(b_0\cdot m_0) = 0$ for all $a_0, b_0 \in A_0$ and $m_0 \in M$.*

Proof. We assume that M is an associative A-module, which means that $ab\cdot m = a\cdot(b\cdot m) = b\cdot(a\cdot m)$ $\forall a, b \in A$, and $m \in M$. The remark 3.2 tells us that $M_{\frac{1}{2}} = M_{-\frac{1}{2}} = \{0\}$ and therefore $M = M_0$. According to lemma 3.3 we have $0 = a_0b_0\cdot m_0 + a_0\cdot(b_0\cdot m_0) + b_0\cdot(a_0\cdot m_0) = 3a_0\cdot(b_0\cdot m_0)$. Since $char(K) \neq 3$ we have $a_0\cdot(b_0\cdot m_0) = 0$ for all $a_0, b_0 \in A_0$ and $m_0 \in M_0$. Conversely we assume that $M = M_0$ and $a_0\cdot(b_0\cdot m_0) = 0$ for all $a_0, b_0 \in A_0$ and $m_0 \in M_0$. Let $a = a_{\frac{1}{2}} + a_0 + a_{-\frac{1}{2}}$ and $b = b_{\frac{1}{2}} + b_0 + b_{-\frac{1}{2}}$ with $a_0, b_0 \in A_0$; $a_{\frac{1}{2}}, b_{\frac{1}{2}} \in A_{\frac{1}{2}}$ and $a_{-\frac{1}{2}}, b_{-\frac{1}{2}} \in A_{-\frac{1}{2}}$. The identity (3.1) in the proof of lemma 3.3 gives $a\cdot(b\cdot m_0) = a\cdot(b_0\cdot m_0) = a_0\cdot(b_0\cdot m_0) = 0$ and $b\cdot(a\cdot m_0) = b\cdot(a_0\cdot m_0) = b_0\cdot(a_0\cdot m_0) = 0$. Taking into account the identity (3.1) in the proof of lemma 3.3, the inclusions $A^2_{-\frac{1}{2}} \subseteq A_{\frac{1}{2}}$, $A_{\frac{1}{2}}A_0 \subseteq A_{\frac{1}{2}} \oplus A_{-\frac{1}{2}}$ and $A_{-\frac{1}{2}}A_0 \subseteq A_{\frac{1}{2}}$ lead to

$$A^2_{-\frac{1}{2}}\cdot M_0 = A_{\frac{1}{2}}A_0\cdot M_0 = A_{-\frac{1}{2}}A_0\cdot M_0 = 0. \tag{3.2}$$

The lemma 3.3 and the identity (3.2) lead to $ab\cdot m_0 = (a_{\frac{1}{2}}b_0 + b_{\frac{1}{2}}a_0 + a_{-\frac{1}{2}}b_0 + b_{-\frac{1}{2}}a_0 + a_{-\frac{1}{2}}b_{-\frac{1}{2}})\cdot m_0 + (a_{\frac{1}{2}}b_{\frac{1}{2}} + a_{\frac{1}{2}}b_{-\frac{1}{2}} + b_{\frac{1}{2}}a_{-\frac{1}{2}})\cdot m_0 + a_0b_0\cdot m_0 = a_0b_0\cdot m_0$. Thus $a\cdot(b\cdot m_0) = b\cdot(a\cdot m_0) = ab\cdot m_0 = 0$ and M is an associative A-module. □

Definition 3.5. Let M be a not necessarily associative A-module and B be a subspace of A. The annulator of B in M is the subspace of M denoted $ann_M(B)$ defined by $ann_M(B) = \{m \in M \mid B\cdot m = 0\}$.

Remark 3.6. If M is an associative A-module, the annulator $ann_M(B)$ is a sub A-module of M. Indeed from $B \cdot (A \cdot m) = A \cdot (B \cdot m)$ it comes that $A \cdot ann_M(B) \subseteq ann_M(B)$. In the example 3.7 $ann_M(A_0)$ is not reduced to $\{0\}$ and is strictly included in M.

Example 3.7. Consider a real commutative algebra A with basis $\{e, e_1, e_2, e_3\}$, whose multiplication table is given by $e^2 = e$, $ee_1 = -\frac{1}{2}e_1$, $ee_2 = \frac{1}{2}e_2$, $e_2e_3 = e_2$ and the other products being zero. Let $\omega : A \to \mathbb{R}$ given by $\omega(e) = 1$, $\omega(e_1) = 0$, $\omega(e_2) = 0$ and $\omega(e_3) = 0$. Then (A, ω) is a baric $\mathbb{R}$-algebra satisfying an identity train of degree 2 and exponent 3. Now let M be a $\mathbb{R}$-vector space of dimension 3. For all $a = \alpha_0 e + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \in A$, we have the linear map $\mu : A \to End_{\mathbb{R}}(M)$ by

$$\mu_a = \begin{pmatrix} \alpha_3 & 0 & -\alpha_3 \\ 0 & 0 & 0 \\ \alpha_3 & 0 & -\alpha_3 \end{pmatrix}. \tag{3.3}$$

Then μ is a representation of A on the vector space $M = \langle m_1, m_2, m_3 \rangle$ defined by $e \cdot M = e_1 \cdot M = e_2 \cdot M = 0$, $e_3 \cdot m_1 = m_1 + m_3$, $e_3 \cdot m_2 = 0$ and $e_3 \cdot m_3 = -m_1 - m_3$ is an associative A-module. Moreover, $ann_M(\langle e_3 \rangle) = \langle m_1 + m_3, m_2 \rangle$.

Proof. The algebra (A, ω) is a baric $\mathbb{R}$-algebra satisfying a train identity of degree 2 and exponent 3. Indeed for all $a = \alpha_0 e + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \in A$ we have $\omega(a) = \alpha_0$, $a^2 = \alpha_0^2 e - \alpha_0 \alpha_1 e_1 + (\alpha_0 \alpha_2 + 2\alpha_2 \alpha_3) e_2$, $a^3 = \alpha_0^3 e + (\alpha_0^2 \alpha_2 + 2\alpha_0 \alpha_2 \alpha_3 + 2\alpha_2 \alpha_3^2) e_2$ and therefore $a^3 a^3 = \omega(a^3) a^3$. Since $\mu_{a^2} = \mu_a^2 = 0$ for all $a \in A$, we have $(2\mu_{a^3} - \omega(a)^3 I_M)(\mu_{a^2} + 2\mu_a^2) = 0$ and therefore μ is a representation of A. We have $\mu_e = \mu_{e_1} = \mu_{e_2} = 0$ and $\mu_{e_3} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, by setting $M = \langle m_1, m_2, m_3 \rangle$ it follows that $e \cdot M = e_1 \cdot M = e_2 \cdot M = 0$, $e_3 \cdot m_1 = m_1 + m_3$, $e_3 \cdot m_2 = 0$ and $e_3 \cdot m_3 = -m_1 - m_3$. We have $M = M_0$ and $A = \mathbb{R}_e \oplus A_{\frac{1}{2}} \oplus A_0 \oplus A_{-\frac{1}{2}}$ where $A_{\frac{1}{2}} = \langle e_2 \rangle$, $A_0 = \langle e_3 \rangle$ and $A_{-\frac{1}{2}} = \langle e_1 \rangle$. Since $e_3 \cdot (e_3 \cdot m) = 0$ then M is an A-module is associative. By definition, $ann_M(\langle e_3 \rangle) = \{m \in M \mid \langle e_3 \rangle \cdot m = 0\}$. Let $m = \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3 \in M$, we have $0 = e_3 \cdot m = \lambda_1 e_3 \cdot m_1 + \lambda_2 e_3 \cdot m_2 + \lambda_3 e_3 \cdot m_3 = \lambda_1(m_1 + m_3) + \lambda_3(-m_1 - m_3) = (\lambda_1 - \lambda_3)(m_1 + m_3)$ results in $\lambda_1 = \lambda_3$. It follows that $ann_M(\langle e_3 \rangle) = \langle m_1 + m_3, m_2 \rangle$. □

Proposition 3.8. *Let (A, ω) be a baric K-algebra satisfying a train identity of degree 2 and exponent 3, and $\mu : A \to End_K(M)$ be a representation of A such that M is an associative A-module and $ann_M(A_0) \neq \{0\}$. Then M is an irreducible A-module if and only if $dim(M) = 1$.*

Proof. According to the remark 3.6 $ann_M(A_0)$ is a sub A-module of M. If M is an A-module irreducible then $M = ann_M(A_0)$, $A.M = \{0\}$ and therefore $dim(M) = 1$. It is obvious that if $dim(M) = 1$, M is an irreducible A-module. □

4. Examples of submodules

In this section we give under certain conditions examples of submodules on a baric K-algebra satisfying a train identity of degree 2 and exponent 3.

Lemma 4.1. *Let (A, ω) be a baric K-algebra satisfying a train identity of degree 2 and exponent 3, and $\mu : A \to End_K(M)$ be a representation of A. If $A_{\frac{1}{2}} = 0$ then $J_\mu(b, b)(m) \in M_{\frac{1}{2}}$ for all $b \in A$ and $m \in M$.*

Proof. We assume that $A_{\frac{1}{2}} = 0$, then $A = Ke \oplus A_0 \oplus A_{-\frac{1}{2}}$. By putting $a = e$ in the identity(2.3) we obtain $2\mu_e J_\mu(e, e) = J_\mu(e, e)$. So for all $m \in M$, we have $J_\mu(e, e)(m) \in M_{\frac{1}{2}}$. In the identity(2.11), by setting $a = e$ and $b \in A$, we obtain $2\mu_e J_\mu(e, b) = J_\mu(e, b)$ since $J(e, e, b) = 0$. It follows that $J_\mu(e, b)(m) \in M_{\frac{1}{2}}$ for all $m \in M$. In the identity (2.12), by setting $a = e$ and $c = b \in A$, we obtain $2\mu_e J_\mu(b, b) = J_\mu(b, b)$ since $J(e, b, b) = 0$. It follows that $J_\mu(b, b)(m) \in M_{\frac{1}{2}}$ for all $m \in M$. $\square$

Proposition 4.2. *Let (A, ω) be a baric K-algebra satisfying a train identity of degree 2 and exponent 3, and $\mu : A \to End_K(M)$ be a representation of A. If $A_{\frac{1}{2}} = 0$ and $a_i \cdot (b_i \cdot m) = b_i \cdot (a_i \cdot m)$ for all $a_i, b_i \in A_i$ where $i \in \{0, -\frac{1}{2}\}$ and $m \in M$, then $L_i = \{m \in M \mid A_i \cdot m \subseteq M_{\frac{1}{2}} \oplus M_i\}$ is a sub-A-module of M.*

Proof. Let $m \in L_i$, $a_i \in A_i$ where $i \in \{0, -\frac{1}{2}\}$, and $b_j \in A_j$ where $j \in \{0, -\frac{1}{2}\}$. According to the proof of lemma 4.1, we have $J_\mu(e, a_i)(m) = ea_i \cdot m + e \cdot (a_i \cdot m) + a_i \cdot (e \cdot m) = ia_i \cdot m + e \cdot (a_i \cdot m) + a_i \cdot (e \cdot m) \in M_{\frac{1}{2}} \subseteq M_{\frac{1}{2}} \oplus M_i$. As $a_i \cdot m \in M_{\frac{1}{2}} \oplus M_i$ then $e \cdot (a_i \cdot m) \in M_{\frac{1}{2}} \oplus M_i$, we deduce that $a_i \cdot (e \cdot m) \in M_{\frac{1}{2}} \oplus M_i$, i.e. $e \cdot L_i \subseteq L_i$. Also according to the proof of lemma 4.1, we have $J_\mu(a_i, b_j)(m) = a_i b_j \cdot m + a_i \cdot (b_j \cdot m) + b_j \cdot (a_i \cdot m) \in M_{\frac{1}{2}}$.

If $i \neq j$ then $a_i b_j \in A_{\frac{1}{2}} = \{0\}$, it follows that $J_\mu(a_i, b_j)(m) = a_i \cdot (b_j \cdot m) + b_j \cdot (a_i \cdot m) \in M_{\frac{1}{2}}$. As $b_j \cdot (a_i \cdot m) \in M_{\frac{1}{2}} \oplus M_i$ since $A_j \cdot M_{\frac{1}{2}} \subseteq M_{\frac{1}{2}} \oplus M_i$ and $A_j \cdot M_i \subseteq M_{\frac{1}{2}}$. We deduce from this that $a_i \cdot (b_j \cdot m) \in M_{\frac{1}{2}} \oplus M_i$, i.e. $A_j \cdot L_i \subseteq L_i$.

If $j = i$ then $a_i \cdot (b_i \cdot m) = b_i \cdot (a_i \cdot m)$, it follows that $J_\mu(a_i, b_i)(m) = a_i b_i \cdot m + 2a_i \cdot (b_i \cdot m) \in M_{\frac{1}{2}}$. Since $a_i b_i \in A_{\frac{1}{2}} \oplus A_i = A_i$ we have $a_i b_i \cdot m \; in M_{\frac{1}{2}} \oplus M_i$. It follows that $a_i \cdot (b_i \cdot m) \in M_{\frac{1}{2}} \oplus M_i$ since $char(K) \neq 2$. Thus $A_i \cdot L_i \subseteq L_i$ and we deduce that L_i is a sub A-module of M. $\square$

The sub A-module L_i is not reduced to $\{0\}$, since it contains M_j where $j \in \{0, -\frac{1}{2}\}$. In the example 4.3, we show that L_i is strictly included in M.

Example 4.3. Let A be a baric real commutative algebra $\{e, e_1, e_2\}$, whose multiplication table is given by $e^2 = e$, $ee_2 = -\frac{1}{2}e_2$ and the other products being zero. Let $\omega : A \to \mathbb{R}$ be given by $\omega(e) = 1$, $\omega(e_1) = 0$ and $\omega(e_2) = 0$. Then (A, ω) is a $\mathbb{R}$- baric algebra satisfying a train identity of degree 2 and exponent 3 [1, theorem 6.1]. Now let M be a $\mathbb{R}$-vector space of dimension 3. For all $a = \alpha_0 e + \alpha_1 e_1 + \alpha_2 e_2 \in A$, we define the linear map $\mu : A \to End_{\mathbb{R}}(M)$ defined by

$$\mu_a = \begin{pmatrix} \frac{1}{2}\alpha_0 & 0 & 0 \\ \alpha_1 & -\frac{1}{2}\alpha_0 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

Then μ is a representation of A. By setting $M = \langle m_1, m_2, m_3 \rangle$, the action of A on M satisfies $e \cdot m_1 = \frac{1}{2}m_1$, $e \cdot m_2 = -\frac{1}{2}m_2$, $e \cdot m_3 = 0$, $e_1 \cdot m_1 = m_2$, $e_1 \cdot m_2 = 0$,

$e_1 \cdot m_3 = 0$, $e_2 \cdot m_1 = m_3$, $e_2 \cdot m_2 = 0$ and $e_2 \cdot m_3 = 0$. Thus we $L_0 = \langle m_2, m_3\rangle$ and $L_{-\frac{1}{2}} = \langle m_2, m_3\rangle$.

Proof. The map μ is a representation of A. Indeed for all $a = \alpha_0 e+\alpha_1 e_1+\alpha_2 e_2 \in A$ we have $\omega(a) = \alpha_0$,

$$2\mu_{a^3} = \begin{pmatrix} \alpha_0^3 & 0 & 0 \\ 0 & -\alpha_0^3 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ 2\mu_{a^2} = \begin{pmatrix} \frac{1}{2}\alpha_0^2 & 0 & 0 \\ 0 & -\frac{1}{2}\alpha_0^2 & 0 \\ -\alpha_0\alpha_2 & 0 & 0 \end{pmatrix},\ 2\mu_a^2 = \begin{pmatrix} \frac{1}{2}\alpha_0^2 & 0 & 0 \\ 0 & \frac{1}{2}\alpha_0^2 & 0 \\ \alpha_0\alpha_2 & 0 & 0 \end{pmatrix} \tag{4.2}$$

and therefore

$$(2\mu_{a^3} - \omega(a)^3 I_M)(\mu_{a^2} + 2\mu_a^2) = 0. \tag{4.3}$$

We have

$$\mu_e = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \mu_{e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mu_{e_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.4}$$

By setting $M = \langle m_1, m_2, m_3\rangle$, it just comes $e.m_1 = \frac{1}{2}m_1$, $e \cdot m_2 = -\frac{1}{2}m_2$, $e \cdot m_3 = 0$, $e_1 \cdot m_1 = m_2$, $e_1 \cdot m_2 = 0$, $e_1 \cdot m_3 = 0$, $e_2 \cdot m_1 = m_3$, $e_2 \cdot m_2 = 0$ and $e_2 \cdot m_3 = 0$.

We have $A = \mathbb{R}e \oplus A_0 \oplus A_{-\frac{1}{2}}$ where $A_0 = \langle e_1\rangle$ and $A_{-\frac{1}{2}} = \langle e_2\rangle$, and $M = M_{\frac{1}{2}} \oplus M_0 \oplus M_{-\frac{1}{2}}$ where $M_{\frac{1}{2}} = \langle m_1\rangle$, $M_0 = \langle m_3\rangle$, $M_{-\frac{1}{2}} = \langle m_2\rangle$. For all $a_i, b_i \in A_i$ where $i \in \{0, -\frac{1}{2}\}$ and $m \in M$, we have $a_i \cdot (b_i \cdot m) = b_i \cdot (a_i \cdot m)$. By definition $L_0 = \{m \in M \mid \langle e_1\rangle .m \subseteq \langle m_1\rangle \oplus \langle m_3\rangle\}$ and $L_{-\frac{1}{2}} = \{m \in M \mid \langle e_2\rangle \cdot m \subseteq \langle m_1\rangle \oplus \langle m_2\rangle\}$. Let $m = \lambda_1 m_1+\lambda_2 m_2+\lambda_3 m_3 \in M$, we have $e_1 \cdot m = \lambda_1 e_1 \cdot m_1+\lambda_2 e_1 \cdot m_2+\lambda_3 e_1 \cdot m_3 = \lambda_1 m_2 \in \langle m_2\rangle$, $e_1 \cdot m \in \langle m_1\rangle \oplus \langle m_3\rangle$ leads to $\lambda_1 = 0$ and therefore $L_0 = \langle m_2, m_3\rangle$. Similarly, we have $e_2 \cdot m = \lambda_1 e_2 \cdot m_1 + \lambda_2 e_2 \cdot m_2 + \lambda_3 e_2 \cdot m_3 = \lambda_1 m_3 \in \langle m_3\rangle$, $e_2 \cdot m \in \langle m_1\rangle \oplus \langle m_2\rangle$ implies that $\lambda_1 = 0$ and thus $L_{-\frac{1}{2}} = \langle m_2, m_3\rangle$. □

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