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CHARACTERIZATION OF A HYPERBOLIC RICCI SOLITON ON SEQUENTIAL WARPED PRODUCT MANIFOLD

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Abstract. The purpose of the paper is to analyze a hyperbolic Ricci soliton on sequential warped product manifold. We investigate the necessary condition for a hyperbolic Ricci soliton on sequential warped product to become an Einstein manifold when the potential vector field is admitted as a Killing or a conformal vector field. We also arrive conditions under which a sequential warped product becomes hyperbolic Ricci soliton. Later we characterize hyperbolic Ricci soliton on a sequential standard static space-time and a sequential generalized Robertson-Walker space-time.

1. Introduction

Geometric flow is an important concept in the mathematical area of differential geometry. The geometric flow is a type of geometric evolution equation, a partial differential equation applied to a geometric entity such as a Riemannian metric. In the early 1980s, R. Hamilton [10] introduced the notion of the Ricci flow encouraged by Eells and Sampson's project on harmonic map heat flow [9]. The Ricci flow is an evolution equation on a smooth manifold M with Riemannian metric $g(t)$ given by

$$\frac{\partial}{\partial t}g(t) = -2Ric.$$

Ricci solitons appear as self-similar solutions to Hamiltons's Ricci flow and often arise as limits of dilations of singularities in the Ricci flow [10]. Ricci solitons are natural generalizations of Einstein metrics. A Ricci soliton is defined on a Riemannian manifold (M, g) by

$$Ric + \frac{1}{2}\pounds_V g = \lambda g$$

where $\pounds_V g$ is the Lie derivative along the vector field V, Ric is the Ricci tensor of (M, g) and λ is a real constant. In [12], Kong and Liu introduced the notion of hyperbolic Ricci flow which is a system of second degree non-linear partial differential equations. The wave features of metrices and curvatures are described

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by this flow. Consider $g_{ij}(t)$ as a family of Riemannian metrics defined on the on the manifold (M^n, g_0). The hyperbolic Ricci flow is mathematically defined as

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}$$

with $g(0) = g_0$, $\frac{\partial g_{ij}}{\partial t} = h_{ij}$, where h_{ij} is a symmetric 2-tensor field.

A self-similar solution $g(t)$ of the hyperbolic Ricci flow on M^n becomes a hyperbolic Ricci soliton if there exists a 1-parameter family of diffeomorphisms $\phi(t) : M \to M$ and a positive function $\sigma(t)$ such that

$$g(t) = \sigma(t)\phi(t)^*(g_0),\ \forall t \in (a, b). \tag{1.1}$$

Differentiating (1.1) twice we have,

$$-2Ric(g(t)) = \sigma''(t)\phi(t)^*(g_0) + 2\sigma'(t)\phi(t)^*(\pounds_V g_0) + \sigma(t)\phi(t)^*(\pounds_V(\pounds_V g_0)), \tag{1.2}$$

where Ric denotes the Ricci curvature on M, V indicates the time dependent vector field and $\pounds$ is the Lie derivative. The solution $g(t)$ is shrinking, steady or expanding at a time t_0 if σ' is negative, zero or positive, respectively.

Putting $\sigma''(0) = -2\mu$, $\sigma(0) = 2$ and $\sigma'(0) = \lambda$ in the equation (1.2) we have

$$Ric(g_0) + \lambda\pounds_V g_0 + (\pounds_V \circ \pounds_V)(g_0) = \mu g_0.$$

Omitting the subscript on g, a hyperbolic Ricci soliton structure is a Riemannian manifold (M^n, g) if there exists a vector field V on M such that

$$Ric + \lambda\pounds_V g + \pounds_V(\pounds_V g) = \mu g, \tag{1.3}$$

for some real scalars μ and λ. The parameter μ reflects the rate of change in the solutions. A hyperbolic Ricci soliton (M, g, V, λ, μ) will be a gradient hyperbolic structure if $V = \nabla f$ for the potential function f. A hyperbolic Ricci soliton becomes a Ricci soliton if V is a 2-killing vector field.

S. Azami constructed hyperbolic Ricci-Bourguignon flow in 2021 [5]. The hyperbolic Ricci-Bourguignon flow is defined on Riemannian manifold M^n equipped with the Riemannian metric g and a symmetric tensor h in the following equation

$$\frac{\partial^2 g}{\partial t^2} = -2Ric + 2\rho Rg,$$

with $g(0) = g_0$, $\frac{\partial g}{\partial t} = h$, where R stands for the scalar curvature of the Riemannian metric g and $\rho \in \mathbb{R}$.

A Riemannian manifold (M^n, g) is called a hyperbolic Ricci-Bourguignon soliton structure if there exists a vector field V on M and real scalars μ, λ such that

$$Ric + \lambda\pounds_V g + \pounds_V(\pounds_V g) = (\mu + \rho R)g, \tag{1.4}$$

where $\rho \in \mathbb{R}$.

The idea of warped products was initiated by O'Neill and Bishop to build Riemannian manifolds with negative sectional curvature [6]. The study of warped products provides some significant insights in differential geometry and in the area of physics [1, 2, 13]. S. Shenawy established a current category of warped product manifolds, specifically sequential warped products which can be considered as a generalization of singly warped products [17]. Let (M_i, g_i) $i = 1, 2, 3$ be three Riemannian manifolds with two smooth positive functions $f : M_1 \to (0, \infty)$ on M_1 and $h : M_1 \times M_2 \to (0, \infty)$ on $M_1 \times M_2$. Then the sequential warped product manifold is the product manifold $M = (M_1 \times M_2) \times M_3$ equipped with the metric tensor $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$, identified by $(M_1 \times_f M_2) \times_h M_3$. The functions f and h are mentioned as warping functions.

H. I. Yoldaş studied various types of Ricci soliton endowed with various vector fields [18, 19, 20]. In 2020, S. Pahan derived to characterize the relation between the warping function and the squared norm of mean curvature of warped product pseudo-slant submanifolds of generalized Sasakian space forms under some certain conditions [16]. In 2023, M. Altunbaş established Ricci solitons on tangent bundles on tangent bundles with respect to the complete lift of a projective semi-symmetric connection using vertical and complete lifts of torqued vector fields[3].

Motivated by the above researches, the paper establishes the necessary condition for a hyperbolic Ricci soliton on sequential warped product manifold to become an Einstein manifold when we consider a potential vector field as a 2-Killing or a conformal vector field, and discovers a hyperbolic Ricci soliton from sequential warped product under certain given conditions. The paper goes on to characterize hyperbolic Ricci solitons on sequential standard static space-times and sequential generalized Robertson-Walker space-times.

2. **Preliminaries**

Now we contemplate the following propositions from [7, 17], which will be used in the proof of the main results from this section.

Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product manifold furnished with metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ and let $X_i, Y_i, Z_i \in \chi(M_i)$.

Proposition 2.1. *The Levi-Civita connection's components on (M, g) are specified by :*
1) $\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1$,
2) $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = X_1(lnf) X_2$,
3) $\nabla_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2) grad^1 f$,
4) $\nabla_{X_3} X_1 = \nabla_{X_1} X_3 = X_1(lnh) X_3$,
5) $\nabla_{X_2} X_3 = \nabla_{X_3} X_2 = X_2(lnh) X_3$,
6) $\nabla_{X_3} Y_3 = \nabla^3_{X_3} Y_3 - h g_3(X_3, Y_3) gradh$,
where $X_i, Y_i, Z_i \in \chi(M_i)$.

Proposition 2.2. *The Riemannian curvature's non-zero components of (M, g) are provided by :*
1) $Ric(X_1, Y_1) = Ric^1(X_1, Y_1) - \frac{n_2}{f} H^f_1(X_1, Y_1) - \frac{n_3}{h} H^h(X_1, Y_1)$,
2) $Ric(X_2, Y_2) = Ric^2(X_2, Y_2) - f^2 g_2(X_2, Y_2) f^* - \frac{n_3}{h} H^h(X_2, Y_2)$,
3) $Ric(X_3, Y_3) = Ric^3(X_3, Y_3) - h^2 g_3(X_3, Y_3) h^*$,
4) $Ric(X_i, Y_j) = 0,\ i \neq j$,
where $f^* = \frac{\Delta^1 f}{f} + (n_2 - 1)\frac{|grad^1 f|^2}{f^2}$ *and* $h^* = \frac{\Delta h}{h} + (n_1 + n_2 - 1)\frac{|gradh|^2}{h^2}$.

Lemma 2.3. *For a vector field $\xi \in \chi(M)$, the following equation is satisfied by :*

$$\begin{aligned}(\pounds_\xi g)(X,Y) &= (\pounds^1{}_{\xi_1} g_1)(X_1,Y_1) + f^2(\pounds^2{}_{\xi_2} g_2)(X_2,Y_2) + h^2(\pounds^3{}_{\xi_3} g_3)(X_3,Y_3)\\ &+ 2f\xi_1(f)g_2(X_2,Y_2) + 2h(\xi_1+\xi_2)(h)g_3(X_3,Y_3). \end{aligned} \tag{2.1}$$

Corollary 2.4. *Let $(M = (M_1 \times_f M_2) \times_h M_3, g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3)$ be a n-dimensional sequential warped product. Then,*

1) $(\pounds_Z \pounds_Z g)(X_1,Y_1) = (\pounds^1{}_{Z_1}\pounds^1{}_{Z_1} g_1)(X_1,Y_1)$,

2) $(\pounds_Z \pounds_Z g)(X_2,Y_2) = f^2(\pounds^2{}_{Z_2}\pounds^2{}_{Z_2} g_2)(X_2,Y_2) + 2Z_1(f^2)(\pounds^2{}_{Z_2} g_2)(X_2,Y_2) + Z_1(Z_1(f^2))g_2(X_2,Y_2)$,

3) $(\pounds_Z \pounds_Z g)(X_3,Y_3) = h^2(\pounds^3{}_{Z_3}\pounds^3{}_{Z_3} g_3)(X_3,Y_3) + 2(Z_1+Z_2)(h^2)(\pounds^3{}_{Z_3} g_3)(X_3,Y_3) + (Z_1+Z_2)((Z_1+Z_2)(h^2))g_3(X_3,Y_3)$,

4) $(\pounds_Z \pounds_Z g)(X_1,Y_2) = -(\frac{X_1 f}{f})[f^2(\pounds^2{}_{Z_2} g_2)(Y_2,Z_2) + Z_1(f^2)g_2(Y_2,Z_2)]$,

5) $(\pounds_Z \pounds_Z g)(X_1,Y_3) = -(\frac{X_1 h}{h})[h^2(\pounds^3{}_{Z_3} g_3)(Y_3,Z_3) + (Z_1+Z_2)(h^2)g_2(Y_3,Z_3)]$,

6) $(\pounds_Z \pounds_Z g)(X_2,Y_3) = 0$,

for all vector fields $X = X_1 + X_2 + X_3$, $Y = Y_1 + Y_2 + Y_3$, $Z = Z_1 + Z_2 + Z_3$ on M and $X_i,\ Y_i,\ Z_i \in \chi(M_i)$.

Proof. The proof of the above corollary is easily seen in [4]. □

3. Different vector fields on hyperbolic Ricci soliton with sequential warped product

Here, we characterize applications of different vector fields on hyperbolic Ricci soliton with sequential warped product. The following definitions will assist to prove our main results.

Definition 3.1. A vector field ξ on a Riemannian or pseudo Riemannian manifold M is called a Killing vector field if it satisfies

$$\pounds_\xi g = 0. \tag{3.1}$$

Definition 3.2. On a (pseudo) Riemannian manifold M, a vector field ξ is called a 2-Killing vector field if it satisfies [14]

$$\pounds_\xi \pounds_\xi g = 0. \tag{3.2}$$

Definition 3.3. A vector field ξ on a (pseudo) Riemannian manifold M is called a conformal vector field [8] if

$$\pounds_\xi g = \rho g, \tag{3.3}$$

for a smooth function $\rho \in C^\infty(M)$. In particular, ξ is called conformal Killing if $\rho = 0$.

In 2024, Kaya and Özgür worked on hyperbolic Ricci solitons on sequential warped product manifolds[11]. They explored that a hyperbolic Ricci soliton with sequential warped product becomes an Einstein manifold if the potential field is a Killing vector field and a conformal vector field. In this paper, on a hyperbolic Ricci soliton with sequential warped product we will characterize the manifolds M_1, M_2 and M_3, separately when a potential vector field is to be considered as a 2-Killing or a conformal vector field. Also, we will obtain a sequential warped product manifold to be a hyperbolic Ricci soliton under certain conditions.

Theorem 3.4. *Let $(M, g, \xi, \lambda, \mu)$ be a hyperbolic Ricci soliton with potential vector field of the form $\xi = \xi_1 + \xi_2 + \xi_3$, where $(M = (M_1 \times_f M_2) \times_h M_3, g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3)$ is a sequential warped product manifold.*

(a) If $H^f = \sigma g$ and $H^h = \psi g$ then $(M_1, g_1, \xi_1, \lambda, \mu_1)$ becomes a hyperbolic Ricci soliton, where $\mu_1 = \mu + \frac{n_2}{f}\sigma + \frac{n_3}{h}\psi$.

(b) If ξ_1 is a conformal vector field on M_1 with factor ρ_1 satisfying $H^f = \sigma g$ and $H^h = \psi g$ then (M_1, g_1) is an Einstein manifold.

(c) If ξ_1 is a 2-Killing vector field with $H^f = \sigma g$ and $H^h = \psi g$ then $(M_1, g_1, h_1, \gamma_1)$ is h_1-almost Ricci soliton where $h_1 = \lambda$ and $\gamma_1 = (\mu + \frac{n_2}{f}\sigma + \frac{n_3}{h}\psi)$.

(d) If ξ_2 is a 2-Killing vector field and $H^h = \psi g$ then $(M_2, g_2, h_2, \gamma_2)$ is an h_2-almost Ricci soliton where $h_2 = \lambda f^2 + 2\xi_1(f^2)$ and $\gamma_2 = \mu f^2 + f^2 f^* + \frac{n_3}{h}\psi - 2\lambda f \xi_1(f) - \xi_1(\xi_1(f^2))$.

(e) If ξ_2 is a conformal vector field on M_2 with factor ρ_2 and $H^h = \psi g$ then (M_2, g_2) is an Einstein manifold.

(f) If ξ_3 is a 2-Killing vector field then $(M_3, g_3, h_3, \gamma_3)$ is an h_3-almost Ricci soliton where $h_3 = \lambda h^2 + 2(\xi_1 + \xi_2)(h^2)$ and $\gamma_3 = [\mu h^2 + h^2 h^* - 2h\lambda(\xi_1 + \xi_2)(h) - (\xi_1 + \xi_2)((\xi_1 + \xi_2)(h^2))]$.

(g) If ξ_3 is a conformal vector field on M_3 with factor ρ_3 then (M_3, g_3) is an Einstein manifold.

Proof. Let us consider that $(M, g, \xi, \lambda, \mu)$ is a hyperbolic Ricci soliton with the structure of sequential warped product. For $X, Y \in \chi(M)$, the equation (1.3) becomes

$$Ric(X, Y) + \lambda(\pounds_\xi g)(X, Y) + (\pounds_\xi \circ \pounds_\xi)g(X, Y) = \mu g(X, Y). \tag{3.4}$$

For vector fields X,Y if we take $X=X_1+X_2+X_3$, $Y=Y_1+Y_2+Y_3$ and using proposition 2.2, lemma 2.3 and corollary 2.4 we observe

$$\begin{aligned}
Ric^1(X_1,Y_1) &- \frac{n_2}{f}H_1^f(X_1,Y_1)-\frac{n_3}{h}H^h(X_1,Y_1)\\
&+ Ric^2(X_2,Y_2)-f^2g_2(X_2,Y_2)f^*-\frac{n_3}{h}H^h(X_2,Y_2)\\
&+ Ric^3(X_3,Y_3)-h^2g_3(X_3,Y_3)h^*\\
&+ \lambda[(\pounds^1{}_{\xi_1}g_1)(X_1,Y_1)+f^2(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2)+h^2(\pounds^3{}_{\xi_3}g_3)(X_3,Y_3)\\
&+ 2f\xi_1(f)g_2(X_2,Y_2)+2h(\xi_1+\xi_2)(h)g_3(X_3,Y_3)]\\
&+ (\pounds^1{}_{\xi_1}\pounds^1{}_{\xi_1}g_1)(X_1,Y_1)\\
&+ f^2(\pounds^2{}_{\xi_2}\pounds^2{}_{\xi_2}g_2)(X_2,Y_2)+2\xi_1(f^2)(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2)+\xi_1(\xi_1(f^2))g_2(X_2,Y_2)\\
&+ h^2(\pounds^3{}_{\xi_3}\pounds^3{}_{\xi_3}g_3)(X_3,Y_3)+2(\xi_1+\xi_2)(h^2)(\pounds^3{}_{\xi_3}g_3)(X_3,Y_3)\\
&+ (\xi_1+\xi_2)((\xi_1+\xi_2)(h^2))g_3(X_3,Y_3)\\
&= \mu g_1(X_1,Y_1)+\mu f^2g_2(X_2,Y_2)+\mu h^2g_3(X_3,Y_3). \qquad (3.5)
\end{aligned}$$

Let $X=X_1$, $Y=Y_1$. Then, the equation (3.5) becomes

$$\begin{aligned}
Ric^1(X_1,Y_1) &- \frac{n_2}{f}H_1^f(X_1,Y_1)-\frac{n_3}{h}H^h(X_1,Y_1)\\
&+ \lambda(\pounds^1{}_{\xi_1}g_1)(X_1,Y_1)+(\pounds^1{}_{\xi_1}\pounds^1{}_{\xi_1}g_1)(X_1,Y_1)\\
&= \mu g_1(X_1,Y_1). \qquad (3.6)
\end{aligned}$$

If we put $H^f=\sigma g$ and $H^h=\psi g$ the above equation (3.6) can be rewritten as

$$\begin{aligned}
Ric^1(X_1,Y_1) &+ \lambda(\pounds^1{}_{\xi_1}g_1)(X_1,Y_1)\\
&+ (\pounds^1{}_{\xi_1}\pounds^1{}_{\xi_1}g_1)(X_1,Y_1)=(\mu+\frac{n_2}{f}\sigma+\frac{n_3}{h}\psi)g_1(X_1,Y_1). \qquad (3.7)
\end{aligned}$$

We consider ξ_1 as a conformal vector field on M_1 with factor ρ_1 then we get $\pounds^1{}_{\xi_1}g_1=\rho_1 g_1$. It follows that

$$\pounds^1{}_{\xi_1}\pounds^1{}_{\xi_1}g_1=(\xi_1(\rho_1)+\rho_1^2)g_1.$$

If ξ_1 is a 2-Killing vector field, $\pounds^1{}_{\xi_1}\pounds^1{}_{\xi_1}g_1=0$.

From the above we can easily prove (a)-(c).

We assume that $X=X_2$, $Y=Y_2$. The equation (3.5) can be written as

$$\begin{aligned}
Ric^2(X_2,Y_2) &- f^2g_2(X_2,Y_2)f^*-\frac{n_3}{h}H^h(X_2,Y_2)\\
&+ \lambda[f^2(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2)+2f\xi_1(f)g_2(X_2,Y_2)]\\
&+ f^2(\pounds^2{}_{\xi_2}\pounds^2{}_{\xi_2}g_2)(X_2,Y_2)+2\xi_1(f^2)(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2)+\xi_1(\xi_1(f^2))g_2(X_2,Y_2)\\
&= \mu f^2g_2(X_2,Y_2). \qquad (3.8)
\end{aligned}$$

If we put $H^h = \psi g$ the above equation (3.8) gives

$$\begin{aligned} Ric^2(X_2,Y_2) &- f^2 g_2(X_2,Y_2)f^* \\ &+ \lambda[f^2(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2) + 2f\xi_1(f)g_2(X_2,Y_2)] \\ &+ f^2(\pounds^2{}_{\xi_2}\pounds^2{}_{\xi_2}g_2)(X_2,Y_2) + 2\xi_1(f^2)(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2) + \xi_1(\xi_1(f^2))g_2(X_2,Y_2) \\ &= (\mu f^2 + \frac{n_3}{h}\psi)g_2(X_2,Y_2). \end{aligned} \tag{3.9}$$

The results (d) and (e) follow from the equations (3.8) and (3.9), respectively.

Let $X = X_3$, $Y = Y_3$. We attain from the equation (3.5)

$$\begin{aligned} Ric^3(X_3,Y_3) &- h^2 g_3(X_3,Y_3)h^* + \lambda[h^2(\pounds^3{}_{\xi_3}g_3)(X_3,Y_3) + 2h(\xi_1+\xi_2)(h)g_3(X_3,Y_3)] \\ &+ h^2(\pounds^3{}_{\xi_3}\pounds^3{}_{\xi_3}g_3)(X_3,Y_3) + 2(\xi_1+\xi_2)(h^2)(\pounds^3{}_{\xi_3}g_3)(X_3,Y_3) \\ &+ (\xi_1+\xi_2)((\xi_1+\xi_2)(h^2))g_3(X_3,Y_3) = \mu h^2 g_3(X_3,Y_3). \end{aligned} \tag{3.10}$$

The results (f) and (g) can be seen from the equation above. □

Theorem 3.5. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product manifold endowed with metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. Then $(M, g, \xi, \lambda, \mu)$ will be a hyperbolic Ricci soliton where $\mu_1 = \mu + \frac{n_2}{f}\phi + \frac{n_3}{h}\psi$ if*

a) $(M_1, g_1, \xi_1, \lambda, \mu_1)$ is a hyperbolic Ricci soliton with potential vector field of the form $\xi = \xi_1 + \xi_2 + \xi_3$,

b) ξ_i is conformal vector field on M_i with factor ρ_i for $i = 2, 3$ satisfying $H^f = \phi g$ and $H^h = \psi g$,

c) M_2, M_3 are Einstein manifolds i.e. $Ric^2 = \tau_2 g_2$ and $Ric^3 = \tau_3 g_3$ where τ_2, τ_3 are scalar functions,

d) $(\mu_1 - \frac{n_2}{f}\phi - \frac{n_3}{h}\psi)f^2 = \tau_2 - f^* - n_3 h\psi + f^2\rho_2 + 2\rho(\xi_1 f) + f^2(\xi_2(\rho_2) + {\rho_2}^2) + 2\rho_2\xi_1(f^2) + \xi_1(\xi_1(f^2))$ and $(\mu_1 - \frac{n_2}{f}\phi - \frac{n_3}{h}\psi)h^2 = \tau_3 - h^* + h^2\rho_3 + 2h(\xi_1+\xi_2)(h) + h^2(\xi_3(\rho_3) + {\rho_3}^2) + 2\rho_3(\xi_1+\xi_2)(h^2) + (\xi_1+\xi_2)((\xi_1+\xi_2)(h^2))$.

Proof. Since ξ_i is conformal vector field on M_i with factor ρ_i for $i = 2, 3$, we have $\pounds^2_{\xi_2}g_2 = \rho_2 g_2$ and $\pounds^2_{\xi_3}g_3 = \rho_3 g_3$ such that

$$\pounds^2_{\xi_2}\pounds^2_{\xi_2}g_2 = (\xi_2(\rho_2) + {\rho_2}^2)g_2, \tag{3.11}$$

$$\pounds^3_{\xi_3}\pounds^3_{\xi_3}g_3 = (\xi_3(\rho_3) + {\rho_3}^2)g_3. \tag{3.12}$$

Since $(M_1, g_1, \xi_1, \lambda_1, \mu)$ is a hyperbolic Ricci soliton, one has

$$Ric^1(X_1,Y_1) + \lambda_1 \pounds_{\xi_1}{}^1 g_1(X_1,Y_1) + \pounds_{\xi_1}{}^1\pounds_{\xi_1}{}^1 g_1(X_1,Y_1) = \mu_1 g_1(X_1,Y_1), \tag{3.13}$$

for any vector fields X_1, Y_1 on M_1.

From lemma 2.3 we obtain

$$\begin{aligned} (\pounds_\xi g)(X,Y) &= (\pounds^1_\xi g_1)(X_1,Y_1) + [f^2\rho_2 + 2f\xi_1(f)]g_2(X_2,Y_2) \\ &+ [h^2\rho_3 + 2h(\xi_1+\xi_2)(h)]g_3(X_3,Y_3) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
(\pounds_\xi \pounds_\xi g)(X,Y) &= (\pounds_\xi{}^1 \pounds_\xi{}^1 g_1)(X_1,Y_1) + f^2(\xi_2(\rho_2) + {\rho_2}^2) g_2(X_2,Y_2) \\
&+ 2\xi_1(f^2)\rho_2 g_2(X_2,Y_2) + \xi_1(\xi_1(f^2)) g_2(X_2,Y_2) \\
&+ \xi_1(\xi_1(f^2)) g_2(X_2,Y_2) + h^2(\xi_3(\rho_3) + {\rho_3}^2) g_3(X_3,Y_3) \\
&+ 2(\xi_1+\xi_2)(h^2)\rho_3 g_3(X_3,Y_3) \\
&+ (\xi_1+\xi_2)((\xi_1+\xi_2)(h^2)) g_3(X_3,Y_3) \qquad (3.15)
\end{aligned}$$

Also we can see that

$$\begin{aligned}
Ric(X,Y) &+ \lambda(\pounds_\xi g)(X,Y) + (\pounds_\xi \pounds_\xi g)(X,Y) = Ric^1(X_1,Y_1) - \frac{n_2}{f}\phi g_1(X_1,Y_1) - \frac{n_3}{f}\psi g_1(X_1,Y_1) \\
&+ \tau_2 g_2(X_2,Y_2) - f^* g_2(X_2,Y_2) - n_3 h\psi g_2(X_2,Y_2) + \tau_3 g_3(X_3,Y_3) \\
&- h^* g_3(X_3,Y_3) + \lambda_1 \pounds_{\xi_1}{}^1 g_1(X_1,Y_1) + [f^2\rho_2 + 2f(\xi_1(f))] g_2(X_2,Y_2) \\
&+ [h^2\rho_3 + 2h(\xi_1+\xi_2)(h)] g_3(X_3,Y_3) + (\pounds_{\xi_1}{}^1 \pounds_{\xi_1}{}^1 g_1)(X_1,Y_1) \\
&+ f^2(\xi_2(\rho_2) + {\rho_2}^2) g_2(X_2,Y_2) + 2\xi_1(f^2)\rho_2 g_2(X_2,Y_2) + \xi_1(\xi_1(f^2)) g_2(X_2,Y_2) \\
&+ h^2(\xi_3(\rho_3) + {\rho_3}^2) g_3(X_3,Y_3) + 2(\xi_1+\xi_2)(h^2)\rho_3 g_3(X_3,Y_3) \\
&+ (\xi_1+\xi_2)((\xi_1+\xi_2)(h^2)) g_3(X_3,Y_3). \qquad (3.16)
\end{aligned}$$

Using equations (3.11) – (3.15) we find from equation (3.16) that

$$\begin{aligned}
Ric(X,Y) + \lambda(\pounds_\xi g)(X,Y) &+ (\pounds_\xi \pounds_\xi g)(X,Y) = [\mu_1 - \frac{n_2}{f}\phi - \frac{n_3}{h}\psi] g_1(X_1,Y_1) \\
&+ [\tau_2 - f^* - n_3 h\psi + f^2\rho_2 + 2\rho(\xi_1(f)) + f^2(\xi_2(\rho_2) \\
&+ {\rho_2}^2) + 2\rho_2\xi_1(f^2) + \xi_1(\xi_1(f^2))] g_2(X_2,Y_2) \\
&+ [\tau_3 - h^* + h^2\rho_3 + 2h(\xi_1+\xi_2)(h) + h^2(\xi_3(\rho_3) + {\rho_3}^2) \\
&+ 2\rho_3(\xi_1+\xi_2)(h^2) \\
&+ (\xi_1+\xi_2)((\xi_1+\xi_2)(h^2))] g_3(X_3,Y_3). \qquad (3.17)
\end{aligned}$$

Using the condition (d), the equation (3.17) yields $(M, g, \xi, \lambda, \mu)$ is a hyperbolic Ricci soliton where $\mu = \mu_1 - \frac{n_2}{f}\phi - \frac{n_3}{h}\psi$. □

4. **Hyperbolic Ricci solitons on Sequential Standard Static Space-times**

The following propositions, taken from [7, 17] will be effective to determine the main results of the following section. Let $({M_1}^{n_1}, g_1)$ and $({M_2}^{n_2}, g_2)$ be two Riemannian manifolds equipped with two smooth positive functions $h : M_1 \times M_2 \to (0,\infty)$ and $f : M_1 \to (0,\infty)$. Then $M = (M_1 \times_f M_2) \times_h I$, is a $(n_1 + n_2 + 1)$-dimensional product manifold endowed with the metric tensor $g = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2)$. This product manifold is called a standard static space-time, where dt^2 is the Euclidean metric tensor on a connected and open subinterval I of $\mathbb{R}$.

Let $M = (M_1 \times_f M_2) \times_h I$ be a sequential standard static space-time equipped with metric $g = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2)$ and also let $X_i, Y_i, Z_i \in \chi(M_i)$.

Proposition 4.1. *The Levi-Civita connection's components on (M, g) are specified by :*
1) $\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1$,
2) $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = X_1(lnf) X_2$,
3) $\nabla_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2) grad^1 f$,
4) $\nabla_{X_i} \partial_t = \nabla_{\partial_t} X_i = X_i(lnh)\partial_t \ i = 1, 2$,
5) $\nabla_{\partial_t} \partial_t = hgradh$.

Proposition 4.2. *The Riemannian curvature's non-zero components of (M, g) are provided by :*
1) $Ric(X_1, Y_1) = Ric^1(X_1, Y_1) - \frac{n_2}{f} H^f_1(X_1, Y_1) - \frac{n_3}{h} H^h(X_1, Y_1)$,
2) $Ric(X_2, Y_2) = Ric^2(X_2, Y_2) - f^2 g_2(X_2, Y_2) f^{\#} - \frac{1}{h} H^h(X_2, Y_2)$,
3) $Ric(\partial_t, \partial_t) = h\Delta h$,
4) $Ric(X_i, Y_j) = 0, \ i \neq j$,
where $f^{\#} = f\Delta^1 f + (n_2 - 1)|grad^1 f|^2$.

Lemma 4.3. *For a vector field $Z \in \chi(M_i)$, the following equation is satisfied by :*

$$
\begin{aligned}
(\pounds_Z g)(X, Y) &= (\pounds^1{}_{Z_1} g_1)(X_1, Y_1) + f^2 (\pounds^2{}_{Z_2} g_2)(X_2, Y_2) - 2uv\frac{\partial \omega}{\partial t} \\
&+ 2fZ(f) g_2(X_2, Y_2) - 2uvh(Z_1 + Z_2)(h), \qquad (4.1)
\end{aligned}
$$

for all vector fields $X = X_1 + X_2 + \omega\partial_t$, $Y = Y_1 + Y_2 + u\partial_t$, $Z = Z_1 + Z_2 + v\partial_t$ on M and $X_i, Y_i, Z_i \in \chi(M_i)$.

Corollary 4.4. *Let $M = (M_1 \times_f M_2) \times_h I$ be a sequential standard static space-time endowed with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2)$. Then*

$$
\begin{aligned}
(\pounds_\xi \pounds_\xi g)(\omega\partial_t, u\partial_t) &= -2h^2 y(\omega \frac{\partial u}{\partial t}\frac{\partial y}{\partial t} + u\frac{\partial \omega}{\partial t}\frac{\partial y}{\partial t} + u\omega\frac{\partial^2 y}{\partial t^2} \\
&- 4h^2(\xi_1 + \xi_2)(h^2) u\omega \frac{\partial y}{\partial t} - (\xi_1 + \xi_2)(\xi_1 + \xi_2)(h^2)), (4.2)
\end{aligned}
$$

for all vector fields $X = X_1 + X_2 + \omega\partial_t$, $Y = Y_1 + Y_2 + u\partial_t$, $\xi = \xi_1 + \xi_2 + y\partial_t$ on M and $X_i, Y_i, \xi_i \in \chi(M_i)$.

Theorem 4.5. *Let $(M, g, \xi, \lambda, \mu)$ be a hyperbolic Ricci soliton with potential vector field of the form $\xi = \xi_1 + \xi_2 + y\partial_t$, where $(M = (M_1 \times_f M_2) \times_h I, g = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2))$ is a sequential standard static space-time.*

(a) If $H^f = \sigma g$ and $H^h = \psi g$, then $(M_1, g_1, \xi_1, \lambda, \mu_1)$ is a hyperbolic Ricci soliton.

(b)If ξ_1 is a conformal vector field on M_1 with factor ρ_1 with $H^f = \sigma g$ and $H^h = \psi g$ then (M_1, g_1) is an Einstein manifold.

(c) If ξ_1 is a Killing vector field with $H^f = \sigma g$ and $H^h = \psi g$ then (M_1, g_1)

is an Einstein manifold.

(d) If ξ_1 is a 2-Killing vector field with $H^f = \sigma g$ and $H^h = \psi g$ then $(M_1, g_1, h_1, \gamma_1)$ is an h_1-almost Ricci soliton where $h_1 = \lambda_1$ and $\gamma_1 = \mu + \frac{n_2}{f}\sigma + \frac{1}{h}\psi$.

(e) If ξ_2 is a 2-Killing vector field and $H^h = \psi g$ then $(M_2, g_2, h_2, \gamma_2)$ is an h_2-almost Ricci soliton where $h_2 = \lambda f^2 + 2\xi(f^2)$ and $\gamma_2 = \mu f^2 + f^{\#} + h\psi - 2\lambda f \xi_1(f^2) - \xi_1(\xi_1(f^2))$.

(f) If ξ_2 is conformal vector field on M_2 with factor ρ_2 and $H^h = \psi g$ then (M_2, g_2) will be an Einstein manifold.

(g)$h\Delta h - 2u\lambda\omega\frac{\partial y}{\partial t} - 2\lambda h(\xi_1+\xi_2)(h) - 2h^2 y(\omega\frac{\partial u}{\partial t}\frac{\partial y}{\partial t} + u\frac{\partial \omega}{\partial t}\frac{\partial y}{\partial t} + u\omega\frac{\partial^2 y}{\partial t^2} + 4h^2 u\omega(\xi_1+\xi_2)(h^2)\frac{\partial y}{\partial t} - (\xi_1+\xi_2)(\xi_1+\xi_2)(h^2)) = -\mu h^2$.

Proof. We assume that $(M, g, \xi, \lambda, \mu)$ becomes a hyperbolic Ricci soliton with the structure of sequential warped product manifold. For $X, Y \in \chi(M)$, the equation (1.3) becomes

$$Ric(X,Y) + \lambda(\pounds_\xi g)(X,Y) + (\pounds_\xi \circ \pounds_\xi)g(X,Y) = \mu g(X,Y), \tag{4.3}$$

where $X = X_1 + X_2 + \omega\partial_t$, $Y = Y_1 + Y_2 + u\partial_t$, $X_i, Y_i, \xi_i \in \chi(M_i)$, $i = 1, 2, 3$. Using proposition 4.2, lemma 4.3 and corollary 4.4 we observe

$$\begin{aligned}
Ric^1(X_1,Y_1) &- \frac{n_2}{f}H_1^f(X_1,Y_1) - \frac{1}{h}H^h(X_1,Y_1) \\
&+ Ric^2(X_2,Y_2) - g_2(X_2,Y_2)f^* - \frac{1}{h}H^h(X_2,Y_2) \\
&+ h\Delta h + \lambda(\pounds^1{}_{\xi_1}g_1)(X_1,Y_1) + f^2\lambda(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2) - 2\lambda u\omega\frac{\partial y}{\partial t} \\
&+ 2f\xi_1(f)g_2(X_2,Y_2) - 2h(\xi_1+\xi_2)(h) + (\pounds^1{}_{\xi_1}\pounds^1{}_{\xi_1}g_1)(X_1,Y_1) \\
&+ f^2(\pounds^2{}_{\xi_2}\pounds^2{}_{\xi_2}g_2)(X_2,Y_2) + 2\xi_1(f^2)(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2) + \xi_1(\xi_1(f^2))g_2(X_2,Y_2) \\
&- 2h^2 y(\omega\frac{\partial u}{\partial t}\frac{\partial y}{\partial t} + u\frac{\partial \omega}{\partial t}\frac{\partial y}{\partial t} + u\omega\frac{\partial^2 y}{\partial t^2} - 4h^2(\xi_1+\xi_2)(h^2)u\omega\frac{\partial y}{\partial t} \\
&- (\xi_1+\xi_2)(\xi_1+\xi_2)(h^2)) = \mu g_1(X_1,Y_1) + \mu f^2 g_2(X_2,Y_2) - \mu h^2.
\end{aligned} \tag{4.4}$$

Putting $X = X_1$, $Y = Y_1$ we have

$$\begin{aligned}
Ric^1(X_1,Y_1) &- \frac{n_2}{f}H_1^f(X_1,Y_1) - \frac{1}{h}H^h(X_1,Y_1) \\
&+ \lambda[(\pounds^1{}_{\xi_1}g_1)(X_1,Y_1) + (\pounds^1{}_{\xi_1}\pounds^1{}_{\xi_1}g_1)(X_1,Y_1) \\
&= \mu g_1(X_1,Y_1).
\end{aligned} \tag{4.5}$$

Taking $X = X_2$, $Y = Y_2$ we obtain

$$\begin{aligned} Ric^2(X_2,Y_2) &- g_2(X_2,Y_2)f^{\#} - \frac{1}{h}H^h(X_2,Y_2) \\ &+ f^2\lambda(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2) + 2f\xi_1(f)g_2(X_2,Y_2) - 2h(\xi_1+\xi_2)(h) \\ &+ f^2(\pounds^2{}_{\xi_2}\pounds^2{}_{\xi_2}g_2)(X_2,Y_2) + 2\xi_1(f^2)(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2). \\ &+ \xi_1(\xi_1(f^2))g_2(X_2,Y_2) = \mu f^2 g_2(X_2,Y_2) \end{aligned} \tag{4.6}$$

Also from the equation (4.4) we derive that

$$\begin{aligned} h\Delta h &- 2u\lambda\omega\frac{\partial y}{\partial t} - 2\lambda h(\xi_1+\xi_2)(h) \\ &- 2h^2 y(\omega\frac{\partial u}{\partial t}\frac{\partial y}{\partial t} + u\frac{\partial \omega}{\partial t}\frac{\partial y}{\partial t} \\ &+ u\omega\frac{\partial^2 y}{\partial t^2} + 4h^2 u\omega(\xi_1+\xi_2)(h^2)\frac{\partial y}{\partial t} \\ &- (\xi_1+\xi_2)(\xi_1+\xi_2)(h^2)) = -\mu h^2. \end{aligned} \tag{4.7}$$

The equations (4.5), (4.6) and (4.7) give the results (a)-(g). □

5. Hyperbolic Ricci Solitons on Sequential Generalized Robertson-Walker Space-times

If $M = (I \times_f M_2) \times_h M_3$ is a $(n_2 + n_3 + 1)$-dimensional product manifold furnished with the metric tensor $g = (-dt^2 \oplus f^2 g_2) \oplus h^2 g_3$ with two smooth functions $h : I \times M_2 \to (0,\infty)$ and $f : I \to (0,\infty)$ then this product manifold is a generalized Robertson-Walker space-time, where dt^2 is the Euclidean metric tensor on a connected and open subinterval I of $\mathbb{R}$. Let $M = (I \times_f M_2) \times_h M_3$ be a sequential generalized Robertson-Walker space-time with metric $g = (-dt^2 \oplus f^2 g_2) \oplus h^2 g_3$ and also let $X_i, Y_i \in \chi(M_i)$.

Proposition 5.1. [7, 17] *The Levi-Civita connection's components on* (M, g) *are specified by :*
1) $\nabla_{\partial_t}\partial_t = 0$,
2) $\nabla_{X_i}\partial_t = \nabla_{\partial_t}X_i = \frac{\dot f}{f}X_i$, $i = 2, 3$,
3) $\nabla_{X_2}Y_2 = \nabla^2_{X_2}Y_2 - f\dot f g_2(X_2,Y_2)\partial_t$,
4) $\nabla_{X_2}X_3 = \nabla_{X_3}X_2 = X_2(lnh)X_3$ $i = 1, 2$,
5) $\nabla_{X_3}Y_3 = \nabla^3_{X_3}Y_3 - hg_3(X_3,Y_3)gradh$.

Proposition 5.2. [7, 17] *The Riemannian curvature's non-zero components of* (M, g) *are provided by :*
1) $Ric(\partial_t,\partial_t) = \ddot f\frac{n_2}{f} + \frac{n_3}{h}\frac{\partial^2 h}{\partial t^2}$,
2) $Ric(X_2,Y_2) = Ric^2(X_2,Y_2) - f^2 g_2(X_2,Y_2)f^{\diamond} - \frac{1}{h}H^h(X_2,Y_2)$,
3) $Ric(X_3,Y_3) = Ric^3(X_3,Y_3) - h^{\#}g_3(X_3,Y_3)$,
4) $Ric(X_i,Y_j) = 0$, $i \neq j$,
where $f^{\diamond} = -f\ddot f + (n_2 - 1)\dot f^2$ *and* $h^{\#} = h\Delta h + (n_3 - 1)|gradh|^2$.

Lemma 5.3. *For a vector field $\xi \in \chi(M_i)$, the following equation is stated by :*

$$
\begin{aligned}
(\pounds_\xi g)(X,Y) &= -2u\omega\frac{\partial y}{\partial t} + f^2(\pounds^2{}_{\xi_2}g_2)(X_2,Y_2) + h^2(\pounds^3{}_{\xi_3}g_3)(X_3,Y_3) \\
&+ 2fy\frac{\partial f}{\partial t}g_2(X_2,Y_2) + 2h[y\frac{\partial h}{\partial t} + \xi_2(h)]g_3(X_3,Y_3), \qquad (5.1)
\end{aligned}
$$

for all vector fields $X = \omega\partial_t + X_2 + X_3$, $Y = u\partial_t + Y_2 + Y_3$, $\xi = y\partial_t + \xi_2 + \xi_3$ on M and $X_i, Y_i, \xi_i \in \chi(M_i)$.

Corollary 5.4. *Let $(M = (I \times_f M_2) \times_h M_3, g = (-dt^2 \oplus f^2 g_2) \oplus h^2 g_3)$ be a sequential generalized Robertson-Walker space-time. Then $(\pounds_\xi \pounds_\xi g)(\omega\partial_t, u\partial_t) = -2y(\omega\frac{\partial u}{\partial t}\frac{\partial y}{\partial t} + u\frac{\partial \omega}{\partial t}\frac{\partial y}{\partial t} + u\omega\frac{\partial^2 y}{\partial t^2})$,
for vector field $\xi = y\partial_t + \xi_2 + \xi_3$ on M.*

Theorem 5.5. *Let $(M, g, \xi, \lambda, \mu)$ be a hyperbolic Ricci soliton with potential vector field of the form $\xi = y\partial_t + \xi_2 + \xi_3$, where $(M = (I \times_f M_2) \times_h M_3, g = (-dt^2 \oplus f^2 g_2) \oplus h^2 g_3)$ is a sequential generalized Robertson-Walker space-time.*

(a) If ξ_2 is a 2-Killing vector field and $H^h = \psi g$, then $(M_2, g_2, h_2, \gamma_2)$ is h_2-almost Ricci soliton, where $h_2 = f^2\lambda + 4f\omega\frac{\partial f}{\partial t}$ and $\gamma_2 = \mu f^2 - f\ddot{f} + (n_2-1)f^2 + n_3 h\psi - 2f\omega\lambda\frac{\partial f}{\partial t} - 2\omega(\omega(\frac{\partial f}{\partial t})^2 + f\frac{\partial \omega}{\partial t}\frac{\partial f}{\partial t} + f\omega\frac{\partial^2 f}{\partial t^2})$.

(b) If ξ_2 is a conformal vector field on M_2 with factor ρ_2 and $H^h = \psi g$, then (M_2, g_2) turns out to be an Einstein manifold.

(c) If ξ_2 is a 2-Killing vector field, $H^h = \psi g$ and M_2 is an Einstein manifold, then ξ_2 is conformal vector field.

(d) If ξ_3 is a 2-Killing vector field, then $(M_3, g_3, h_3, \gamma_3)$ is a h_3-almost Ricci soliton, where $h_3 = \lambda h^2 + 4h\omega\frac{\partial h}{\partial t} + 2\xi_2(h^2)$ and $\gamma_3 = \mu h^2 + h^\# - 2h\lambda(\omega\frac{\partial h}{\partial t} + \xi_2(h)) - 2\omega(\omega(\frac{\partial h}{\partial t})^2 + h\frac{\partial \omega}{\partial t}\frac{\partial h}{\partial t} + h\omega\frac{\partial^2 h}{\partial t^2}) - \xi_2(\xi_2(h^2))$.

(e) If ξ_3 is a conformal vector field on M_3 with factor ρ_3, then (M_3, g_3) becomes an Einstein manifold.

(f) If ξ_3 is a 2-Killing vector field and M_3 is an Einstein manifold, then ξ_3 is a conformal vector field.

Proof. Let $(M, g, \xi, \lambda, \mu)$ be a hyperbolic Ricci soliton with sequential generalized Robertson-Walker space-time. Then, we have

$$Ric(X,Y) + \lambda\pounds\xi g(X,Y) + \pounds_\xi(\pounds_\xi g(X,Y)) = \mu g(X,Y).$$

Using proposition 5.2, lemma 5.3 and corollary 5.4 the above equation can be written as :

$$\begin{aligned}
\frac{n_2}{f}\ddot{f} &+ \frac{n_3}{h}\frac{\partial^2 h}{\partial t^2} + Ric^2(X_2,Y_2) + (f\ddot{f} - (n_2-1)\dot{f}^2)g_2(X_2,Y_2) - \frac{n_3}{h}H^h(X_2,Y_2) \\
&+ Ric^3(X_3,Y_3) - h^{\#}g_3(X_3,Y_3) - \lambda 2u\omega\frac{\partial y}{\partial t} + \lambda f^2(\pounds_{\xi_2}{}^2 g_2)(X_2,Y_2) + \lambda h^2(\pounds_{\xi_3}{}^3 g_3)(X_3,Y_3) \\
&- 2y\lambda[u\frac{\partial \omega}{\partial t}\frac{\partial y}{\partial t} + \omega\frac{\partial u}{\partial t}\frac{\partial y}{\partial t} + u\omega\frac{\partial^2 y}{\partial t^2}] \\
&+ 2f\omega\lambda\frac{\partial f}{\partial t}g_2(X_2,Y_2) + 2h\lambda[\omega\frac{\partial h}{\partial t} + \xi_2(h)]g_3(X_3,Y_3) \\
&+ f^2(\pounds_{\xi_2}{}^2\pounds_{\xi_2}{}^2 g_2)(X_2,Y_2) + 2(2f\omega\frac{\partial f}{\partial t})(\pounds_{\xi_2}{}^2 g_2)(X_2,Y_2) + 2\omega[\omega(\frac{\partial f}{\partial t})^2 + f\frac{\partial \omega}{\partial t}\frac{\partial f}{\partial t} \\
&+ f\omega\frac{\partial^2 f}{\partial t^2}]g_2(X_2,Y_2) + h^2(\pounds_{\xi_3}{}^3\pounds_{\xi_3}{}^3 g_3)(X_3,Y_3) + 2[2h\omega\frac{\partial h}{\partial t} + \xi_2(h^2)](\pounds_{\xi_3}{}^3 g_3)(X_3,Y_3) \\
&+ 2\omega[\omega(\frac{\partial h}{\partial t})^2 + h\frac{\partial \omega}{\partial t}\frac{\partial h}{\partial t} + h\omega\frac{\partial^2 h}{\partial t^2}]g_3(X_3,Y_3) + \xi_2(\xi_2(h^2))g_3(X_3,Y_3) \\
&= -\mu u\omega + \mu f^2 g_2(X_2,Y_2) + \mu h^2 g_3(X_3,Y_3). \qquad (5.2)
\end{aligned}$$

The results (a)-(g) can be easily proved from the equation (5.2). □

Conclusion 5.6. The curvatures of manifolds and the heat character of the metrics are related by Ricci flow whereas hyperbolic Ricci flow classifies the wave features of the metrics and curvatures of manifolds. Hyperbolic Ricci flow possesses many attractive characteristics of both Ricci flow and the Einstein equation. In our characterization, we have reached the conclusion that hyperbolic Ricci soliton and gradient hyperbolic Ricci soliton with sequential warped product can be transformed into an Einstein manifold with potential vector field taken as 2-Killing vector field and conformal vector field. The ideas from this paper may be taken forward in constructing hyperbolic Ricci-Bourguignon soliton with sequential warped product manifold.

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