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FOURIER TRANSFORM AND THE IDEAL OF GROUP ALGEBRA ON THE NILPOTENT ENGEL-LIE GROUP

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Abstract. We study noncommutative Fourier transform on the nilpotent Engel-Lie group G_4 to solve some interesting problems of noncommutative analysis. In fact the finest structure of G_4 can be shown as a semi-direct product of two real vector groups. This helps us to form our idea by constructing a new larger group in order to define the Fourier transform and to obtain the Plancherel formula on G_4. Moreover we show that our methods lead us to construct several existence theorems for the invariant differential operators on G_4 and on $G_4 \times \mathbb{R}$. Since the heat equation is invariant on $G_4 \times \mathbb{R}$, so a fundametal solution of this equation will be obtained. Finally we establish a theorem that gives a classification of all left ideals of the noncommutative Banach algebra $L^1(G_4)$ of G_4.

1. Introduction and results

1.1 Let G_4 be the nilpotent Engel-Lie group consisting of all matrices of the form

$$\begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & x_4 \\ 0 & 1 & -x_1 & x_3 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

where $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The group G_4 contains the $3-$dimensional real vector group H that consists of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & x_4 \\ 0 & 1 & 0 & x_3 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

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as normal sub-group. Let G be the group consisting of all matrices of the form

$$\begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & 0 \\ 0 & 1 & -x_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.3}$$

Then G_4 can be identified with the group $H \rtimes_\gamma G$ a semi-direct product of G by H where

$$\gamma(X_1)(X) = X_1 X X_1^{-1} \tag{1.4}$$

$$X_1 = \begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & 0 \\ 0 & 1 & -x_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5}$$

and

$$X = \begin{pmatrix} 1 & 0 & 0 & x_4 \\ 0 & 1 & 0 & x_3 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.6}$$

In view of the group isomorphism $\Psi : \mathbb{R} \to G$ defined by

$$\psi(x_1) = \begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} & 0 \\ 0 & 1 & -x_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.7}$$

We can identify the group G with the real vector group $\mathbb{R}$. So G_4 can be identified with the group $K = \mathbb{R}^3 \ltimes_\rho \mathbb{R}(H \simeq \mathbb{R}^3)$ a semi-direct product of $\mathbb{R}^3$ by $\mathbb{R}$, via the group homomorphism$\rho : \mathbb{R} \to Aut(\mathbb{R}^3)$, which is defined by

$$\rho(x_1\ ) = \begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} \\ 0 & 1 & -x_1 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.8}$$

and

$$\begin{aligned} \rho(x_1\ )(x_4, x_3, x_2\ ) &= \begin{pmatrix} 1 & -x_1 & \frac{x_1^2}{2} \\ 0 & 1 & -x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_4 \\ x_3 \\ x_2 \end{pmatrix} \\ &= (x_4 - x_1x_3 + \frac{1}{2}x_1^2x_2, x_3 - x_1x_2, x_2) \end{aligned} \tag{1.9}$$

for any $(x_4, x_3, x_2, x_1) \in \mathbb{R}^4$. The multiplication of two elements $X = (x_4, x_3, x_2, x_1)$ and $Y = (y_4, y_3, y_2, y_1)$ in G_4 is given by

$$\begin{aligned} X \cdot Y &= (x_4, x_3, x_2, x_1)(y_4, y_3, y_2, y_1) \\ &= (x_4 + y_4 - x_1y_3 + \frac{1}{2}x_1^2y_2, x_3 + y_3 - x_1y_2, x_2 + y_2, x_1 + y_1) \end{aligned} \tag{1.10}$$

The inverse X^{-1} of an element X in K is

$$\begin{aligned} X^{-1} &= (x_4, x_3, x_2, x_1)^{-1} \\ &= (-x_4 - x_1 x_3 - \frac{1}{2} x_1^2 x_2 -, -x_3 - x_1 x_2, -x_2, -x_1) \end{aligned} \tag{1.11}$$

1.2 If M is an unimodular Lie group, we denote by $L^1(M)$ the Banach algebra that consists of all complex valued functions on the group M, which are integrable with respect to the Haar measure of M and multiplication is defined by convolution on M, and we denote by $L^2(M)$ its Hilbert space. Let $\mathcal{U}$be the complexified universal enveloping algebra of the real Lie algebra $\underline{g}$ of G_4; which is canonically isomorphic to the algebra of all distributions on G_4 supported by $\{0\}$, where 0 is the identity element of G_4. For any $u \in \mathcal{U}$ one can define a differential operator P_u on G_4 as follows:

$$P_u f(X) = u * f(X) = \int_{G_4} f(Y^{-1}X) u(Y) dY \tag{1.12}$$

for any $f \in C^\infty(G_4)$ where $dY = dy_4 dy_3 dy_2 dy_1$ is the Haar measure on G_4 which is the Lebesgue measure on $\mathbb{R}^4$, $Y = (y_4, y_3, y_2, y_1)$, $X = (x_4, x_3, x_2, x_1)$ and $*$ denotes the convolution product on G_4. The mapping $u \to P_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on G_4. For more details see [7] and [13].

1.3 Let $B = \mathbb{R}^3 \times \mathbb{R}$ be the commutative group of the direct product of $\mathbb{R}^3$ by $\mathbb{R}$. We denote also by $\mathcal{U}$ the complexified enveloping algebra of the real Lie algebra $\underline{b}$ of B. For every $u \in \mathcal{U}$, we can associate a differential operator Q_u on B as follows

$$Q_u f(X) = u *_c f(X) = f *_c u(X) = \int_B f(X - Y) u(Y) dY \tag{1.13}$$

for any $f \in C^\infty(B)$, $X \in B, Y \in B$ where $*_c$ signify the convolution product on the real vector group B and $dY = dy_4 dy_3 dy_2 dy_1$ is the Lebesgue measure on B. The mapping $u \mapsto Q_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on B, which are nothing but the algebra of differential operator with constant coefficients on B. In this paper we will prove the following results:

I- Plancherel Formula (Theorem 2.7).

II- Existence theorems for any Invariant differential operator on G_4 (Theorems 3.2, 3.6 and 3.7).

III- An theorem for all left ideals of $L^1(G_4)$ (Theorem 4.2 and Corollary 4.3).

2. Invariant Functions and Fourier Transform

Let $L = \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ be the group with law:

$$\begin{aligned} XY &= (x_4, x_3, x_2, x_1, t)(y_4, y_3, y_2, y_1, s) \\ &= (((x_4, x_3, x_2)(\rho_1(t)(y_4, y_3, y_2))), x_1 + y_1, t + s) \\ &= (((x_4, x_3, x_2)(\rho_1(t)(y_4, y_3, y_2))), x_1 + y_1, t + s) \\ &= (x_4 + y_4 - ty_3 + \frac{1}{2}t^2 y_2, x_3 + y_3 - ty_2, x_2 + y_2, x_1 + y_1, t + s) \quad (2.1) \end{aligned}$$

for all $X = (x_4, x_3, x_2, x_1, t) \in L$ and $Y = (y_4, y_3, y_2, y_1, s) \in L$. In this case the group G_4 can be identified with the closed subgroup $\mathbb{R}^3 \times \{0\} \times \mathbb{R}$ of L and B with the subgroup $\mathbb{R}^3 \times \mathbb{R} \times \{0\}$ of L.

Definition 2.1. For every $f \in C^\infty(G_4)$, one can define a function $\widetilde{f} \in C^\infty(L)$ as follows:

$$\widetilde{f}(x_4, x_3, x_2, x_1, t) = f(\rho(x_1)(x_4, x_3, x_2), x_1 + t) \tag{2.2}$$

for all $(x_4, x_3, x_2, x_1, t) \in L$.

So every function $\psi(x_4, x_3, x_2, x_1)$ on G_4 extends uniquely as an invariant function $\widetilde{\psi}(x_4, x_3, x_2, x_1, t)$ on K.

Remark 2.2. The function $\widetilde{f}$ is invariant in the following sense:

$$\widetilde{f}(\rho(s)(x_4, x_3, x_2), x_1 - s, t + s) = \widetilde{f}(x_4, x_3, x_2, x_1, t) \tag{2.3}$$

for any $(x_4, x_3, x_2, x_1, t) \in L$ and $s \in R$.

Lemma 2.3. *For every function $F \in C^\infty(L)$ invariant in the sense of formula (2.3) and for every $u \in U$, we have*

$$u \ * F(x_4, x_3, x_2, x_1, t) = u \ *_c F(x_4, x_3, x_2, x_1, t) \tag{2.4}$$

for every $(x_4, x_3, x_2, x_1, t) \in L$, where $*$ signifies the convolution product on G_4 with respect to the variables (x_4, x_3, x_2, t) and $\star_c$ signifies the commutative convolution product on B with respect to the variables (x_4, x_3, x_2, x_1).

Proof. In fact we have

$$\begin{aligned} & P_u F(x_4, x_3, x_2, x_1, t) = u * F(x_4, x_3, x_2, x_1, t) \\ = & \int_{G_4} F\left[(y_4, y_3, y_2, s)^{-1}(x_4, x_3, x_2, x_1, t)\right] u(y_4, y_3, y_2, s) dy_4 dy_3 dy_2 ds \\ = & \int_{G_4} F\left[(\rho(s^{-1})(-y_4, -y_3 - y_2,)(x_4, x_3, x_2)), x_1, t - s\right] u(y_4, y_3, y_2, s) dy_4 dy_3 dy_2 ds \\ = & \int_{G_4} F\left[(-y_4, -y_3 - y_2,) + (x_4, x_3, x_2), x_1 - s, t\right] u(y_4, y_3, y_2, s) dy_4 dy_3 dy_2 ds \\ = & \ u *_c (x_4, x_3, x_2, x_1, t) = Q_u F(x_4, x_3, x_2, x_1, t) \qquad (2.5) \end{aligned}$$

where P_u and Q_u are the invariant differential operators on G_4 and B respectively. $\square$

As in [5], we will define the Fourier transform on G_4. Therefor let $\mathcal{S}(G_4)$ be the Schwartz space of G_4 which can be considered as the Schwartz space of $\mathcal{S}(\mathbb{R}^3\times\mathbb{R})$ and let $\mathcal{S}'(G_4)$ be the space of all tempered distributions on G_4. The action ρ of the group $\mathbb{R}$ on $\mathbb{R}^3$ defines a natural action ρ of the dual group $(\mathbb{R})^*$ of the group $\mathbb{R}$ $((\mathbb{R})^* \simeq \mathbb{R})$ on $(\mathbb{R}^3)^\star$ which is given by :

$$\rho(x_1\ ) = \begin{pmatrix} 1 & 0 & 0 \\ -x_1 & 1 & 0 \\ \frac{x_1^2}{2} & -x_1 & 1 \end{pmatrix} \tag{2.6}$$

and

$$\begin{aligned} & \rho\left(x_1\right)\left(\xi_4,\xi_3,\xi_2\right) \\ = & \begin{pmatrix} 1 & 0 & 0 \\ -x_1 & 1 & 0 \\ \frac{x_1^2}{2} & -x_1 & 1 \end{pmatrix} \begin{pmatrix} \xi_4 \\ \xi_3 \\ \xi_2 \end{pmatrix} \\ = & \left(\xi_4, -x_1\xi_4+\xi_3, \frac{1}{2}x_1^2\xi_4 - x_1\xi_3, \xi_2\right) \end{aligned} \tag{2.7}$$

for any $(\xi_4,\xi_3,\xi_2) \in \mathbb{R}^3$ and $x_1 \in \mathbb{R}$

Definition 2.4. If $f \in S(G_4)$, one can define its Fourier transform of f by:

$$\mathcal{F}f\ \ (\xi) = \int_{G_4} f(X)\ e^{-\ i\ \langle\xi,X\rangle}\ dX \tag{2.8}$$

for any $\xi = (\xi_4,\xi_3,\xi_2,\xi_1) \in \mathbb{R}^4$ and $X = (x_4,x_3,x_2,x_1) \in \mathbb{R}^4$, where $\langle\xi,X\rangle = \xi_4x_4+\xi_3x_3+\xi_2x_2+\xi_1x_1$.

It is clear that $\mathcal{F}f \in S(\mathbb{R}^4)$ and the mapping $f \to\ \mathcal{F}f$ is an isomorphism of the topological vector space $S(G_4)$ onto $S(\mathbb{R}^4)$.

Definition 2.5. If $f \in S(G_4)$, we define the Fourier transform of its invariant $\widetilde{f}$ as follows

$$\mathcal{F}\widetilde{f}(\xi,0) = \int_{L\times\mathbb{R}} \widetilde{f}(X,t)e^{-\ i\ \langle\xi,X\rangle}e^{-\ i\ \langle\mu,t\rangle}\ dX\ dtd\mu \tag{2.9}$$

where $(\mu,t) \in \mathbb{R}^2$ and$\langle\mu,t\rangle = \mu t$.

Corollary 2.6. *For every $u \in \mathcal{S}(G_4)$, and $f \in \mathcal{S}(G_4)$, we have*

$$\mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi,0) = \mathcal{F}(\widetilde{f})(\xi,0)\mathcal{F}(\overset{\vee}{u})(\xi) \tag{2.10}$$

for any $\xi = (\xi_4,\xi_3,\xi_2,\xi_1) \in \mathbb{R}^4$ and $\mu \in \mathbb{R}$.

Proof. By Lemma 2.3, we have

$$\overset{\vee}{u} * \widetilde{f}(x_4,x_3,x_2,x_1,t) =\overset{\vee}{} u \star_c \widetilde{f}(x_4,x_3,x_2,x_1,t) \tag{2.11}$$

So, we get

$$\mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi,0) = \mathcal{F}(\overset{\vee}{u} *_c \widetilde{f})(\xi,0) = \mathcal{F}(\widetilde{f})(\xi,0)\mathcal{F}(\overset{\vee}{u})(\xi) \tag{2.12}$$

□

Theorem 2.7. *(Plancherel's formula)*
For any $f \in L^1(G_4) \cap L^2(G_4)$, we get

$$\begin{aligned}
&\int\limits_{G_4} |f(x_4, x_3, x_2, x_1)|^2 \, dx_4 dx_3 dx_2 dx_1 \\
= &\int\limits_{\mathbb{R}^4} |\mathcal{F}f(\xi_4, \xi_3, \xi_2, \xi_1)|^2 \, d\xi_4 d\xi_3 d\xi_2 d\xi_1
\end{aligned} \tag{2.13}$$

Proof. For each function $f \in C_0^\infty(G_4)$, define a function $\tilde{\check{f}}$ as

$$\tilde{\check{f}}(x_4, x_3, x_2, x_1) = \overline{f(\rho(x_1)(x_4, x_3, x_2), x_1)^{-1}} \tag{2.14}$$

Then, we have

$$\begin{aligned}
& f \star \tilde{\check{f}}(0,0,0,0,0) \\
= & \int\limits_{G_4} \tilde{\check{f}}\left[(x_4, x_3, x_2, x_1)^{-1}(0,0,0,0,0)\right] f(x_4, x_3, x_2, x_1) dx_4 dx_3 dx_2 dx_1 \\
= & \int\limits_{G_4} \tilde{\check{f}}\left[(\rho(x_1^{-1})((-x_4, -x_3, -x_2) + (0,0,0)), 0, -x_1)\right] f(x_4, x_3, x_2, x_1) dx_4 dx_3 dx_2 dx_1 \\
= & \int\limits_{G_4} \check{f}\left[(\rho(x_1^{-1})((-x_4, -x_3, -x_2) + (0,0,0)), -x_1)\right] f(x_4, x_3, x_2, x_1) dx_4 dx_3 dx_2 dx_1 \\
= & \int\limits_{G_4} \check{f}\left[(\rho(x_1^{-1})(-x_4, -x_3, -x_2)), -x_1)\right] f(x_4, x_3, x_2, x_1) dx_4 dx_3 dx_2 dx_1 \\
= & \int\limits_{G_4} \overline{f(x_4, x_3, x_2, x_1)} f(x_4, x_3, x_2, x_1) dx_4 dx_3 dx_2 dx_1 \\
= & \int\limits_{G_4} |f(x_4, x_3, x_2, x_1)|^2 \, dx_4 dx_3 dx_2 dx_1
\end{aligned} \tag{2.15}$$

We get the Plancherel's formula on G_4

$$\begin{aligned} & f \star \widetilde{\overset{\vee}{f}}(0,0,0,0,0) \\ = & \int\limits_{\mathbb{R}^5} \mathcal{F}(f \star \widetilde{\overset{\vee}{f}})(\xi,\mu) d\xi d\mu = \int\limits_{\mathbb{R}^5} \mathcal{F}(f \star_c \widetilde{\overset{\vee}{f}})(\xi,\mu) d\xi d\mu \\ = & \int\limits_{\mathbb{R}^4} \mathcal{F}(\widetilde{\overset{\vee}{f}})(\xi,0)\mathcal{F}(f)(\xi) d\xi = \int\limits_{\mathbb{R}^4} \overline{\mathcal{F}(f)}\,(\xi)\mathcal{F}(f)(\xi) d\xi \\ = & \int\limits_{\mathbb{R}^4} \left|\mathcal{F}f(\xi_4,\xi_3,\xi_2,\xi_1)\right|^2 d\xi_4 d\xi_3 d\xi_2 d\xi_1 \\ = & \int\limits_{G_4} \left|f(x_4,x_3,x_2,x_1)\right|^2 dx_4 dx_3 dx_2 dx_1 \end{aligned}$$

The Fourier transform can be extended to an isometry of $L^2(G_4)$ onto $L^2(\mathbb{R}^4)$. $\square$

Corollary 2.8. *In equation 2.15, replace the first f by g we obtain the Parseval formula on G_4*

$$\begin{aligned} & \int\limits_{G_4} \overline{f(x_4,x_3,x_2,x_1)} g(x_4,x_3,x_2,x_1) dx_4 dx_3 dx_2 dx_1 \\ = & \int\limits_{\mathbb{R}^4} \overline{\mathcal{F}f(\xi_4,\xi_3,\xi_2,\xi_1))} \mathcal{F}g((\xi_4,\xi_3,\xi_2,\xi_1) d\xi_4 d\xi_3 d\xi_2 d\xi_1 \end{aligned} \tag{2.16}$$

3. Tempered Fundamental solution.

If we consider the group G_4 as a subgroup of L, then $\widetilde{f} \in \mathcal{S}(G_4)$ where x_1 is fixed, and if we consider B as a subgroup of L, then $\widetilde{f} \in \mathcal{S}(B)$ where t is fixed. Denote by $\mathcal{S}_E(L)$ the space of all functions $\phi(x_4,x_3,x_2,x_1,t) \in C^\infty(L)$ such that $\phi(x_4,x_3,x_2,x_1,t) \in \mathcal{S}(G_4)$ where x_1 is fixed, and $\phi(x_4,x_3,x_2,x_1,t) \in \mathcal{S}(B)$ where t is fixed. We equip $\mathcal{S}_E(L)$ with the natural topology defined by the seminorms:.

$$\begin{aligned} \phi \to & \sup_{(x_4,x_3,x_2,x_1)\in B} |Q(x_4,x_3,x_2,x_1,t)P(D)\phi(x_4,x_3,x_2,x_1,t)| \qquad t \text{ is fixed.} \\ \phi \to & \sup_{(x_4,x_3,x_2,t)\in K} |R(x_4,x_3,x_2,x_1,t)\ S(D)\phi(x_4,x_3,x_2,x_1,t)| \qquad x_1 \text{ is fixed.} \end{aligned} \tag{3.1}$$

where P, Q, R and S run over the family of all complex polynomial in four variables. Let $\mathcal{S}_E^I(L)$ be the subspace of all functions $F \in \mathcal{S}_E(L)$, which are invariant in the sense of Remark 2.2, then we have the following result.

Proposition 3.1. *Let $u \in \mathcal{U}$ and Q_u be the invariant differential operator on the group B which is associated to u, then we have*

(i) The mapping $f \mapsto \widetilde{f}$ is a topological isomorphism of $\mathcal{S}(G_4)$ onto $\mathcal{S}_E^I(L)$.

(ii) The mapping $F \mapsto Q_u F$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image, where Q_u acts on the variables $(x_4,x_3,x_2,x_1) \in B$.

Proof. (i) In fact the map is continuous and the restriction mapping $F \mapsto RF$ on G_4 is continuous from $\mathcal{S}_E^I(L)$ into $\mathcal{S}(G_4)$ that satisfies $R\circ \sim = Id_{\mathcal{S}(G_4)}$ and $\sim \circ R = Id_{\mathcal{S}_E^I(L)}$, where $Id_{\mathcal{S}(G_4)}$ (resp. $Id_{\mathcal{S}_E^I(L)}$) is the identity mapping of $\mathcal{S}(G_4)$ (resp. $\mathcal{S}_E^I(L)$) and G_4 is considered as a subgroup of L.

To prove (ii) we refer to [15] and his famous result that is: *Any invariant differential operator on B is a topological isomorphism of $S(B)$ onto its image.* From this result, we obtain that

$$Q_u : \mathcal{S}_E(L) \to \mathcal{S}_E(L) \tag{3.2}$$

is a topological isomorphism and its restriction on $\mathcal{S}_E^I(L)$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image. Hence the theorem is proved. $\square$

In the following we will prove that every invariant differential operator on $G_4 = \mathbb{R}^3 \times \{0\} \times \mathbb{R}$ has a tempered fundamental solution. As stated in the introduction, we will consider the two invariant differential operators P_u and Q_u, the first on the group $G_4 = \mathbb{R}^3 \times \{0\} \times \mathbb{R}$ and the second on the abelien group $B = \mathbb{R}^3 \times \mathbb{R} \times \{0\}$. Our main result is:

Theorem 3.2. *Every nonzero invariant differential operator P_u on G_4 associated to $\mathcal{U}$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image*

Proof. By equation (2.4 we have for every $u \in \mathcal{U}$ and $F \in \mathcal{S}_E^I(L)$

$$\begin{aligned}
& P_uF(x_4,x_3,x_2,x_1,t) \\
= \;& u * F(x_4,x_3,x_2,x_1,t) \\
= \;& \int_{G_4} F\left[(y_4,y_3,y_2,s)^{-1}(x_4,x_3,x_2,x_1,t)\right] u(z',y',r',s')dy_4dy_3dy_2ds \\
= \;& \int_{G_4} F\left[\rho(s^{-1})(-y_4,-y_3-y_2,)(x_4,x_3,x_2),x_1,t-s)\right] u(y_4,y_3,y_2,s)dy_4dy_3dy_2ds \\
= \;& \int_{G_4} F\left[(-y_4,-y_3-y_2,)+(x_4,x_3,x_2),x_1-s,t)\right] u(y_4,y_3,y_2,s)dy_4dy_3dy_2ds \\
= \;& u *_c (x_4,x_3,x_2,x_1,t) = Q_uF(x_4,x_3,x_2,x_1,t)
\end{aligned} \tag{3.3}$$

This shows that:

$$P_uF(x_4,x_3,x_2,x_1,t) = Q_uF(x_4,x_3,x_2,x_1,t) \tag{3.4}$$

for all $(x_4,x_3,x_2,x_1,t) \in L$, where $\star$ is the convolution product on $G_4 = \mathbb{R}^3 \times \{0\} \times \mathbb{R}$ and $\star_c$ is the convolution product on the group $B = \mathbb{R}^3 \times \mathbb{R} \times \{0\}$. By Proposition 3.1 the mapping $F \mapsto Q_uF$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image, then the mapping $F \mapsto P_uF$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image. Since

$$R(P_uF)(x_4,x_3,x_2,x_1,t) = P_u(RF)(x_4,x_3,x_2,x_1,t) \tag{3.5}$$

the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S}_E^I(L) & \xrightarrow{P_u} & P_u\mathcal{S}_E^I(L) \\ \sim\Big\downarrow R & & \Big\downarrow R \\ \mathcal{S}(G_4) & \xrightarrow{P_u} & P_u\mathcal{S}(G_4) \end{array}$$

Hence the mapping $F \mapsto P_uF$ is a topological isomorphism of $\mathcal{S}(G_4)$ onto its image. □

Corollary 3.3. *Every nonzero invariant differential operator on G_4 has a tempered fundamental solution.*

Proof. The transpose ${}^tP_u$ of P_u is a continuous mapping of $\mathcal{S}'(G_4)$ onto $\mathcal{S}'(G_4)$. This means that for every tempered distribution T on G_4 there is a tempered distribution E on G_4 such that

$$P_uE = T \tag{3.6}$$

□

Indeed the Dirac measure δ belongs to $\mathcal{S}'(G_4)$.

Definition 3.4. For every $f \in D(L)$, one can define a function $\widehat{f} \in C^\infty(L)$ as follows:

$$\begin{aligned} \widehat{f}(x_4, x_3, x_2, x_1, t) &= f(\rho(x_1)(x_4, x_3, x_2), 0, x_1 + t) \\ &= (x_4 - tx_3 + \frac{1}{2}t^2x_2, x_3 - tx_2, x_2\ , 0, x_1 + t) \end{aligned} \tag{3.7}$$

for any $(x_4, x_3, x_2, x_1, t) \in L$.

Remark 3.5. The function $\widehat{f}$ is invariant in the following sense:

$$\widehat{f}((\rho(s)(x_4, x_3, x_2)), x_1 - s, t + s) = \widehat{f}(x_4, x_3, x_2, x_1, t) \tag{3.8}$$

for any $(x_4, x_3, x_2, x_1, t) \in L$ and $s \in R$.

Theorem 3.6. *For every $u \in \mathcal{U}$, one can find a distribution $T \in \mathcal{D}'(G_4)$ such that*

$$u * T = \delta_G \tag{3.9}$$

where δ_{G_4} is the Dirac measure on G_4 at the identity element of G_4 and $*$ denotes the convolution product on G_4.

Proof. Let P_ζ be the polynomial

$$\zeta \mapsto \mathcal{F}(\overset{\vee}{u})(\xi + \zeta)$$

which is obtained by translation, where $\xi = (\xi_4, \xi_3, \xi_2, \xi_1) \in \mathbb{R}^4$, and $\zeta = (\zeta_4, \zeta_3, \zeta_2, \zeta_1) \in \mathbb{C}^4$. Let T be a distribution on L defined by

$$\langle T, f\rangle = \int\limits_L \int\limits_\Omega \frac{\mathcal{F}f(\xi + \zeta, \lambda)}{\mathcal{F}(\overset{\vee}{u})(\xi + \zeta)} \Phi(P_\zeta, \xi) d\zeta d\xi d\lambda$$

for any $f \in \mathcal{D}(L)$, where Ω is a ball in $\mathbb{C}^4$ with center 0, Φ is the Hormander function [12] and $d\zeta = d\zeta_1 d\zeta_2\ d\zeta_3 d\zeta_4$ is the Lebesgue measure on $\mathbb{C}^4$. Now we can define a distribution $\widehat{T}$ on L, which is invariant in the sense of equation (3.8), as follows:

$$\left\langle \widehat{T}, f \right\rangle = \left\langle T, \widehat{f} \right\rangle = \int\limits_{L} \int\limits_{\Omega} \frac{\mathcal{F}\widehat{f}(\xi+\zeta,\lambda)}{\mathcal{F}(\overset{\vee}{u})(\xi+\zeta)} \Phi(P_\zeta,\xi) d\zeta d\xi d\lambda \tag{3.10}$$

By Hormander construction, and Lemma 2.3, we obtain for any $u \in \mathcal{U}$

$$\begin{aligned} \left\langle \widehat{u * T}, f \right\rangle &= \left\langle u * T, \widehat{f} \right\rangle = \left\langle T, \overset{\vee}{u} *_c \widehat{f} \right\rangle \\ &= \int\limits_{L} \int\limits_{\Omega} \frac{\mathcal{F}(\overset{\vee}{u} *_c \widehat{f})(\xi+\zeta,v)}{\mathcal{F}\overset{\vee}{u}(\xi+\zeta)} \Phi(P_\zeta,\xi) d\zeta d\xi dv \\ &= \int\limits_{L} \int\limits_{\Omega} \frac{\mathcal{F}(\widehat{f})(\xi+\zeta,v)\mathcal{F}(\overset{\vee}{u})(\xi+\zeta)}{\mathcal{F}(\overset{\vee}{u})(\xi+\zeta)} \Phi(P_\zeta,\xi) d\zeta d\xi dv \\ &= \int\limits_{G_4} \int\limits_{\Omega} \mathcal{F}(\widehat{f})(\xi+\zeta,0)\Phi(P_\zeta,\xi) d\zeta d\xi \\ &= \widehat{f}((0,0,0,0,0) = \left\langle \delta_L, \widehat{f} \right\rangle = \left\langle \widehat{\delta_L}, f \right\rangle \end{aligned} \tag{3.11}$$

where δ_L is the Dirac measure at the identity element of L. This gives

$$\widehat{u * T}(x_4,x_3,x_2,x_1,t) = \widehat{\delta_L}(x_4,x_3,x_2,x_1,t).$$

Consequently, we have

$$u * T(x_4,x_3,x_2,0,t) = \delta_{G_4}(x_4,x_3,x_2,0,t) \tag{3.12}$$

Hence the theorem. □

Theorem 3.7. *([1]) Every invariant differential operator onG_4 which is not identically 0 has a tempered fundamental solution.*

Proof. For each complex number s with positive real part, we can define a distribution T^s on L by:

$$\langle T^s, f \rangle = \int\limits_{\mathbb{R}^5} \left[\left| \mathcal{F}(\overset{\vee}{u})(\xi,\lambda) \right|^2 \right]^s \mathcal{F}(\widehat{f})(\xi,\lambda) d\xi d\lambda d\mu dv \tag{3.13}$$

for each $f \in \mathcal{S}(L)$. By Atiyah's theorems ([1]), the function $s \mapsto T^s$ has a meromorphic continuation in the whole complex plan, which is analytic at $s = 0$ and its value at this point is the Dirac measure on the group L. Now we can define another distribution, $\widehat{T^s}$, as follows:

$$\begin{aligned} \left\langle \widehat{T^s}, f \right\rangle &= \left\langle T^s, \widehat{f} \right\rangle \\ &= \int\limits_{\mathbb{R}^6} \left[\left| \mathcal{F}(\overset{\vee}{u})(\xi,\lambda) \right|^2 \right]^s \mathcal{F}(\widehat{f})(\xi,\lambda) d\xi d\lambda d\mu \; dv \end{aligned} \tag{3.14}$$

for any $f \in \mathcal{S}(L)$ and $s \in \mathbb{C}$, with Re $(s) \geq 0$.

Note that the distribution $\widehat{T^s}$ is invariant in the sense of equation (3.8) , and we have

$$\begin{aligned} & \left\langle u * \widehat{\widetilde{u} *_c T^s}, f \right\rangle \\ = & \left\langle u * \widetilde{u} *_c T^s, \widehat{f} \right\rangle \\ = & \left\langle T^s, \overset{\vee}{\widetilde{u}} *_c \overset{\vee}{u} * \widehat{f} \right\rangle \\ = & \int\limits_{\mathbb{R}^6} \left[\left| \mathcal{F}(\overset{\vee}{u})(\xi,\lambda) \right|^2 \right]^s \mathcal{F}(\overset{\vee}{\widetilde{u}} *_c \overset{\vee}{u} * \widehat{f})(\xi,\lambda) d\xi d\lambda \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \overset{\vee}{\widetilde{u}}(x_4,x_3,x_2,x_1) &= \overline{u(-x_4,-x_3,-x_2,-x_1)} \\ \overset{\vee}{u}(x_4,x_3,x_2,x_1) &= \overline{(x_4,x_3,x_2,x_1)^{-1}} \end{aligned}$$

and

$$\overset{\vee}{\widetilde{u}} *_c f(x_4,x_3,x_2,x_1) = \int\limits_{\mathbb{R}^4} f(x_4-a, x_3-b, x_2-c, x_1-r) \overset{\vee}{\widetilde{u}}((a,b,c,r) dadbdcdr \tag{3.16}$$

is the commutative convolution product on G_4. We get:

$$\left\langle u * \widehat{\widetilde{u} *_c T^s}, f \right\rangle = \int\limits_{\mathbb{R}^6} \left[\left| \mathcal{F}(\overset{\vee}{u})(\xi,\lambda) \right|^2 \right]^s \mathcal{F}(\widehat{f})(\xi,\lambda) d\xi d\lambda$$

hence

$$u * \widehat{\widetilde{u} *_c T^s} = \widehat{T^{s+1}} \tag{3.17}$$

In view of the invariance in equation (3.8), the restriction of the distributions $u * \widehat{\widetilde{u} *_c T^s} = \widehat{T^{s+1}}$ on the sub-group $\mathbb{R}^2 \times \{0\} \times \mathbb{R} \times \{1\} \times \mathbb{R}_+^* \simeq G_4$ are nothing but the distributions

$$u * \widetilde{u} *_c T^s = T^{s+1}. \tag{3.18}$$

The distribution T^s can be expanded around $s = -1$ in the form

$$T^s = \sum_{j=-5}^{\infty} \alpha_j (s+1)^j \tag{3.19}$$

where each α_j is a distribution on G_4. But $u * \widetilde{u} *_c T^s = T^{s+1}$ can not have a pole at $s = -1$ (since $T^{\,0} = \delta_{G_4}$) and so we must have:

$$\begin{aligned} u * \widetilde{u} *_c \alpha_j &= 0 \quad \text{for} \quad j < 0 \\ u * \widetilde{u} *_c \alpha_0 &= \delta_{G_4} \end{aligned} \tag{3.20}$$

Hence the theorem. □

4. Ideals of Group algebra $L^1(G_4)$

In the following we will establish a classification of all left ideals in the Banach algebra $L^1(G_5)$.

Proposition 4.1. *(i) The mapping* Γ *from* $\widetilde{L^1(G_4)}|_B$ *to* $\widetilde{L^1(G_4)}|_{G_4}$ *defined by*

$$\begin{aligned} \widetilde{F}|_B\ (x_4,x_3,x_2,x_1,0) &\rightarrow \Gamma(\widetilde{F}|_B\ )(x_4,x_3,x_2,0,x_1) \\ &= \widetilde{F}|_{G_4}(x_4,x_3,x_2,0,x_1) \end{aligned} \tag{4.1}$$

is a topological isomorphism.

(ii) For every $u \in L^1(G_4)$ *and* $F \in L^1(G_4)$, *we obtain*

$$\Gamma(u\ *_c \widetilde{F}|_B)(x_4,x_3,x_2,0,x_1) = u\ * \widetilde{F}|_{G_4}(x_4,x_3,x_2,0,x_1) \tag{4.2}$$

where

$$\begin{aligned} & (u *_c \widetilde{F}|_B)(x_4,x_3,x_2,x_1,0) \\ = & \int_B \widetilde{F}\,[x_4-y_4,x_3-y_3,x_2-y_2,x_1-y_1,0]\,u(y_5,y_4,y_3,y_2,y_1) \\ & dy_5dy_4dy_3dy_2dy_1 \end{aligned} \tag{4.3}$$

Proof. It is enough to see

$$\begin{aligned} & \Gamma(u\ *_c \widetilde{F}|_B)(x_4,x_3,x_2,0,x_1) \\ = & \int_B \widetilde{F}\,[x_4-y_4,x_3-y_3,x_2-y_2,-y_1,x_1]\,u(y_5,y_4,y_3,y_2,y_1) \\ & dy_5dy_4dy_3dy_2dy_1 \\ = & \int_{G_4} F\,[(\rho_1(-y_1)(x_4-y_4,x_3-y_3\ ,x_2-y_2)),x_1-y_1)]\,(y_5,y_4,y_3,y_2,y_1) \\ & dy_5dy_4dy_3dy_2dy_1 \\ = & \ u\ * \widetilde{F}|_{G_4}(x_5,x_4,x_3,0,x_2,0,x_1) \end{aligned} \tag{4.4}$$

for every $F \in L^1(G_4)$. So it is easy to show that

$$\Gamma : \widetilde{L^1(G_4)}|_B\ \rightarrow \widetilde{L^1(G_4)}|_{G_4} \tag{4.5}$$

is an topological isomorphism, and we obtain

$$\begin{aligned} \widetilde{F}|_{G_4}\ (x_4,x_3,x_2,x_1,0) &\rightarrow \Gamma^{-1}(\widetilde{F}|_{G_4}\ )(x_4,x_3,x_2,0,x_1) \\ &= \widetilde{F}|_B\ (x_4,x_3,x_2,x_1,0) \end{aligned} \tag{4.6}$$

$\square$

Now if I is a subset of $L^1(G_4)$, we denote by $\widetilde{I}$ its image by the mapping $\sim$. Let $J = \widetilde{I}\ |_B$. Our main result is:

Theorem 4.2. *Let I be a subset of $L^1(G_4)$, then the following conditions are equivalent.*
(i) *$J = \widetilde{I}\ |_B$ is an ideal in the Banach algebra $L^1(B)$.*
(ii) *I is a left ideal in the Banach algebra $L^1(G_4)$.*

Proof. (i) implies (ii) Let I be a subset of the algebra $L^1(B)$ such that $J = \widetilde{I}|_B$ is an ideal in $L^1(B)$, then we have:

$$u *_c \widetilde{I}\ |_B(x_4, x_3, x_2, x_1, 0) \subseteq \widetilde{I}\ |_B(x_4, x_3, x_2, x_1, 0) \tag{4.7}$$

for any $u \in L^1(B)$ and $(x_4, x_3, x_2, x_1) \in B$, where

$$\begin{aligned} & u *_c \widetilde{I}\ |_B(x_4, x_3, x_2, 0, x_1) \\ = & \left\{ \begin{array}{c} \int\limits_B \widetilde{f}|_B\ [x_4 - y_4, x_3 - y_3, x_2 - y_2, 0, x_1 - y_1] u(y_5, y_4, y_3, y_2, y_1) \\ dy_5 dy_4 dy_3 dy_2 dy_1,\ \ f \in\ I \end{array} \right\} \end{aligned} \tag{4.8}$$

It shows that

$$u *_c \widetilde{f}\ |_B(x_4, x_3, x_2, 0, x_1) \in \widetilde{I}\ |_B(x_4, x_3, x_2, 0, x_1) \tag{4.9}$$

for any $\widetilde{f} \in \widetilde{I}$. Then we get

$$\begin{aligned} & \Gamma(u *_c \widetilde{f}|_B)(x_5, x_4, x_3, x_2, 0, x_1, 0) \\ = & \ u\ * \widetilde{f}|_{G_4}(x_4, x_3, x_2, 0, x_1) \in \Gamma(\widetilde{I}\ |_B)(x_4, x_3, x_2, 0, x_1) \\ = & \ \widetilde{I}\ |_{G_4}(x_4, x_3, x_2, 0, x_1) = I(x_4, x_3, x_2, x_1) \end{aligned} \tag{4.10}$$

(ii) implies (i) If I is an ideal in $L^1(G_4)$, then we get

$$\begin{aligned} & u * \widetilde{I}\ |_{G_4}(x_4, x_3, x_2, 0, x_1) \\ = & \ u * I\ (x_4, x_3, x_2, x_1) \subseteq \widetilde{I}\ |_{G_4}(x_4, x_3, x_2, 0, x_1) \\ = & \ I\ (x_4, x_3, x_2, x_1) \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} & u * \widetilde{I}\ |_{G_4}(x_4, x_3, x_2, 0, x_1) \\ = & \left\{ \begin{array}{c} \int\limits_{G_4} F\left[(\rho_1(-y_1)(x_4 - y_4, x_3 - y_3, x_2 - y_2)), x_1 - y_1)\right] \\ (y_4, y_3, y_2, y_1)\ dy_5 dy_4 dy_3 dy_2 dy_1 \end{array} \right\} \\ = & \ u\ \star \widetilde{F}|_{G_5}(x_4, x_3, x_2, 0, x_1) \\ = & \ u\ \star F(x_4, x_3, x_2, x_1) \end{aligned} \tag{4.12}$$

This leads us

$$\begin{aligned} & \Gamma^{-1}(u * \widetilde{F}\ |_{G_4})(x_4, x_3, x_2, x_1, 0) \\ = & \ u *_c \widetilde{F}|_B(x_4, x_3, x_2, x_1, 0) \in \Gamma^{-1}(u * \widetilde{I}\ |_{G_4})(x_4, x_3, x_2, 0, x_1) \\ = & \ u * \widetilde{I}\ |_B(x_4, x_3, x_2, x_1, 0) \end{aligned} \tag{4.13}$$

$\square$

Corollary 4.3. *Let I be a subset of the Banach algebra $L^1(G_4)$ and $\widetilde{I}$ its image by the mapping $\sim$ such that $J = \widetilde{I}|_B$ is an ideal in $L^1(B)$, then the following conditions are verified:*

(i) J is a maximal ideal in the algebra $L^1(B)$ if and only if I is a left ideal closed in the algebra $L^1(G_4)$.

(ii) J is a closed ideal in the algebra $L^1(B)$ if and only if I is a left prime ideal in the algebra $L^1(G_4)$.

(iii) J is a maximal ideal in the algebra $L^1(B)$ if and only if I is a left maximal ideal in the algebra $L^1(G_4)$.

(iv) J is a dense ideal in the algebra $L^1(B)$ if and only if I is a left dense ideal in the algebra $L^1(G_4)$.

5. Conclusion

The noncommutative Fourier transform does exist, and is being used, in the representation theory of non-abelian compact groups. But still, from the Fourier analysis point of view, it is not really satisfactory. In this paper we show how the classical Fourier transform on $\mathbb{R}^n$ can be defined on the nilpotent Engel-Lie group G_4, which is noncommutative and non-compact, to obtain the Plancherel formula and two interesting results.

The first one is the solvability of the Lewy and Mizohata operators, see [3] and [10].

The second is the solvability of the following heat equation on the Lie group $G_4 \times \mathbb{R}$ direct product of G_4 and the real vector group $\mathbb{R}$

$$(\frac{\partial}{\partial x_1})^2 + (\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{x_1^2 \partial}{2\partial x_4})^2 = \frac{\partial}{\partial t} \tag{5.1}$$

U. Boscain, in [2], wrote that there is no general solution. Since equation (5.1) is among the elements of the enveloping algebra $\mathcal{U}$ of the group $G_4 \times \mathbb{R}$, so we obtain the existence of the fundamental solutions of this equation.

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